PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 00, Number 0, Pages 000–000
S 0002-9939(XX)0000-0

# SUBELLIPTIC CORDES ESTIMATES

ANDRÁS DOMOKOS AND JUAN J. MANFREDI

Abstract. We prove Cordes type estimates for subelliptic linear partial differential operators in non-divergence form with measurable coefficients in the Heisenberg group. As an application we establish interior horizontal $W^{2,2}$-regularity for p-harmonic functions in the Heisenberg group $\mathbb{H}^1$ for the range $\frac{\sqrt{17}-1}{2} \le p < \frac{5+\sqrt{5}}{2}$.

## 1. Introduction

The main goal of this paper is to prove some estimates of Cordes type for subelliptic partial differential operators in non-divergence form with measurable coefficients in the Heisenberg group, including the linearized p-Laplacian. To show the applicability of our methods let us state the following theorem that constitutes a special case of our results.

**Theorem 1.1.** *Let* $\frac{\sqrt{17}-1}{2} \le p < \frac{5+\sqrt{5}}{2}$*. Then any p-harmonic function in the Heisenberg group* $\mathbb{H}^1$ *initially in* $HW^{1,p}_{\mathrm{loc}}$ *is in* $HW^{2,2}_{\mathrm{loc}}$*.*

We build on previous regularity results obtained by Marchi [7, 8] and extended by the first author [3], which give non-uniform bounds of the $HW^{2,2}$ (or $HW^{2,p}$) norm of the approximate p-harmonic functions. Using the Cordes condition [2, 11] and Strichartz's spectral analysis [10] we establish $HW^{2,2}$ estimates for linear subelliptic partial differential operators with measurable coefficients. As an application we obtain uniform $HW^{2,2}$ bounds for the approximate p-harmonic functions for p in a range that depends on the dimension of the Heisenberg group $\mathbb{H}^n$.

Consider the Heisenberg group $\mathbb{H}^n$; that is $\mathbb{R}^{2n+1}$ with the group multiplication

$$(x_1,...,x_{2n},t)\cdot(y_1,...,y_{2n},u) = (x_1+y_1,...,x_{2n}+y_{2n},t+u-\frac{1}{2}\sum_{i=1}^{n}(x_i y_{n+i}-x_{n+i}y_i)) .$$

For $i \in \{1,...,n\}$ consider the vector fields

$$X_i = \frac{\partial}{\partial x_i} - \frac{x_{n+i}}{2}\frac{\partial}{\partial t} , \quad X_{n+i} = \frac{\partial}{\partial x_{n+i}} + \frac{x_i}{2}\frac{\partial}{\partial t} , \quad T = \frac{\partial}{\partial t} .$$

The nontrivial commutators are $[X_i, X_{n+i}] = T$, otherwise $[X_i, X_j] = 0$.
Let $\Omega \subset \mathbb{H}^n$ be a domain. Consider the following Sobolev space with respect to the horizontal vector fields $X_i$ as

$$HW^{2,2}(\Omega) = \{u \in L^2(\Omega) \ : \ X_i X_j u \in L^2(\Omega) \, , \text{ for all } i,j \in \{1,...,2n\}\}$$

---

2000 *Mathematics Subject Classification.* 35H20, 35J70.

*Key words and phrases.* Cordes conditions, subelliptic equations, p-Laplacian.
The authors were partially supported by NSF award DMS-0100107.

endowed with the inner-product

$$(u,v)_{HW^{2,2}(\Omega)} = \int_\Omega \Big( u(x)v(x) + \sum_{i,j=1}^{2n} X_iX_ju(x)\cdot X_iX_jv(x)\Big)dx\,.$$

$HW^{2,2}(\Omega)$ is a Hilbert space and let $HW_0^{2,2}(\Omega)$ be the closure of $C_0^\infty(\Omega)$ in this Hilbert space.

We denote by $X^2u$ the matrix of second order horizontal derivatives whose entries are $(X^2u)_{ij} = X_j(X_iu)$, and by $\Delta_H u = \sum_{i=1}^{2n} X_iX_iu$ the subelliptic Laplacian associated to the horizontal vector fields $X_i$.

