Notes by
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Einstein, "On the Electrodynamics of
moving Bodies"

§8 Transformation of the Energy of Light Rays

main result

Propagating
light
complex

How does energy transform between different inertial frames of reference?

By the same rule as applies to the frequency of light

Theory of the Pressure of Radiation Exerted on Perfect Reflectors

main result

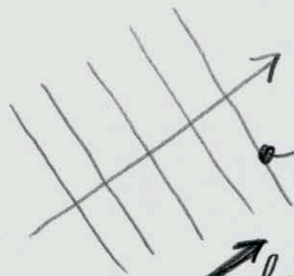
compute radiation pressure on a moving mirror by transforming to its rest frame.

"All problems in the optics of moving bodies can be solved by the method here employed."

"It is remarkable that..."

Transformation of the energy of Light waves

Propagating plane wave

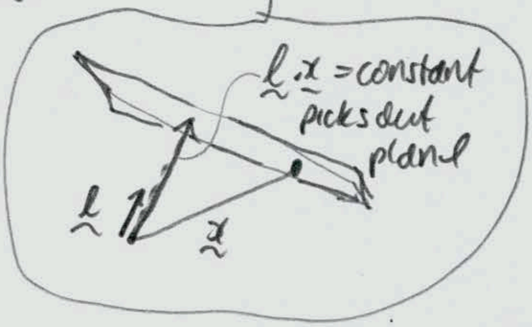


$\underset{\sim}{x}(t) = (x(t), y(t), z(t))$

$\underset{\sim}{l} = (l, m, n)$
unit vector in direction of propagation

Locus of constant phase at

$$t - \frac{1}{c}\underset{\sim}{l}\cdot\underset{\sim}{x} = \underbrace{t - \frac{1}{c}(lx + my + nz)} = \Phi = \text{constant}$$



$$\underbrace{\underset{\sim}{l}.\underset{\sim}{x}} = ct$$

scalar distance of plane from origin grows at c

components (X, Y, Z)

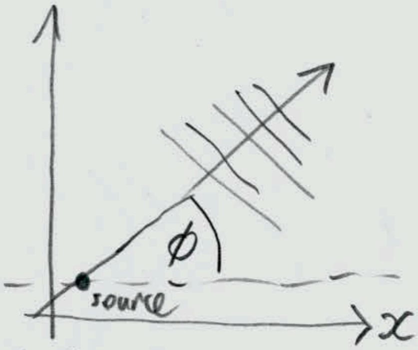
Electromagnetic wave is

$$\underset{\sim}{E}(t) = \underset{\sim}{E}_0 \sin\Phi$$

$$\underset{\sim}{H}(t) = \underset{\sim}{H}_0 \sin\Phi$$

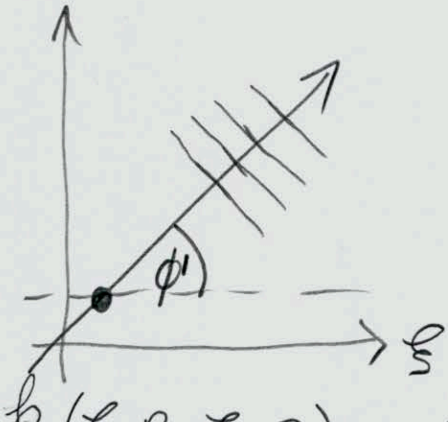
components (L, M, N)

Two inertial frames of reference



$K(x, y, z, t)$
(unprimed)

Frame whose origin moves at v in $+x$ direction $\Longrightarrow$



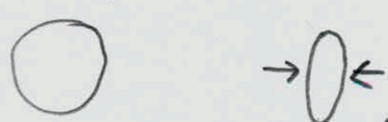
$k(\xi, \eta, \zeta, \tau)$ (primed)

Energy density $\propto \underbrace{|E^2| + |H^2|}_{\text{amplitude } A^2 \text{ of wave}}$ $\Longrightarrow$ Energy density $\propto (A')^2$

$$(A')^2 = A^2 \frac{(1 - \cos\phi \, ^{v}/_{c})^2}{1 - v^2/c^2}$$

Tempting mistake: Energy of light $=$ (Energy density) $\times$ $\underbrace{\text{(volume enclosing light)}}$

