

# REARRANGEMENTS IN CARNOT GROUPS

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ABSTRACT. In this paper we extend the notion of rearrangement of nonnegative functions to the setting of Carnot groups. We define rearrangement with respect to a given family of anisotropic balls  $B_r$ , or equivalently with respect to a gauge  $\|x\|$ , and prove basic regularity properties of this construction. If  $u$  is a bounded nonnegative real function with compact support, we denote by  $u^*$  its rearrangement. Then, the radial function  $u^*$  is of bounded variation. In addition, if  $u$  is continuous then  $u^*$  is continuous, and if  $u$  belongs to the horizontal Sobolev space  $W_h^{1,p}$ , then  $\frac{D_h u^*(x)}{|D_h(\|x\|)|}$  is in  $L^p$ . Moreover, we found a generalization of the inequality of Pólya and Szegő

$$\int \frac{|D_h u^*|^p}{|D_h(\|x\|)|^p} dx \leq C \int |D_h u|^p dx,$$

where  $p \geq 1$ .

## 1. INTRODUCTION

Let  $u: \mathbb{R}^n \mapsto \mathbb{R}$  be a non-negative measurable real function with compact support. The rearrangement of  $u$  is the radial function  $u^*$  that has the same distribution function as  $u$  with respect to the Lebesgue measure  $\mathcal{L}^n$ . That is, for every  $\lambda > 0$  we have

$$\mathcal{L}^n(\{x: u^*(x) > \lambda\}) = \mathcal{L}^n(\{x: u(x) > \lambda\}).$$

In particular, for any non-negative Borel measurable real function  $\phi$  we have

$$\int_{\mathbb{R}^n} \phi(u^*(x)) d\mathcal{L}^n(x) = \int_{\mathbb{R}^n} \phi(u(x)) d\mathcal{L}^n(x).$$

Pólya and Szegő proved in [PS] that if  $u \in W^{1,p}(\mathbb{R}^n)$ , where  $p \geq 1$ , then so is  $u^*$  and we have the inequality

$$\int_{\mathbb{R}^n} |Du^*(x)|^p d\mathcal{L}^n(x) \leq \int_{\mathbb{R}^n} |Du(x)|^p d\mathcal{L}^n(x).$$

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If the Lebesgue measure  $\mathcal{L}^n$  is replaced by another measure we get a different rearrangement. The motivation for this article originated from a result of Schulz and Vera de Serio [SV] concerning rearrangements in  $\mathbb{R}^2$  relative to an absolutely continuous measure with respect to  $\mathcal{L}^2$  with a density  $\rho$ . The rearrangement  $u_\rho^*$  of a non-negative function  $u$  is determined by the condition

$$\int_{\{u_\rho^* > \lambda\}} \rho(x) d\mathcal{L}^2(x) = \int_{\{u > \lambda\}} \rho(x) d\mathcal{L}^2(x)$$

for every  $\lambda \geq 0$ . One of the main results in [SV] states that if  $\log \rho$  is a nonnegative sub-harmonic function in  $\mathbb{R}^2$ , then for every non-negative  $u \in W^{1,2}(\mathbb{R}^2)$  the rearrangement  $u_\rho^* \in W^{1,2}(\mathbb{R}^2)$  and we have the inequality

$$\int_{\mathbb{R}^2} |Du_\rho^*(x)|^2 d\mathcal{L}^2(x) \leq \int_{\mathbb{R}^2} |Du(x)|^2 d\mathcal{L}^2(x).$$

In this paper we define rearrangements in general spaces that include Carnot-Carathéodory spaces, and prove inequalities of Pólya-Szegő type in the case of Carnot groups.

## 2. REAL VARIABLE STRUCTURES FOR REARRANGEMENTS

We are given a family  $\{B_r\}$  of non-empty bounded open sets, “balls centered at 0”, in  $\mathbb{R}^n$  indexed by  $r > 0$  satisfying the following conditions:

$$(2.1) \quad r < s \implies B_r \subset B_s,$$

$$(2.2) \quad \bigcap_{r>0} B_r = \{0\},$$

$$(2.3) \quad \bigcup_{r>0} B_r = \mathbb{R}^n, \quad \text{and}$$

$$(2.4) \quad \bigcup_{0 < r < s} B_r = B_s.$$

We also set  $B_0 = \emptyset$ . For  $x \in \mathbb{R}^n$  we define the gauge

$$\|x\| = \inf\{r > 0 : x \in B_r\}$$

and assume that

$$(2.5) \quad x \mapsto \|x\| \quad \text{is a continuous function.}$$

It follows easily that

$$(2.6) \quad B_r = \{x : \|x\| < r\}.$$

We are also given a non-negative Borel measure  $\mu$  in  $\mathbb{R}^n$  such that the volume function

$$V(r) = \mu(B_r)$$

satisfies the following properties:

$$(2.7) \quad V(0) = \lim_{r \rightarrow 0^+} V(r) = 0,$$

$$(2.8) \quad V(\infty) = \mu(\mathbb{R}^n),$$

$$(2.9) \quad V: [0, \infty] \mapsto [0, \mu(\mathbb{R}^n)] \quad \text{is an absolutely continuous bijection.}$$