**Lemma 1.1.** *For all $u \in HW_0^{2,2}(\Omega)$ we have*

$$||X^2u||_{L^2(\Omega)} \le c_n ||\Delta_H u||_{L^2(\Omega)}\,,$$

*where*

$$c_n = \sqrt{1+\frac{2}{n}}\,.$$

*The constant $c_n$ is sharp when $\Omega = \mathbb{H}^n$.*

*Proof.* We follow the spectral analysis of $\Delta_H$ developed by Strichartz [10]. Let us recall the fact that $-\Delta_H$ and $iT$ commute, and share the same system of eigenvectors

$$\begin{aligned}\Phi_{\lambda,k,l}(z,t) \quad = \quad & \frac{\lambda^n}{(2\pi)^{n+1}(n+2k)^{n+1}}\cdot \exp\left(-\frac{il\lambda t}{n+2k}\right) \\ & \cdot \exp\left(-\frac{\lambda|z|^2}{4(n+2k)}\right)\cdot L_k^{n-1}\left(\frac{\lambda|z|^2}{2(n+2k)}\right)\,,\end{aligned}$$

where $l = \pm 1$, $k \in \{0,1,2,...\}$ and $L_k^{n-1}$ is the Laguerre polynomial

$$L_k^{n-1}(t) = \frac{e^t}{t^{n-1}}\cdot\frac{1}{k!}\cdot\frac{d^k}{dt^k}\left(e^{-t}t^{k+n-1}\right)\,.$$

For the eigenvalues, we have the following relations

$$iTu*\Phi_{\lambda,k,l} \quad = \quad \frac{l\lambda}{n+2k}u*\Phi_{\lambda,k,l} \tag{1.1}$$

$$-\Delta_H u*\Phi_{\lambda,k,l} \quad = \quad \lambda u*\Phi_{\lambda,k,l}\,, \tag{1.2}$$

where $*$ denotes the group convolution. Therefore, the spectral decomposition of $\Delta_H u$ for $u \in C_0^\infty(\Omega)$, the Plancherel formula, and relations (1.1)-(1.2) give

$$\begin{aligned}||\Delta_H u||^2_{L^2(\Omega)} \quad &= \quad 2\pi\sum_{k=0}^{\infty}\sum_{l=\pm1}(n+2k)\int_0^\infty\int_{\mathbb{C}^n}|\Delta_H u*\Phi_{\lambda,k,l}(z,0)|^2\,dzd\lambda \\ &= \quad 2\pi\sum_{k=0}^{\infty}\sum_{l=\pm1}(n+2k)\int_0^\infty\int_{\mathbb{C}^n}\left|\frac{n+2k}{l}iTu*\Phi_{\lambda,k,l}(z,0)\right|^2 dzd\lambda \\ &\ge \quad n^2||Tu||^2_{L^2}(\Omega)\end{aligned}$$

Therefore, for all $u \in C_0^\infty(\Omega)$ we have

$$||Tu||_{L^2(\Omega)} \le \frac{1}{n}||\Delta_H u||_{L^2(\Omega)}\,. \tag{1.3}$$

In the following we will use the fact that the formal adjoint of $X_k$ is $-X_k$. Let $u \in C_0^\infty(\Omega)$. For $k \in \{1, ..., n\}$ and $j \neq k+n$, $X_k$ and $X_j$ commute, therefore

$$\int_\Omega (X_k X_j u(x))^2 \, dx = \int_\Omega X_k X_k u(x) \cdot X_j X_j u(x) dx\,.$$

For $j = k+n$ we have

$$\begin{aligned}
&\int_\Omega (X_k X_j u(x))^2 \, dx = \int_\Omega X_k X_j u(x) \cdot (X_j X_k u(x) + Tu(x)) \, dx \\
= &\int_\Omega X_k X_j u(x) \cdot X_j X_k u(x) \, dx + \int_\Omega X_k X_j u(x) \cdot Tu(x) \, dx \\
= &-\int_\Omega X_j u(x) \cdot X_k X_j X_k u(x) \, dx + \int_\Omega X_k X_j u(x) \cdot Tu(x) \, dx \\
= &-\int_\Omega X_j u(x) \cdot (X_j X_k + T) \, X_k u(x) \, dx + \int_\Omega X_k X_j u(x) \cdot Tu(x) \, dx \\
= &-\int_\Omega X_j u(x) \cdot X_j X_k X_k u(x) \, dx + 2\int_\Omega X_k X_j u(x) \cdot Tu(x) \, dx \\
= &\int_\Omega X_k X_k u(x) \cdot X_j X_j u(x) \, dx + 2\int_\Omega X_k X_j u(x) \cdot Tu(x) \, dx\,.
\end{aligned}$$