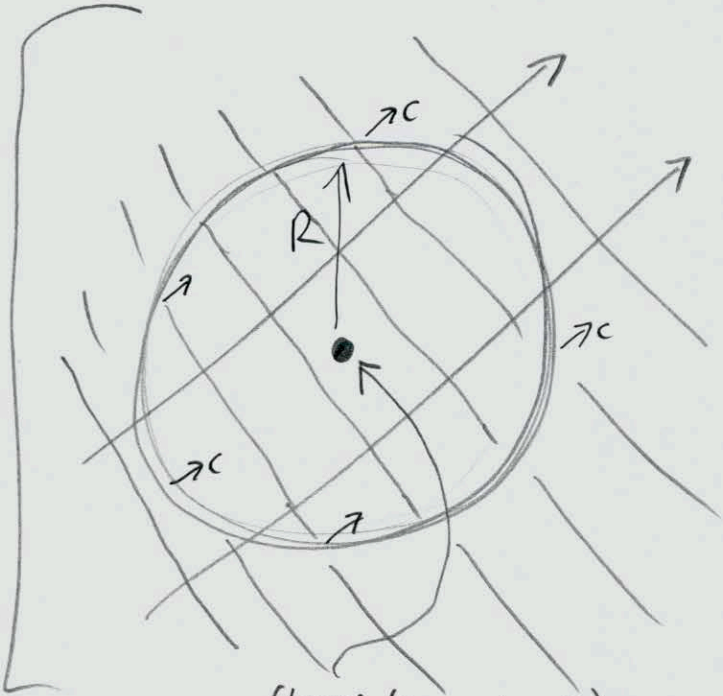
In K   In k



contracts by $\sqrt{1 - v^2/c^2}$

$$V' = \sqrt{1 - v^2/c^2}\, V$$

NO!
must use volume that propagates at c with the light.
Does not transform by this rule!



(locus center (lct, mct, nct) moves at c)

In K

sphere radius R propagates at c with the wave.

||| No energy crosses boundary. ||| key condition

sphere is

$$(x-lct)^2+(y-mct)^2+(z-nct)^2=R^2$$

$$\text{Volume}=\frac{4}{3}\pi R^3 = S$$

same sphere in k
at instant $\tau=0$

substitute for $x, y, \dots$ but NOT l, m, n

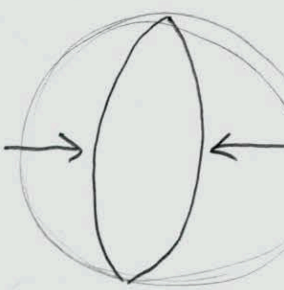
$$x=\beta(\xi+v\overset{0}{\tau})=\beta\xi$$

$$y=\eta \quad z=\zeta \quad t=\beta\left(\overset{0}{\tau}+\frac{v}{c^2}\xi\right)=\beta\frac{v}{c^2}\xi$$

$$\boxed{\underbrace{\left(\beta\xi-l\beta\xi\frac{v}{c}\right)^2}+\left(\eta-m\beta\xi\frac{v}{c}\right)^2+\left(\zeta-n\beta\xi\frac{v}{c}\right)^2=R^2}$$

$$=\underset{\left(\frac{1}{\sqrt{1-v^2/c^2}}\right)^2}{\beta^2}\left(1-l\frac{v}{c}\right)^2\xi^2=\frac{1-\overbrace{\cos\phi}^{l}\frac{v}{c}}{(1-v^2/c^2)}\xi^2$$

$$\frac{\left(1-\cos\phi\frac{v}{c}\right)^2}{1-v^2/c^2}\xi^2+(\eta-\square)^2+(\zeta-\square)^2=R^2$$



sphere reduced in this direction by $\sqrt{\dfrac{1-v^2/c^2}{\left(1-\cos\phi\frac{v}{c}\right)^2}}$

$$\text{Volume}=\frac{\sqrt{1-v^2/c^2}}{1-\cos\phi\frac{v}{c}}\,\frac{4\pi}{3}R^3=S'$$

From A   From S

$$\frac{\text{Energy in } k}{\text{Energy in } K} = \frac{E'}{E} = \frac{A'^2}{A^2} \cdot \frac{S'}{S} = \frac{\left(1-\cos\phi \frac{v}{c}\right)^2}{1-\frac{v^2}{c^2}} \cdot \frac{\sqrt{1-\frac{v^2}{c^2}}}{1-\cos\phi \frac{v}{c}}$$

$$= \frac{1-\cos\phi \frac{v}{c}}{\sqrt{1-\frac{v^2}{c^2}}}$$

Main result

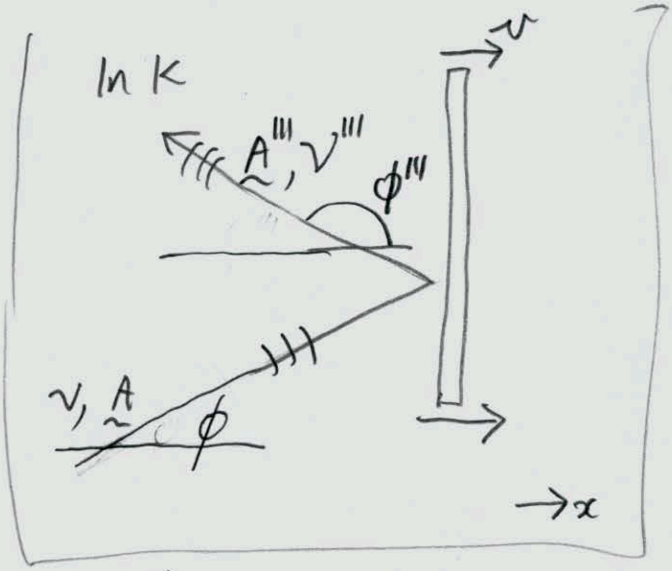
For $\phi = 0$
$\cos\phi = 1$
Propagation in direction of motion observer