Let  $u: \mathbb{R}^n \mapsto [0, \infty]$  be a non-negative  $\mu$ -measurable function whose support has finite  $\mu$ -measure. For each  $t \geq 0$  define

$$(2.10) \quad E_u(t) = \{x \in \mathbb{R}^n \mid u(x) > t\},$$

$$(2.11) \quad \nu_u(t) = \mu(E_u(t)), \quad \text{and}$$

$$(2.12) \quad \tilde{\nu}_u(r) = \sup\{t: \nu_u(t) > V(r)\}.$$

We follow the convention  $\sup \emptyset = 0$ . We are ready for our general definition of rearrangement.

**Definition 1.** Given a family of non-empty bounded open sets  $\{B_r\}_{r>0}$  and a Borel measure  $\mu$  such that properties (2.1) through (2.9) hold, the rearrangement of a  $\mu$ -measurable function  $u: \mathbb{R}^n \mapsto [0, \infty]$  is the “radial” function  $u^*: \mathbb{R}^n \mapsto [0, \infty]$  defined by

$$u^*(x) = \tilde{\nu}_u(\|x\|).$$

The following lemma is elementary.

**Lemma 1.** *If the support of  $u$  has finite  $\mu$ -measure, then the function  $\tilde{\nu}_u$  is finite, non-increasing and continuous from the right on  $(0, \infty)$ . Moreover, we have*

$$\tilde{\nu}_u(0) = \tilde{\nu}_u(0^+) = \text{ess sup } u.$$

Observe that the equivalence

$$\nu_u(t) > V(r) \iff \mu(E_u(t)) > \mu(B_r)$$

always holds.

**Corollary 1.** *If the gauge  $x \mapsto \|x\|$  is differentiable  $\mu$ -a. e., then the gradient  $Du^*(x)$  exists  $\mu$ -a. e. and satisfies*

$$Du^*(x) = \tilde{\nu}'_u(\|x\|) \cdot D(\|x\|).$$

*Proof.* It is enough to observe that the absolute continuity of the volume function gives  $\mu(\{x: \|x\| \in A\}) = 0$  whenever  $A$  is a set of measure zero in  $\mathbb{R}$ .  $\square$

**Lemma 2.** *For every  $t \geq 0$  we have*

$$\mu(E_{u^*}(t)) = \mu(E_u(t)).$$

*Therefore  $u$  and  $u^*$  have the same distribution function with respect to the measure  $\mu$ .*

*Proof.* Let us start observing that the level set  $E_{u^*}(t)$  is the ball  $B_r$  where  $r = V^{-1}(\nu_u(t))$ . Indeed, we have

$$E_{u^*}(t) = \{u^* > t\} = \bigcup_{s>t} B_{V^{-1}(\nu_u(s))} = B_{V^{-1}(\nu_u(t))},$$

because  $V^{-1} \circ \nu_u$  is right continuous and property (2.4). The lemma follows from the following chain of equalities

$$\mu(E_{u^*}(t)) = \mu(B_r) = V(r) = V(V^{-1}(\nu_u(t))) = \nu_u(t) = \mu(E_u(t)).$$

□

**Corollary 2.** *For any non-negative Borel measurable real function  $\phi$  we have*

$$\int_{\mathbb{R}^n} \phi(u^*(x)) d\mu(x) = \int_{\mathbb{R}^n} \phi(u(x)) d\mu(x).$$

**Lemma 3.** *If  $u$  is continuous and has compact support then  $\nu_u$  is strictly decreasing on the interval  $[0, \text{ess sup } u]$  and  $V^{-1} \circ \nu_u$  is a right inverse of  $\tilde{\nu}_u$ .*

*Proof.* The proof is identical to the proof of Lemma 1.4.1 and Lemma 1.5.1 in [SV], since only properties (2.1) through (2.9) are used. □

**Theorem 1.** *If  $u$  is continuous with compact support so is  $u^*$ .*

*Proof.* Once again, the proof is identical to the proof of Theorem 1.4.4 in [SV], since only properties (2.1) through (2.9) are used. □

**Lemma 4.** *If  $u$  is continuous with compact support then*

- (i)  $\tilde{\nu}_u$  is continuous and,
- (ii) if  $\nu'_u(r) \neq 0$  for a. e.  $r \in [0, \text{ess sup } u]$  then  $\tilde{\nu}_u$  is absolutely continuous.