Similarly,

$$\begin{aligned}
&\int_\Omega (X_j X_k u(x))^2 \, dx \\
= &\int_\Omega X_k X_k u(x) \cdot X_j X_j u(x) \, dx - 2\int_\Omega X_j X_k u(x) \cdot Tu(x) \, dx\,.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&||X^2 u||^2_{L^2(\Omega)} = \sum_{k,j=1}^{2n} ||X_k X_j u||^2_{L^2(\Omega)} = \\
= &\sum_{k,j=1}^{2n} \int_\Omega X_k X_k u(x) \cdot X_j X_j u(x) \, dx + 2\sum_{k=1}^{n} \int_\Omega [X_k, X_{k+n}] u(x) \cdot Tu(x) \, dx \\
= &\int_\Omega \left( \sum_{k=1}^{2n} X_k X_k u(x) \right)^2 \, dx + 2n \int_\Omega (Tu(x))^2 \, dx \\
\leq &\left(1 + 2n\frac{1}{n^2}\right) ||\Delta_H u||^2_{L^2(\Omega)} = \left(1 + \frac{2}{n}\right) ||\Delta_H u||^2_{L^2(\Omega)}\,.
\end{aligned}$$

The constant $\sqrt{1+\frac{2}{n}}$ is sharp when $\Omega = \mathbb{H}^n$, because for $v = \Phi_{\lambda,0,1}$ we have $Tv = \frac{i}{n}\Delta_H v$. □

## 2. Cordes conditions for second order subelliptic PDE operators in non-divergence forms with measurable coefficients

Let us consider now

$$\mathcal{A}u = \sum_{i,j=1}^{2n} a_{ij}(x) X_i X_j u$$

where the functions $a_{ij} \in L^\infty(\Omega)$. Let us denote by $A = (a_{ij})$ the $2n \times 2n$ matrix of coefficients.

**Definition 2.1.** [2, 11] We say that $A$ satisfies the Cordes condition $K_{\varepsilon,\sigma}$ if there exists $\varepsilon \in (0, 1]$ and $\sigma > 0$ such that

$$0 < \frac{1}{\sigma} \leq \sum_{i,j=1}^{2n} a_{ij}^2(x) \leq \frac{1}{2n-1+\varepsilon} \left( \sum_{i=1}^{2n} a_{ii}(x) \right)^2 , \text{ a.e. } x \in \Omega \,. \tag{2.1}$$

**Theorem 2.1.** *Let $0 < \varepsilon \leq 1$, $\sigma > 0$ such that $\gamma = \sqrt{1-\varepsilon}\, c_n < 1$ and $A$ satisfies the Cordes condition $K_{\varepsilon,\sigma}$. Then for all $u \in HW_0^{2,2}(\Omega)$ we have*

$$||X^2 u||_{L^2} \leq \sqrt{1 + \frac{2}{n}} \, \frac{1}{1-\gamma} \, ||\alpha||_{L^\infty} ||\mathcal{A} u||_{L^2} \,, \tag{2.2}$$

*where*

$$\alpha(x) = \frac{\langle A(x), I\rangle}{||A(x)||^2} \,.$$

*Proof.* We denote by $I$ the identity $2n \times 2n$ matrix, by $\langle A, B\rangle = \sum_{i,j=1}^{2n} a_{ij} b_{ij}$ the inner product and by $||A|| = \sqrt{\sum_{i,j=1}^{2n} a_{ij}^2}$ the Euclidean norm in $\mathbb{R}^{2n\times 2n}$ for matrices $A$ and $B$. The Cordes condition $K_{\varepsilon,\sigma}$ implies that

$$\frac{\langle A(x), I\rangle^2}{||A(x)||^2} \geq 2n - (1-\varepsilon) \tag{2.3}$$

for all $x \in \Omega' \subset \Omega$, where the Lebesgue measure of $\Omega \setminus \Omega'$ is 0.
Let be now $x \in \Omega'$ arbitrary, but fixed. Consider the quadratic polynomial

$$P(\alpha) = ||A(x)||^2 \alpha^2 - 2\langle A(x), I\rangle \alpha + 2n - (1-\varepsilon) \,.$$

Inequality (2.3) shows that

$$\min_{\alpha\in\mathbb{R}} P(\alpha) = P\left( \frac{\langle A(x), I\rangle}{||A(x)||^2} \right) \leq 0 \,. \tag{2.4}$$

