EXAMPLE 6a Probability and a paradox

Suppose that we possess an infinitely large urn and an infinite collection of balls labeled ball number 1, number 2, number 3, and so on. Consider an experiment performed as follows: At 1 minute to 12 P.M., balls numbered 1 through 10 are placed in the urn and ball number 10 is withdrawn. (Assume that the withdrawal takes no time.) At $\frac{1}{2}$ minute to 12 P.M., balls numbered 11 through 20 are placed in the urn and ball number 20 is withdrawn. At $\frac{1}{4}$ minute to 12 P.M., balls numbered 21 through 30 are placed in the urn and ball number 30 is withdrawn. At $\frac{1}{8}$ minute to 12 P.M., and so on. The question of interest is, How many balls are in the urn at 12 P.M.?

The answer to this question is clearly that there is an infinite number of balls in the urn at 12 P.M., since any ball whose number is not of the form $10n$, $n \geq 1$, will have been placed in the urn and will not have been withdrawn before 12 P.M. Hence, the problem is solved when the experiment is performed as described.

However, let us now change the experiment and suppose that at 1 minute to 12 P.M. balls numbered 1 through 10 are placed in the urn and ball number 1 is withdrawn; at $\frac{1}{2}$ minute to 12 P.M., balls numbered 11 through 20 are placed in the urn and ball number 2 is withdrawn; at $\frac{1}{4}$ minute to 12 P.M., balls numbered 21 through 30 are placed in the urn and ball number 3 is withdrawn; at $\frac{1}{8}$ minute to 12 P.M., balls numbered 31 through 40 are placed in the urn and ball number 4 is withdrawn, and so on. For this new experiment, how many balls are in the urn at 12 P.M.?

Surprisingly enough, the answer now is that the urn is *empty* at 12 P.M. For, consider any ball—say, ball number n. At some time prior to 12 P.M. [in particular, at $\left(\frac{1}{2}\right)^{n-1}$ minutes to 12 P.M.], this ball would have been withdrawn from the urn. Hence, for each n, ball number n is not in the urn at 12 P.M.; therefore, the urn must be empty at that time.

We see then, from the preceding discussion that the manner in which the balls are withdrawn makes a difference. For, in the first case only balls numbered $10n, n \geq 1$, are ever withdrawn, whereas in the second case all of the balls are eventually withdrawn. Let us now suppose that whenever a ball is to be withdrawn, that ball is randomly selected from among those present. That is, suppose that at 1 minute to 12 P.M. balls numbered 1 through 10 are placed in the urn and a ball is randomly selected and withdrawn, and so on. In this case, how many balls are in the urn at 12 P.M.?

Solution. We shall show that, with probability 1, the urn is empty at 12 P.M. Let us first consider ball number 1. Define E_n to be the event that ball number 1 is still in the urn after the first n withdrawals have been made. Clearly,

$$P(E_n) = \frac{9 \cdot 18 \cdot 27 \cdots (9n)}{10 \cdot 19 \cdot 28 \cdots (9n + 1)}$$

[To understand this equation, just note that if ball number 1 is still to be in the urn after the first n withdrawals, the first ball withdrawn can be any one of 9, the second any one of 18 (there are 19 balls in the urn at the time of the second withdrawal, one of which must be ball number 1), and so on. The denominator is similarly obtained.]

Now, the event that ball number 1 is in the urn at 12 P.M. is just the event $\bigcap_{n=1}^{\infty} E_n$. Because the events $E_n, n \geq 1$, are decreasing events, it follows from Proposition 6.1 that

$$
\begin{aligned}
P\{\text{ball number 1 is in the urn at 12 P.M.}\} & \\
&= P\left(\bigcap_{n=1}^{\infty} E_n\right) \\
&= \lim_{n\to\infty} P(E_n) \\
&= \prod_{n=1}^{\infty}\left(\frac{9n}{9n+1}\right)
\end{aligned}
$$

We now show that

$$
\prod_{n=1}^{\infty} \frac{9n}{9n+1} = 0
$$

Since

$$
\prod_{n=1}^{\infty}\left(\frac{9n}{9n+1}\right) = \left[\prod_{n=1}^{\infty}\left(\frac{9n+1}{9n}\right)\right]^{-1}
$$

this is equivalent to showing that

$$
\prod_{n=1}^{\infty}\left(1+\frac{1}{9n}\right) = \infty
$$

Now, for all $m \geq 1$,

$$
\begin{aligned}
\prod_{n=1}^{\infty}\left(1+\frac{1}{9n}\right) \geq \prod_{n=1}^{m}\left(1+\frac{1}{9n}\right) \\
&= \left(1+\frac{1}{9}\right)\left(1+\frac{1}{18}\right)\left(1+\frac{1}{27}\right)\cdots\left(1+\frac{1}{9m}\right) \\
&> \frac{1}{9}+\frac{1}{18}+\frac{1}{27}+\cdots+\frac{1}{9m} \\
&= \frac{1}{9}\sum_{i=1}^{m}\frac{1}{i}
\end{aligned}
$$

Hence, letting $m\rightarrow\infty$ and using the fact that $\sum_{i=1}^{\infty} 1/i = \infty$ yields

$$
\prod_{n=1}^{\infty}\left(1+\frac{1}{9n}\right) = \infty
$$

Thus, letting F_i denote the event that ball number i is in the urn at 12 P.M., we have shown that $P(F_1) = 0$. Similarly, we can show that $P(F_i) = 0$ for all i.

(For instance, the same reasoning shows that $P(F_i) = \prod_{n=2}^{\infty}[9n/(9n+1)]$ for $i = 11, 12, \ldots, 20$.) Therefore, the probability that the urn is not empty at 12 P.M., $P\left(\bigcup_{1}^{\infty} F_i\right)$, satisfies

$$P\left(\bigcup_{1}^{\infty} F_i\right) \leq \sum_{1}^{\infty} P(F_i) = 0$$

by Boole's inequality. (See Self-Test Exercise 14.)

Thus, with probability 1, the urn will be empty at 12 P.M. ■

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