*Proof.* The continuity of  $\tilde{\nu}_u$  follows from the continuity of  $u^*$ . The argument for (ii) is the same as in the proof of Proposition 1.5.2 in [SV] using the absolute continuity of  $V$ . □

## 3. REARRANGEMENTS IN CARNOT GROUPS.

Consider a collection of  $m$  smooth vector fields in  $\mathbb{R}^n$

$$\{X_1, X_2, \dots, X_m\}$$

satisfying Hörmander's condition

$$\text{Rank Lie}[X_1, X_2, \dots, X_m](x) = n$$

at every  $x \in \mathbb{R}^n$ . We will also assume that the horizontal tangent space

$$T_h(x) = \text{Linear span}[X_1, X_2, \dots, X_m](x)$$

has dimension  $m \leq n$  for all  $x \in \mathbb{R}^n$ .

A piecewise smooth curve  $t \mapsto \gamma(t) \in \mathbb{R}^n$  is horizontal if its tangent vector  $\gamma'(t)$  is in  $T_h(\gamma(t))$ . The Carnot-Carathéodory distance between the points  $p$  and  $q$  is defined as follows:

$$d_{CC}(p, q) = \inf\{\text{length}(\gamma) \mid \gamma \in \Gamma\}$$

where the set  $\Gamma$  is the set of all horizontal curves  $\gamma$  such that  $\gamma(0) = p$  and  $\gamma(1) = q$ . To measure the length of a curve we use the metric in  $T_h(x)$  determined by requiring that the vector fields  $\{X_1, X_2, \dots, X_m\}$  form an orthonormal basis. We can always extend this metric to a full Riemannian metric in  $\mathbb{R}^n$  so that its volume element is the Lebesgue measure  $\mathcal{L}^n$ .

By Chow's theorem (see, for example, [BR]) any two points can be connected by a horizontal curve, which makes  $d_{CC}$  a metric on  $\mathbb{R}^n$ . A Carnot-Carathéodory ball of radius  $r$  centered at a point  $p_0$  is given by

$$B(p_0, r) = \{p \in \mathbb{R}^n : d_{CC}(p, p_0) < r\}.$$

Observe that properties (2.1) to (2.6) always hold in an arbitrary metric space if  $B_r = B(x_0, r)$  is the family of balls centered at some fixed point  $x_0$ , where 0 in property (2.2) is replaced by  $x_0$ .

Given a Borel measure  $\mu$  property (2.8) always holds and so does (2.7) if  $\mu$  is non-atomic. Property (2.9) follows easily if  $\mu$  is absolutely continuous with respect to  $\mathcal{L}^n$ .

From now on we will consider the case of a Carnot group  $\mathcal{G}$  of dimension  $n$  and homogenous dimension  $Q$  as defined, for example, in [FS]. The vector fields  $\{X_1, X_2, \dots, X_m\}$  are left-invariant so that we think of them as elements in the Lie algebra  $\mathfrak{g}$ . The Haar measure of the group is  $\mathcal{L}^n$  and we have a family of group homomorphisms  $\delta_r$  indexed by  $r > 0$ , called dilations, satisfying

$$\delta_r \circ \delta_s = \delta_{rs}.$$

The volume of a ball is given by

$$V(B(p_0, R)) = \text{constant} \cdot R^Q.$$

In order to determine a real variable structure for rearrangements we have to single out a gauge  $x \rightarrow \|x\|$  and set  $B_R = \{x \mid \|x\| < R\}$ . There are many choices of gauges which are smooth away from the origin, see [FS]. A gauge that is usually non-smooth but natural in our setting is the Carnot gauge

$$\|x\|_C = d_{CC}(x, 0).$$

We occasionally identify  $\mathcal{G}$  with the underlying space  $\mathbb{R}^n$ .

**Theorem 2.** *A Carnot group  $\mathcal{G}$  endowed with the Carnot gauge  $\|x\|_C$ , or with a smooth gauge  $x \mapsto \|x\|$  together with the Lebesgue measure  $\mathcal{L}^n$  forms a real variable rearrangement structure. That is, properties (2.1) through (2.9) hold.*

In particular Theorem (1) applies to a Carnot group endowed with an arbitrary gauge.

The horizontal gradient of a function  $u: \mathcal{G} \mapsto \mathbb{R}$  is the projection of the full gradient onto the horizontal tangent space

$$D_h u = (X_1 u)X_1 + (X_2 u)X_2 + \dots + (X_m u)X_m.$$

For  $p \geq 1$  the horizontal Sobolev space is defined by

$$W_h^{1,p}(\mathcal{G}) = \{u \in L^p(\mathcal{G}) \mid D_h u \in L^p(\mathcal{G})\}.$$

Endowed with the norm

$$\|u\|_{W_h^{1,p}(\mathcal{G})} = \|u\|_{L^p(\mathcal{G})} + \|D_h u\|_{L^p(\mathcal{G})},$$

the class  $W_h^{1,p}(\mathcal{G})$  is a Banach space (see [GN] and [L]).

The horizontal divergence  $\text{div}_h(F)$  of a horizontal vector field  $F$  ( $F(x) = \sum_{i=1}^m F^i(x)X_i(x)$ ) is defined by requiring that for every compactly supported smooth function  $\phi$  the equality

$$\int_{\mathcal{G}} \phi \text{div}_h(F) d\mathcal{L}^n = - \int_{\mathcal{G}} \langle D_h \phi, F \rangle d\mathcal{L}^n$$

holds.