Therefore there exists

$$\alpha(x) = \frac{\langle A(x), I\rangle}{||A(x)||^2} \tag{2.5}$$

such that $P(\alpha(x)) \leq 0$. Observing that

$$||I - \alpha(x)A(x)||^2 = ||A(x)||^2 \alpha^2(x) - 2\langle A(x), I\rangle \alpha(x) + 2n$$

we get that (2.4) implies that

$$||I - \alpha(x)A(x)||^2 \leq 1 - \varepsilon \,,$$

which is equivalent to

$$|\langle I - \alpha(x)A(x), M\rangle| \leq \sqrt{1-\varepsilon} ||M|| \,, \text{ for all } M \in \mathcal{M}_{2n}(\mathbb{R}) \,. \tag{2.6}$$

Condition (2.6) can be written also as

$$\left| \sum_{i=1}^{n} m_{ii} \; - \; \alpha(x) \sum_{i,j=1}^{n} \; a_{ij}(x) m_{ij} \right| \; \leq \; \sqrt{1-\varepsilon} \left( \sum_{i,j=1}^{n} m_{ij}^2 \right)^{1/2} \tag{2.7}$$

for all $M \in \mathcal{M}_{2n}(\mathbb{R})$.
Formula (2.7) and Lemma 1.1 imply that for all $u \in HW_0^{2,2}(\Omega)$ we have

$$\int_\Omega |\Delta_H u(x) - \alpha(x)\mathcal{A}u(x)|^2\, dx \le (1-\varepsilon)\int_\Omega \sum_{i,j=1}^{2n} (X_iX_ju(x))^2\, dx \le$$

$$\le (1-\varepsilon)c_n^2 \int_\Omega |\Delta_H u(x)|^2\, dx\,.$$

Therefore, for $\gamma = \sqrt{1-\varepsilon}\, c_n < 1$ we get

$$||\Delta_H u - \alpha\mathcal{A}u||_{L^2(\Omega)} \le \gamma ||\Delta_H u||_{L^2(\Omega)}$$

which shows that

$$||X^2 u||_{L^2(\Omega)} \le c_n ||\Delta_H u||_{L^2(\Omega)} \le$$

$$\le \frac{c_n}{1-\gamma}||\alpha\mathcal{A}u||_{L^2(\Omega)} \le \frac{c_n}{1-\gamma}||\alpha||_{L^\infty(\Omega)}||\mathcal{A}u||_{L^2(\Omega)}\,.$$

$\square$

## 3. $HW^{2,2}$-interior regularity for p-harmonic functions in $\mathbb{H}^n$

Let $\Omega \in \mathbb{H}^n$ be a domain, $h \in HW^{1,p}(\Omega)$ and $p > 1$. Consider the problem of minimizing the functional

$$\Phi(u) = \int_\Omega |Xu(x)|^p\, dx$$

over all $u \in HW^{1,p}(\Omega)$ such that $u - h \in HW_0^{1,p}(\Omega)$. The Euler equation for this problem is the $p$-Laplace equation

$$\sum_{i=1}^{2n} X_i\left(|Xu|^{p-2}\, X_iu\right) = 0\,,\ \text{in } \Omega\,. \tag{3.1}$$

A function $u \in HW^{1,p}(\Omega)$ is called a weak solution for (3.1) if

$$\sum_{i=1}^{2n}\int_\Omega |Xu(x)|^{p-2}X_iu(x)\cdot X_i\varphi(x)dx = 0\,,\ \forall\, \varphi \in HW_0^{1,p}(\Omega)\,. \tag{3.2}$$

$\Phi$ is a convex functional on $HW^{1,p}$, therefore weak solutions are minimizers for $\Phi$ and vice-versa.

For $m \in \mathbb{N}$ let us define now the approximating problems of minimizing functionals

$$\Phi_m(u) = \int_\Omega \left(\frac{1}{m} + |Xu(x)|^2\right)^{\frac{p}{2}}$$

and the corresponding Euler equations

$$\sum_{i=1}^{2n} X_i\left(\left(\frac{1}{m} + |Xu|^2\right)^{\frac{p-2}{2}}\ X_iu\right) = 0\,,\ \text{in } \Omega\,. \tag{3.3}$$