Next, we recall the definition of horizontal bounded variation from [GN]. We say that  $u \in BV_h(\Omega)$  if

$$\|u\|_{BV_h(\Omega)} = \sup \left\{ \int_{\Omega} u \text{div}_h F d\mathcal{L}^n \right\} < \infty.$$

where the supremum is taken among all  $F \in C_0^\infty(\Omega, \mathfrak{g})$  such that  $\sum_{i=1}^m |F^i(x)|^2 \leq 1$ . If the function  $u$  is smooth, the horizontal bounded variation is just the  $L^1$ -norm of the length of the horizontal gradient

$$\|u\|_{BV_h(\Omega)} = \int_{\Omega} |D_h u| d\mathcal{L}^n.$$

A measurable set  $E \subset \mathcal{G}$  has finite horizontal perimeter relative to a domain  $\Omega \subset \mathcal{G}$  if  $\chi_E \in BV_h(\Omega)$  in which case we write

$$\mathcal{P}_h(E, \Omega) = \|\chi_E\|_{BV_h(\Omega)}.$$

We shall denote  $\mathcal{P}_h(E, \mathcal{G})$  simply by  $\mathcal{P}_h(E)$ .

Using the anisotropic dilations, it is easy to see that

$$(3.1) \quad \mathcal{P}_h(B_R) = R^{Q-1} \mathcal{P}_h(B_1).$$

**Theorem 3.** *Suppose that  $\mathcal{G}$  is a Carnot group endowed with a gauge so that the unit ball  $B_1$  is regular enough to have finite horizontal perimeter. Let  $u \in L^\infty(\mathcal{G})$  be a nonnegative function with compact support. Then  $u^* \in BV_h(\mathcal{G})$ .*

**Remark 1.** *The finiteness of the horizontal perimeter of a ball certainly holds for smooth gauges and also for the Carnot gauge  $\|x\|_C$  in a general Carnot group. See Remark 4.3 in [MSC].*

*Proof.* We will use integration in polar coordinates. For  $r > 0$  set

$$\phi(r) = \int_{B_r} \operatorname{div}_h F(y) d\mathcal{L}^n(y).$$

It follows from Proposition 1.15 in [FS], that there exists a Radon measure  $\sigma$  on  $\partial B_1$  such that

$$\phi'(r) = \int_{\partial B_r} \operatorname{div}_h F(y) d\sigma_r(y),$$

where  $d\sigma_r$  is the image of the measure  $d\sigma$  under the dilation  $x \mapsto \delta_r(x)$ .

Let  $F \in C_0^\infty(\mathcal{G}, \mathfrak{g})$  be a test field satisfying  $|F| \leq 1$ . We have

$$\begin{aligned} \int_{\mathcal{G}} u^*(x) \operatorname{div}_h F(x) d\mathcal{L}^n(x) &= \int_0^\infty \int_{\partial B_r} u^*(y) \operatorname{div}_h F(y) d\sigma_r(y) dr \\ &= \int_0^\infty \tilde{\nu}_u(r) \left( \int_{\partial B_r} \operatorname{div}_h F(y) d\sigma_r(y) \right) dr \\ &= \int_0^\infty \tilde{\nu}_u(r) \phi'(r) dr \\ &= - \int_0^\infty \phi(r) d\tilde{\nu}_u(r) \end{aligned}$$

Observe next that  $\phi(R) \leq \mathcal{P}_h(B_R)$ . Using (3.1) we get

$$\phi(R) \leq \mathcal{P}_h(B_1) \cdot R^{Q-1}.$$

Since  $-d\tilde{\nu}_u$  is a positive measure we see that

$$\int_{\mathcal{G}} u^*(x) \operatorname{div}_h F(x) d\mathcal{L}^n x \leq -\mathcal{P}_h(B_1) \int_0^\infty r^{Q-1} d\tilde{\nu}_u(r) < \infty.$$

Therefore  $u^* \in BV_h(\mathcal{G})$ . □

A basic result that we shall use several times is the isoperimetric inequality for horizontal perimeters. (See Garofalo and Nhieu [GN] and Franchi, Gallot and Wheeden [FGW]). Recall that  $Q$  is the homogeneous dimension of our Carnot group. For every set  $E$  with finite horizontal perimeter  $\mathcal{P}_h(E) < \infty$  we have

$$(3.2) \quad (\mathcal{L}^n(E))^{\frac{Q-1}{Q}} \leq C_{\text{iso}} \mathcal{P}_h(E),$$

where  $C_{\text{iso}}$  is a constant independent of the set  $E$ . Inequality (3.2) follows from Theorem 1.18 in [GN] by taking the domain  $\Omega$  in this theorem to be a metric ball of radius  $R$  and letting  $R \rightarrow \infty$ .

Garofalo and Nhieu [GN] and Franchi, Gallot and Wheeden [FGW] also extended Federer's classical co-area formula to the subelliptic setting.

**Horizontal Co-area Formula:** Let  $\Omega \subset \mathcal{G}$  be a domain and let  $u \in BV_h(\Omega)$ . Then, for a. e.  $t \in \mathbb{R}$ , the set

$$E_u(t) = \{x \in \mathcal{G} \mid u(x) > t\}$$

has finite horizontal perimeter relative to  $\Omega$  and the co-area formula

$$(3.3) \quad \|u\|_{BV_h(\Omega)} = \int_{\mathbb{R}} \mathcal{P}_h(E_u(t), \Omega) dt.$$

holds. Conversely, for  $u \in L^1(\Omega)$ , if for a. e.  $t \in \mathbb{R}$  the set  $E_u(t)$  has finite horizontal perimeter relative to  $\Omega$ , and

$$\int_{\mathbb{R}} \mathcal{P}_h(E_u(t), \Omega) dt < \infty,$$

then  $u \in BV_h(\Omega)$  and we have (3.3).

For a function  $u \in BV_h(\mathcal{G})$  recall the variation measure  $\|D_h u\|$  defined by

$$\|D_h u\|(U) = \sup \left\{ \int_{\mathcal{G}} u \operatorname{div}_h F d\mathcal{L}^n \right\},$$

where  $U$  is an open set in  $\mathcal{G}$  and the supremum is taken with respect to  $F \in C_0^\infty(U, \mathfrak{g})$  such that  $\sum_{i=1}^m |F^i(x)|^2 \leq 1$ . With this notation,



the horizontal perimeter of a set  $E$  relative to a domain  $\Omega$  is just  $\|D_h \chi_E\|(\Omega)$ . We can also write (3.3) as follows

$$(3.4) \quad \|D_h u\|(\Omega) = \int_{\mathbb{R}} \mathcal{P}_h(E_u(t), \Omega) dt.$$

Hence, for any nonnegative Borel measurable  $g$  we have

$$(3.5) \quad \int_{\mathcal{G}} g d\|D_h u\| = \int_{\mathbb{R}} \int_{\mathcal{G}} g d\mathcal{P}_h(E_u(t)) dt.$$

**Lemma 5.** *Consider a function  $u$  in the horizontal Sobolev space  $W_h^{1,1}(\mathcal{G})$ . Given a number  $t < \|u\|_{\infty}$  and  $s > t$  we have*

$$\mathcal{L}^n(u^{-1}(t, s)) > 0.$$

*Proof.* Suppose that  $\mathcal{L}^n(u^{-1}(t, s)) = 0$ . Let  $g$  be a smooth function with compact support bounded by 1. Write

$$\begin{aligned} \int_{\mathcal{G}} X_i u \cdot g d\mathcal{L}^n &= \int_{u \leq t} X_i u \cdot g d\mathcal{L}^n + \int_{u \geq s} X_i u \cdot g d\mathcal{L}^n \\ &= - \int_{\mathcal{G}} X_i (t - u)^+ \cdot g d\mathcal{L}^n + \int_{\mathcal{G}} X_i (u - s)^+ \cdot g d\mathcal{L}^n \\ &= \int_{u \leq t} (t - u) \cdot X_i g d\mathcal{L}^n + \int_{u \geq s} (s - u) \cdot X_i g d\mathcal{L}^n, \end{aligned}$$

where we have used the lattice properties of  $W_h^{1,1}(\mathcal{G})$  (Lemma 3.5 in [GN]) and integration by parts.

On the other hand we also have

$$\begin{aligned} \int_{\mathcal{G}} X_i u \cdot g d\mathcal{L}^n &= \int_{\mathcal{G}} X_i (u - t) \cdot g d\mathcal{L}^n \\ &= - \int_{\mathcal{G}} (u - t) \cdot X_i g d\mathcal{L}^n \\ &= \int_{u \leq t} (t - u) \cdot X_i g d\mathcal{L}^n + \int_{u \geq s} (t - u) \cdot X_i g d\mathcal{L}^n. \end{aligned}$$

We conclude that

$$(3.6) \quad \int_{u \geq s} X_i g d\mathcal{L}^n = 0.$$

If we call  $E = \{u \geq s\}$ , it follows from (3.6) that  $\mathcal{P}_h(E) = 0$ . Since sets of horizontal perimeter zero have  $\mathcal{L}^n$  measure zero as it follows from the horizontal isoperimetric inequality (3.2), we deduce that  $u(x) \leq t$  for a. e.  $x \in \mathcal{G}$  contradicting the hypothesis  $t < \|u\|_{\infty}$ .  $\square$

**Theorem 4.** *If  $u \in W_h^{1,1}(\mathcal{G}) \cap L^{\infty}$  is a nonnegative function with compact support, then  $u^* \in W_h^{1,1}(\mathcal{G})$ .*

*Proof.* Once we have Lemma 5, the isoperimetric inequality (3.2) and the coarea formula (3.5), the proof is identical to the proof of Theorem 1.6.7 in [SV].  $\square$

#### 4. ENERGY INEQUALITY FOR $p = 1$

We assume in this section the finiteness of the horizontal perimeter of ball. We begin with a lemma showing a quasi-monotonicity property of the horizontal perimeter under rearrangements.