The weak form of this equation is

$$\sum_{i=1}^{2n}\int_\Omega \left(\frac{1}{m} + |Xu(x)|^2\right)^{\frac{p-2}{2}}\ X_iu(x)\cdot X_i\varphi(x)dx = 0\,,\ \text{ for all }\ \varphi \in HW_0^{1,p}(\Omega)\,. \tag{3.4}$$

The differentiated version of equation (3.3) has the form

$$\sum_{i,j=1}^{2n} a_{ij}^m \, X_i X_j u = 0 \,, \text{ in } \Omega \tag{3.5}$$

where

$$a_{ij}^m(x) = \delta_{ij} + (p-2) \frac{X_i u(x) \, X_j u(x)}{\frac{1}{m} + |Xu(x)|^2} \,.$$

Let us consider a weak solution $u_m \in HW^{1,p}(\Omega)$ of equation (3.3). Then $a_{ij}^m \in L^\infty(\Omega)$. Define the mapping $L_m : W_0^{2,2}(\Omega) \to L^2(\Omega)$ by

$$L_m(v)(x) \;=\; \sum_{i,j=1}^{2n} a_{ij}^m(x) X_i X_j v(x) \,. \tag{3.6}$$

We will check the validity of Theorem 2.1 for $L_m$. We have

$$\sum_{i=1}^{2n} a_{ii}^m(x) = 2n + (p-2) \frac{|Xu_m|^2}{\frac{1}{m} + |Xu_m|^2} \,,$$

and

$$\sum_{i,j=1}^{2n} \left( a_{ij}^m(x) \right)^2 = 2n + 2(p-2) \frac{|Xu_m|^2}{\frac{1}{m} + |Xu_m|^2} + (p-2)^2 \frac{|Xu_m|^4}{\left( \frac{1}{m} + |Xu_m|^2 \right)^2} \,.$$

Denote

$$(p-2) \frac{|Xu_m|^2}{\frac{1}{m} + |Xu_m|^2} = \Lambda \,.$$

Therefore, for an $\varepsilon \in (1 - \frac{1}{c_n^2}, 1)$ we need

$$2n + 2\Lambda + \Lambda^2 \le \frac{1}{2n - 1 + \varepsilon} \left( 2n + \Lambda \right)^2 \,.$$

This leads to

$$\begin{aligned} (2n-1)\Lambda^2 &\le (1-\varepsilon) \left( 2n + 2\Lambda + \Lambda^2 \right) < \\ &< \frac{1}{c_n^2} \left( 2n + 2\Lambda + \Lambda^2 \right) \,. \end{aligned}$$

Hence,

$$\left( (2n-1) c_n^2 - 1 \right) \Lambda^2 - 2\Lambda - 2n < 0 \,.$$

Solving this inequality we get

$$\Lambda \in \left( \frac{1 - \sqrt{2n \left( (2n-1) \, c_n^2 - 1 \right) + 1}}{(2n-1) c_n^2 - 1} \,,\, \frac{1 + \sqrt{2n \left( (2n-1) \, c_n^2 - 1 \right) + 1}}{(2n-1) c_n^2 - 1} \right) \,. \tag{3.7}$$

Using $c_n^2 = \frac{n+2}{n}$ and the fact that $\frac{|Xu_m|^2}{\frac{1}{m} + |Xu_m|^2} < 1$ we have that for all $m \in \mathbb{N}$ we have

$$p - 2 \in \left( \frac{n - n\sqrt{4n^2 + 4n - 3}}{2n^2 + 2n - 2} \,,\, \frac{n + n\sqrt{4n^2 + 4n - 3}}{2n^2 + 2n - 2} \right) \,, \tag{3.8}$$

and that the operators $L_m$ satisfies the assumptions of Theorem 2.1 uniformly in $m$.

Let us remark that in the case $n = 1$ we have

$$p - 2 \in \left( \frac{1-\sqrt{5}}{2}, \frac{1+\sqrt{5}}{2} \right).$$

**Theorem 3.1.** *Let*

$$2 \leq p < 2 + \frac{n + n\sqrt{4n^2 + 4n - 3}}{2n^2 + 2n - 2}.$$

*If $u \in HW^{1,p}(\Omega)$ is a minimizer for the functional $\Phi$, then $u \in HW^{2,2}_{loc}(\Omega)$.*

*Proof.* The case $p = 2$ it is well known, so let us suppose $p \neq 2$. Let $u \in HW^{1,p}(\Omega)$ be a minimizer for $\Phi$. Consider $x_0 \in \Omega$ and $r > 0$ such that $B_{4r} = B(x_0, 4r) \subset\subset \Omega$. We need a cut-off function $\eta \in C_0^\infty(B_{2r})$ such that $\eta = 1$ on $B_r$. Also consider minimizers $u_m$ for $\Phi_m$ on $HW^{1,p}(B_{2r})$ subject to $u_m - u \in HW_0^{1,p}(B_{2r})$. Then $u_m \to u$ in $HW^{1,p}(B_{2r})$ as $m \to \infty$.

By [3, 7] we get that for $2 \leq p < 4$ we have $u_m \in HW^{2,2}_{loc}(\Omega)$, but with bounds depending on $m$, and also that $u_m$ satisfies equation $L_m(u_m) = 0$ a.e. in $B_{2r}$. So, in $B_{2r}$ we have a.e.

$$X_i X_j(\eta^2 u_m) = X_i X_j(\eta^2) u_m + X_j(\eta^2) X_i u_m + X_i(\eta^2) X_j u_m + \eta^2 X_i X_j u_m$$

and hence

$$L_m(\eta^2 u_m) = u_m \, L_{m,u_m}(\eta^2) + \sum_{i,j=1}^{2n} a_{ij}^m(x) \Big( X_j(\eta^2) X_i u_m + X_i(\eta^2) X_j u_m \Big).$$

By Theorem 2.1 it follows that

$$\begin{aligned} ||X^2 u_m||_{L^2(B_r)} &\leq ||X^2(\eta^2 u_m)||_{L^2(B_{2r})} \leq c||L_m(\eta^2 u_m)||_{L^2(B_{2r})} \\ &\leq c||u_m||_{HW^{1,p}(B_{2r})} \leq c||u||_{HW^{1,p}(B_{2r})} \end{aligned}$$

where $c$ is independent of $m$. Therefore, $u \in HW^{2,2}(B_r)$. □

*Remark* 3.1. Observe that the range for $p$ given by Theorem 3.1 is shrinking from $[2, \frac{5+\sqrt{5}}{2})$ to $[2, 3]$ as $n$ increases from 1 to $\infty$.

For the case $p < 2$ we need the following lemmas. The first lemma is an interpolation result and its proof is based on integration by parts.

**Lemma 3.1.** *For all $u \in C_0^\infty(\Omega)$ and for all $\delta > 0$ there exists $c(\delta) > 0$ such that*

$$||Xu||^2_{L^2(\Omega)} \leq \delta ||X^2 u||^2_{L^2(\Omega)} + c(\delta)||u||^2_{L^2(\Omega)}.$$

*Proof.*

$$\begin{aligned} &||Xu||^2_{L^2(\Omega)} = \sum_{i=1}^{2n} \int_\Omega X_i u(x) \, X_i u(x) \, dx = -\sum_{i=1}^{2n} \int_\Omega u(x) \, X_i X_i u(x) dx = \\ &= -\int_\Omega u(x) \, \Delta_H u(x) \, dx \leq \frac{\delta}{2n} \int_\Omega |\Delta_H u(x)|^2 \, dx + c(\delta) \int_\Omega u^2(x) \, dx \\ &\leq \delta \int_\Omega |X^2 u(x)|^2 \, dx + c(\delta) \int_\Omega u^2(x) \, dx \end{aligned}$$

□

From Lemma 3.1 and the higher order extension results available for the Sobolev spaces on the Heisenberg group [6, 9] we get the following result.

**Lemma 3.2.** *For all $u \in HW^{2,2}(B_r)$ and all $\delta > 0$ there exists $c(\delta) > 0$ such that*

$$||Xu||^2_{L^2(B_r)} \leq \delta ||X^2 u||^2_{L^2(B_r)} + c(\delta)||u||^2_{L^2(B_r)} .$$

By Lemmas 3.1 and 3.2 we can use a method similar to the proof of Theorem 9.11 [5] to get the following result.