**Lemma 6.** *There exists a constant  $C_{per} \geq 1$  such that for all sets  $E \subset \mathcal{G}$  we have*

$$(4.1) \quad \mathcal{P}_h(E^*) \leq C_{per} \mathcal{P}_h(E),$$

where  $E^*$  is the ball  $B_R$  satisfying  $\mathcal{L}^n(B_R) = \mathcal{L}^n(E)$ .

*Proof.* Observe that if  $B_R$  is a ball, then

$$\mathcal{L}^n(B_R) = R^Q \mathcal{L}^n(B_1)$$

and

$$\mathcal{P}_h(B_R) = R^{Q-1} \mathcal{P}_h(B_1).$$

Therefore, we have the following equality for balls

$$(\mathcal{L}^n(B_R))^{\frac{Q-1}{Q}} = C_0 \mathcal{P}_h(B_R),$$

where we have set

$$(4.2) \quad C_0 = \frac{(\mathcal{L}^n(B_1))^{\frac{Q-1}{Q}}}{\mathcal{P}_h(B_1)}.$$

We now combine (4.2) with the isoperimetric inequality (3.2) as follows:

$$\mathcal{P}_h(E^*) = \frac{1}{C_0} (\mathcal{L}^n(E^*))^{\frac{Q-1}{Q}} = \frac{1}{C_0} (\mathcal{L}^n(E))^{\frac{Q-1}{Q}} \leq \frac{C_{iso}}{C_0} \mathcal{P}_h(E).$$

We conclude that

$$(4.3) \quad C_{per} \leq \frac{C_{iso}}{C_0}.$$

$\square$

Note that if  $u \in BV_h(\mathcal{G})$ , we have

$$(4.4) \quad \mathcal{P}_h(E_{u^*}(t)) \leq C_{per} \mathcal{P}_h(E_u(t)).$$

This follows from the fact that the level set  $E_{u^*}(t)$  is the ball  $B_R$  where  $R = V^{-1}(\nu_u(t))$  and the previous Lemma.

**Theorem 5.** *Suppose that  $\mathcal{G}$  is a Carnot group endowed with a gauge so that the unit ball  $B_1$  is regular enough to have finite horizontal perimeter. Then, for all nonnegative  $u \in W_h^{1,1}(\mathcal{G})$  with compact support we have the inequality*

$$(4.5) \quad \int_{\mathcal{G}} |D_h u^*(x)| d\mathcal{L}^n(x) \leq C_{per} \int_{\mathcal{G}} |D_h u(x)| d\mathcal{L}^n(x)$$

*In particular, it follows that  $u^* \in W_h^{1,1}(\mathcal{G})$ .*

*Proof.* Using the co-area formula twice, we get:

$$\begin{aligned} \int_{\mathcal{G}} |D_h u^*(x)| d\mathcal{L}^n(x) &= \int_0^\infty \left( \int_{\mathcal{G}} d\mathcal{P}_h(E_{u^*}(t)) \right) dt \\ &= \int_0^\infty \mathcal{P}_h(E_{u^*}(t)) dt \\ &\leq C_{per} \int_0^\infty \mathcal{P}_h(E_u(t)) dt \\ &= C_{per} \int_{\mathcal{G}} |D_h u(x)| d\mathcal{L}^n(x). \end{aligned}$$

□

## 5. ENERGY INEQUALITY FOR $p \geq 1$

In this section we need to assume that the mapping  $x \mapsto \|x\|$  is differentiable a. e. This is certainly the case for smooth gauges and also for the Carnot gauge. In fact, Monti and Serra Cassano [MSC] have recently established that the Carnot gauge  $\|x\|_C$  is differentiable a. e. and satisfies

$$(5.1) \quad |D_h(\|x\|_C)| = 1 \text{ for a. e. } x \in \mathcal{G}.$$

The key step to obtain the rearrangement energy inequality for  $p \geq 1$  is an integrability property of

$$\frac{1}{|D_h(\|x\|)|}.$$

**Lemma 7.** *For an arbitrary a. e. differentiable gauge in a Carnot group we have*

$$\int_{\mathcal{G}} \frac{1}{|D_h(\|x\|)|} d\mathcal{P}_h(B_R) \leq R^{Q-1} \sigma(B_1),$$

*where  $\sigma$  is the Radon measure supported on  $\partial B_1$  that is used in integration in polar coordinates.*

*Proof.* Let us begin by observing that both sides of the inequality are homogeneous of degree  $Q - 1$ . Therefore, it is enough to prove the lemma when  $R = 1$ . We will write  $B$  for  $B_1$ .