**Lemma 3.3.** *Let us suppose that the operator $\mathcal{A}$ satisfies the assumptions of Theorem 4.1 and that $B_{3r} \subset \Omega$. Then*

$$||X^2 u||_{L^2(B_r)} \leq c\Big( ||\mathcal{A}u||_{L^2(B_{2r})} + ||u||_{L^2(B_{2r})} \Big) ,$$

*for all $u \in HW^{2,2}_{\text{loc}}(B_{3r})$.*

*Proof.* Let $\eta \in C_0^\infty(B_{2r})$, $0 < \sigma < 1$ and $\sigma' = \frac{1+\sigma}{2}$ such that $\eta$ is a cut-off function between $B_{\sigma 2r}$ and $B_{\sigma' 2r}$ satisfying

$$|X\eta| \leq \frac{2}{(1-\sigma)r} \ \text{ and } \ |X^2\eta| \leq \frac{4}{(1-\sigma)^2 r^2} .$$

Then we can use Theorem 2.1 for $\eta u$ to get

$$\begin{aligned}
&||X^2 u||_{L^2(B_{\sigma 2r})} \leq ||X^2(\eta u)||_{L^2(B_{2r})} \leq c||\mathcal{A}(\eta u)||_{L^2(B_{2r})} = \\
&\leq c \left\| \eta \mathcal{A}u + u\, \mathcal{A}(\eta) + \sum_{i,j=1}^{2n} a_{ij} \Big( X_j(\eta) X_i u + X_i(\eta) X_j u \Big) \right\|_{L^2(B_{2r})} \\
&\leq c\Big( ||\mathcal{A}u||_{L^2(B_{2r})} + \frac{1}{(1-\sigma)r}||Xu||_{L^2(B_{\sigma' 2r})} + \frac{1}{(1-\sigma)^2 r^2}||u||_{L^2(B_{\sigma' 2r})} \Big)
\end{aligned}$$

For $k \in \{0, 1, 2\}$ let us use the seminorms

$$|||u|||_k = \sup_{0<\sigma<1} (1-\sigma)^k r^k ||X^k u||_{L^2(B_{\sigma 2r})} .$$

Then

$$|||u|||_2 \leq c\Big( r^2 ||\mathcal{A}u||_{L^2(B_{2r})} + |||u|||_1 + |||u|||_0 \Big) .$$

Lemma 3.2 implies that for $\delta > 0$ small we have

$$|||u|||_1 \leq \delta |||u|||_2 + c(\delta)|||u|||_0 .$$

Therefore,

$$|||u|||_2 \leq c\Big( r^2 ||\mathcal{A}u||_{L^2(B_{2r})} + |||u|||_0 \Big)$$

and hence

$$||X^2 u||_{L^2(B_{\sigma 2r})} \leq \frac{c}{(1-\sigma)^2 r^2} \Big( r^2 ||\mathcal{A}u||_{L^2(B_{2r})} + ||u||_{L^2(B_{2r})} \Big) .$$

For $\sigma = \frac{1}{2}$ we get the desired inequality. □

**Theorem 3.2.** *Let us consider the Heisenberg group $\mathbb{H}^1$ and*

$$\frac{\sqrt{17} - 1}{2} \leq p \leq 2 .$$

*If $u \in HW^{1,p}(\Omega)$ is a minimizer for the functional $\Phi$, then $u \in HW^{2,2}_{loc}(\Omega)$.*

*Proof.* We start the proof in the same way as we did in the proof of Theorem 3.1. Let $u \in HW^{1,p}(\Omega)$ be a minimizer for $\Phi$. Consider $x_0 \in \Omega$ and $r > 0$ such that $B_{4r} = B(x_0, 4r) \subset\subset \Omega$. We need a test function $\eta \in C_0^\infty(B_{3r})$. Also consider minimizers $u_m$ for $\Phi_m$ on $HW^{1,p}(B_{3r})$ subject to $u_m - u \in HW_0^{1,p}(B_{3r})$. Then $u_m \to u$ in $HW^{1,p}(B_{3r})$ as $m \to \infty$. We use the facts that

$$\frac{4}{3} < \frac{5-\sqrt{5}}{2} < \frac{\sqrt{17}-1}{2} < 2,$$

that the homogeneous dimension of $\mathbb{H}^1$ is $Q = 4$, and

$$2 \le \frac{4p}{4-p} \quad \text{for all } \frac{4}{3} \le p < 2\,.$$

The Sobolev embeddings result in the subelliptic setting [1] says that

$$HW_0^{1,p}\big(B_{3r}\big) \hookrightarrow L^q\big(B_{3r}\big)\,, \ \ \text{for } \ 1 \le q \le \frac{4p}{4-p}\,.$$