Let  $U$  be an open set and compute

$$\begin{aligned} \int_U d\mathcal{P}_h(B)(x) &= \mathcal{P}_h(B)(U) \\ &= \sup \left\{ \int_B \operatorname{div}_h F(x) d\mathcal{L}^n(x) \mid F \in C_0^\infty(U, \mathfrak{g}), |F| \leq 1 \right\} \\ &= \sup \left\{ \int_{\partial B} \langle F(x), D_h(\|x\|) \rangle d\sigma(x) \mid F \in C_0^\infty(U, \mathfrak{g}), |F| \leq 1 \right\}, \end{aligned}$$

where the last equality follows from

$$(5.2) \quad \int_B \operatorname{div}_h F(x) d\mathcal{L}^n(x) = \int_{\partial B} \langle F(x), D_h(\|x\|) \rangle d\sigma(x).$$

To prove this formula, consider the continuous function  $\phi_\epsilon(r)$  which takes the value 1 for  $r < 1 - \epsilon$ , vanishes for  $r > 1 + \epsilon$  and is linear otherwise. From the definition of horizontal divergence we get

$$\begin{aligned} \int_{B_{1+\epsilon}} \phi_\epsilon(\|x\|) \operatorname{div}_h F(x) d\mathcal{L}^n(x) &= - \int_{B_{1+\epsilon}} \langle D_h(\phi_\epsilon(\|x\|)), F(x) \rangle d\mathcal{L}^n(x) \\ &= - \int_{B_{1+\epsilon}} \phi'_\epsilon(\|x\|) \langle D_h(\|x\|), F(x) \rangle d\mathcal{L}^n(x) \\ &= \frac{1}{2\epsilon} \int_{B_{1+\epsilon} \setminus B_{1-\epsilon}} \langle D_h(\|x\|), F(x) \rangle d\mathcal{L}^n(x) \\ &= \frac{1}{2\epsilon} \int_{1-\epsilon}^{1+\epsilon} \int_{\partial B} \langle D_h(\|x\|), F(\delta_t(x)) \rangle d\sigma(x) t^{Q-1} dt, \end{aligned}$$

where we have used the fact that  $D_h(\|x\|)$  is homogenous of degree zero. Letting  $\epsilon \rightarrow 0$  we obtain (5.2).

Thus, we have

$$\int_U d\mathcal{P}_h(B)(x) \leq \int_{\partial B \cap U} |D_h(\|x\|)| d\sigma(x).$$

Since this is an inequality between two Radon measures, we conclude that for  $f$  nonnegative and Borel measurable

$$\int_{\mathcal{G}} f(x) d\mathcal{P}_h(B)(x) \leq \int_{\partial B} f(x) |D_h(\|x\|)| d\sigma(x).$$

The lemma follows by applying this formula to

$$f(x) = \frac{1}{|D_h(\|x\|)|}.$$

□

Next, we need to discuss a technical point. It follows from Corollary 1 that

$$D_h u^*(x) = \tilde{\nu}'_u(\|x\|) \cdot D_h(\|x\|).$$

Since  $|\tilde{\nu}'_u|$  is measurable with respect to the  $\sigma$ -algebra generated by  $\tilde{\nu}_u$ , there exists a Borel measurable function  $\Psi: \mathbb{R}^+ \cup \{0\} \rightarrow \mathbb{R}^+ \cup \{0\}$  such that

$$|\tilde{\nu}'_u|(s) = \Psi(\tilde{\nu}_u(s)).$$

Therefore, using the equality  $\tilde{\nu}_u(\|x\|) = u^*(x)$  we can write

$$(5.3) \quad |D_h u^*(x)| = \Psi(u^*(x)) \cdot |D_h(\|x\|)|.$$

Observe that the factor  $\Psi(u^*(x))$  is radial but this is not, in general, the case of the second factor  $|D_h(\|x\|)|$ . Nevertheless, with the choice of the Carnot gauge this factor is identically 1 and  $|D_h u^*(x)|$  is indeed a radial function.

One could possibly think that  $|D_h u^*(x)|$  is measurable with respect to the  $\sigma$ -algebra generated by  $u^*$  so that we had  $|D_h u^*(x)| = \Phi(u^*(x))$  for some Borel function  $\Phi$ . This is actually the case for the Carnot gauge, but it is not for other gauges for which  $|D_h(\|x\|)|$  is not radial. This is why we need Lemma (7).