Therefore, $u_m \to u$ in $L^2\big(B_{3r}\big)$. Also, using that (see [3]) for $\frac{\sqrt{17}-1}{2} \le p \le 2$ we have $u_m \in HW^{2,p}_{\mathrm{loc}}\big(B_{3r}\big)$ we get that $Xu_m \in L^2_{\mathrm{loc}}\big(B_{3r}\big)$. Let us remark that these bounds of $X^2 u_m$ in $L^p$ may depend on $m$ and that $L_m(u_m) = 0$ a.e. in $B_{3r}$. Moreover,

$$||L_m\big(\eta^2 u_m\big)||_{L^2(B_{3r})} =$$

$$= c\left\| u_m\, L_m(\eta^2) + \sum_{i,j=1}^{2n} a_{ij}^{m,u}(x)\Big(X_j(\eta^2)X_i u_m + X_i(\eta^2)X_j u_m\Big)\right\|_{L^2(B_{3r})} \le$$

$$\le c\Big(||u_m||_{L^2(\mathrm{supp}\eta)} + ||Xu_m||_{L^2(\mathrm{supp}\eta)}\Big) < +\infty\,.$$

and hence $u_m \in HW^{2,2}_{\mathrm{loc}}\big(B_{3r}\big)$. By Lemma 3.3 for all $m$ sufficiently large we have

$$||X^2\big(u_m\big)||_{L^2(B_r)} \le c||u_m||_{L^2(B_{2r})} \le 2c||u||_{L^2(B_{2r})}$$

which shows that $X^2 u_m$ is uniformly bounded in $HW^{2,2}(B_r)$, hence $u \in HW^{2,2}(B_r)$. □

In the forthcoming article [4] we establish the $C^{1,\alpha}$ regularity for p-harmonic functions in $\mathbb{H}^n$ when $p$ is in a neighborhood of 2.

**Acknowledgement.** The authors would like to thank Robert S. Strichartz for his valuable hint regarding the spectral analysis of the sublaplacian and estimate (1.3), and Silvana Marchi for making available her papers and for her conversations with the second author.

## References

[1] L. Capogna, D. Danielli and N. Garofalo, *An embedding theorem and the Harnack inequality for nonlinear subelliptic equations*, Comm. Partial Differential Equations, 18(1993), 1765-1794.

[2] H. O. Cordes, *Zero order a priori estimates for solutions of elliptic differential equations*, Proceedings of Symposia in Pure Mathematics IV (1961).

[3] A. Domokos, *Fractional difference quotients along vector fields and applications to second order differentiability of solutions to quasilinear subelliptic equations in the Heisenberg group*, submitted for publication.

[4] A. Domokos and Juan J. Manfredi, *$C^{1,\alpha}$-regularity for p-harmonic functions in the Heisenberg group for p near 2*, submitted for publication.

[5] D. Gilbarg and N.S. Trudinger, *Elliptic Partial Differential Equations of Second Order*, Springer-Verlag, 1983.

[6] G. Lu, *Polynomials, higher order Sobolev extension theorems and interpolation inequalities on weighted Folland-Stein spaces on stratified groups*, Acta Math. Sinica, English Series, 16(2000), 405-444.

[7] S. Marchi, $C^{1,\alpha}$ *local regularity for the solutions of the p-Laplacian on the Heisenberg group for* $2 \leq p \leq 1+\sqrt{5}$, Journal for Analysis and its Applications 20(2001), 617-636. See Erratum, to appear.

[8] S. Marchi, $C^{1,\alpha}$ *local regularity for the solutions of the p-Laplacian on the Heisenberg group. The case* $1 + \frac{1}{\sqrt{5}} < p \leq 2$ , Comment. Math. Univ. Carolinae 44(2003), 33-56. See Erratum, to appear.

[9] D. Nhieu, *Extension of Sobolev spaces on the Heisenberg group*, C. R. Acad. Sci. Ser. I Math., 321(1995), 1559-1564.

[10] R. S. Strichartz, $L^p$ *Harmonic analysis and Radon transforms on the Heisenberg group*, J. Funct. Analysis, 96(1991), 350-406.

[11] G. Talenti, *Sopra una classe di equazioni ellitiche a coeficienti misurabili*, Ann. Mat. Pura. Appl. 69(1965), 285 - 304.

Department of Mathematics, University of Pittsburgh, 301 Thackeray Hall, 15260 Pittsburgh, PA,USA

*E-mail address*: `and36@pitt.edu`

Department of Mathematics, University of Pittsburgh, 301 Thackeray Hall, 15260 Pittsburgh, PA,USA

*E-mail address*: `manfredi@pitt.edu`