**Theorem 6.** *Let  $\mathcal{G}$  be a Carnot group endowed with an a. e. differentiable gauge. Let  $u \in W_h^{1,p}(\mathcal{G})$  be a nonnegative function with compact support and  $p \geq 1$ . There exists a positive constant  $C_{sym}$  such that we have the inequality*

$$(5.4) \quad \int_{\mathcal{G}} \frac{|D_h u^*(x)|^p}{|D_h(\|x\|)|^p} d\mathcal{L}^n(x) \leq (C_{sym})^p \int_{\mathcal{G}} |D_h u(x)|^p d\mathcal{L}^n(x).$$

*In fact, we may take*

$$C_{sym} = \frac{\sigma(B_1)}{\mathcal{P}_h(B_1)} C_{per}.$$

*Proof.* Let  $\Psi_k = \min\{k, \Psi\}$  be the truncation of  $\Psi$  at level  $k$ . By the coarea formula (3.5) we get:

$$\begin{aligned} \int_{\mathcal{G}} \Psi_k^p(u^*(x)) d\mathcal{L}^n(x) &\leq \int_{\mathcal{G}} \frac{\Psi_k^{p-1}(u^*(x))}{|D_h(\|x\|)|} |D_h u^*(x)| d\mathcal{L}^n(x) \\ &= \int_0^\infty \int_{\mathcal{G}} \frac{\Psi_k^{p-1}(u^*(x))}{|D_h(\|x\|)|} d\mathcal{P}_h(E_u^*(t)) dt \\ &= \int_0^\infty \Psi_k^{p-1}(t) \left[ \int_{\mathcal{G}} \frac{1}{|D_h(\|x\|)|} d\mathcal{P}_h(E_u^*(t)) \right] dt \end{aligned}$$

At this time we use the key lemma (7) together with lemma (6) and another application of the coarea formula (3.5) and corollary (2) to get:

$$\begin{aligned} \int_{\mathcal{G}} \Psi_k^p(u^*(x)) d\mathcal{L}^n(x) &\leq C_{sym} \int_0^\infty \Psi_k^{p-1}(t) \left[ \int_{\mathcal{G}} d\mathcal{P}_h(E_u(t)) \right] dt \\ &= C_{sym} \int_0^\infty \int_{\mathcal{G}} \Psi_k^{p-1}(u(x)) d\mathcal{P}_h(E_u(t)) dt \\ &= C_{sym} \int_{\mathcal{G}} \Psi_k^{p-1}(u(x)) |D_h u(x)| d\mathcal{L}^n(x) \\ &\leq C_{sym} \left( \int_{\mathcal{G}} \Psi_k^p(u(x)) d\mathcal{L}^n(x) \right)^{\frac{p-1}{p}} \left( \int_{\mathcal{G}} |D_h u(x)|^p d\mathcal{L}^n(x) \right)^{\frac{1}{p}} \\ &= C_{sym} \left( \int_{\mathcal{G}} \Psi_k^p(u^*(x)) d\mathcal{L}^n(x) \right)^{\frac{p-1}{p}} \left( \int_{\mathcal{G}} |D_h u(x)|^p d\mathcal{L}^n(x) \right)^{\frac{1}{p}}. \end{aligned}$$

Hence, we obtain

$$\left( \int_{\mathcal{G}} \Psi_k^p(u^*(x)) d\mathcal{L}^n(x) \right)^{\frac{1}{p}} \leq C_{sym} \left( \int_{\mathcal{G}} |D_h u(x)|^p d\mathcal{L}^n(x) \right)^{\frac{1}{p}},$$

letting  $k \rightarrow \infty$  and using (5.3) we end the proof.  $\square$

For the Carnot gauge, we can prove a more traditional version of the energy inequality. In this case  $|D_h u^*(x)|$  is radial and from (5.3) it can be written in the form

$$(5.5) \quad |D_h u^*(x)| = \Psi(u^*(x)).$$

**Theorem 7.** *Let  $\mathcal{G}$  be a Carnot group endowed with the Carnot gauge. Let  $u \in W_h^{1,p}(\mathcal{G})$  be a nonnegative function with compact support and*

$p \geq 1$ . Then, we have the inequality

$$(5.6) \quad \int_{\mathcal{G}} |D_{\mathbf{h}} u^*(x)|^p d\mathcal{L}^n(x) \leq (C_{per})^p \int_{\mathcal{G}} |D_{\mathbf{h}} u(x)|^p d\mathcal{L}^n(x).$$

In particular, it follows that  $u^* \in W_{\mathbf{h}}^{1,p}(\mathcal{G})$ .

*Proof.* Let  $\Psi_k = \min\{k, \Psi\}$  be the truncation of  $\Psi$  at level  $k$ . By the coarea formula (3.5) we get:

$$\begin{aligned} \int_{\mathcal{G}} \Psi_k^p(u^*(x)) d\mathcal{L}^n(x) &\leq \int_{\mathcal{G}} \Psi_k^{p-1}(u^*(x)) |D_{\mathbf{h}} u^*(x)| d\mathcal{L}^n(x) \\ &= \int_0^\infty \int_{\mathcal{G}} \Psi_k^{p-1}(u^*(x)) d\mathcal{P}_{\mathbf{h}}(E_u^*(t)) dt \\ &= \int_0^\infty \Psi_k^{p-1}(t) \left[ \int_{\mathcal{G}} d\mathcal{P}_{\mathbf{h}}(E_u^*(t)) \right] dt \end{aligned}$$

Next, we use lemma (6) together with another application of the coarea formula (3.5) and (5.5) to repeat the arguments of the second part of the proof of Theorem (6) to end the proof.  $\square$

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