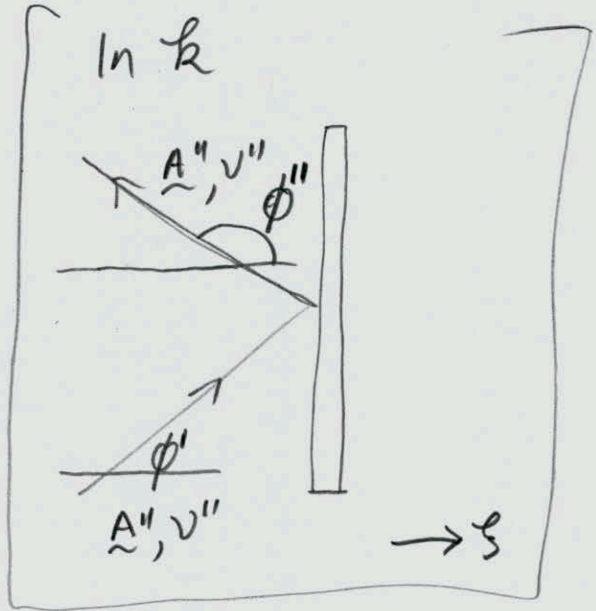
$$\frac{E'}{E} = \frac{1-\frac{v}{c}}{\sqrt{1-\frac{v^2}{c^2}}} = \frac{1-\frac{v}{c}}{\sqrt{\left(1-\frac{v}{c}\right)\left(1+\frac{v}{c}\right)}}$$

$$= \sqrt{\frac{1-\frac{v}{c}}{1+\frac{v}{c}}}$$

special case

Light reflected off $\underbrace{\text{moving}}_{\text{at rest in } k}$ mirror





In K, incident & reflected waves connected by messy complicated formulae

In k, symmetry dictates very simple relations

$A'' = A$

$\cos\phi'' = \cos(\pi - \phi') = -\cos\phi'$

$\nu'' = \nu'$

Recover these by mechanically transforming from k to K

using earlier

$$A' = A\,\frac{1 - \cos\phi\, v/c}{\sqrt{1 - v^2/c^2}}$$

From §7

$$\cos\phi' = \frac{\cos\phi - v/c}{1 - \cos\phi\, v/c}$$

$$\nu' = \nu\,\frac{1 - \cos\phi\, v/c}{\sqrt{1 - v^2/c^2}}$$

Transformation is just a tedious mechanical substitution of variables

e.g. for A

("+" since transforming $k \to K$)

$$A''' = A'' \frac{1 + \cos\phi'' \cdot v/c}{\sqrt{1 - v^2/c^2}}$$

$$A'' = A' = A \frac{1 - \cos\phi\, v/c}{\sqrt{1 - v^2/c^2}}$$

$$\cos\phi'' = -\cos\phi = -\frac{\cos\phi - v/c}{1 - \cos\phi\, v/c}$$

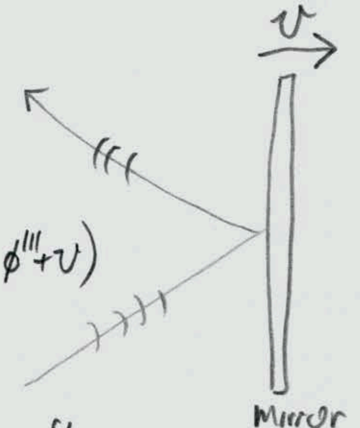
$$A''' = A\left(\frac{1 - \cos\phi\, v/c}{\sqrt{1 - v^2/c^2}}\right) \frac{1 - \left(\frac{\cos\phi - v/c}{1 - \cos\phi\, v/c}\right) v/c}{\sqrt{1 - v^2/c^2}}$$

$$\boxed{\frac{1 - \cos\phi \frac{v}{c} - \left(\frac{v}{c}\right)^2}{1 - \cos\phi\, v/c} = \frac{\left(1 - \cos\phi \frac{v}{c}\right) - \cos\phi \frac{v}{c} - \left(\frac{v}{c}\right)^2}{1 - \cos\phi\, v/c}}$$

$$= A\left(\frac{1 - 2\cos\phi\, v/c + v^2/c^2}{1 - v^2/c^2}\right)$$

compute pressure on moving mirror from $\underbrace{\text{change Energy flux}}_{\text{Energy per unit area}} = \text{Pressure} \times \text{(normal) speed}$

<u>In K</u>



Energy flux reflected

$$= \frac{A'''^2}{8\pi}(-c\cos\phi''' + v)$$

Energy flux incident

$$= \frac{A^2}{8\pi}(c\cos\phi - v)$$

Pressure P:

$$P \cdot v = \frac{A^2}{8\pi}(c\cos\phi - v) - \frac{A'''^2}{8\pi}(-c\cos\phi''' + v)$$

$\Downarrow$ solve for P (Long tedious calculation...!)

$$P = 2\frac{A^2}{8\pi}\frac{(\cos\phi - v/c)^2}{1 - v^2/c^2}$$

$v/c \ll 1$

$$P \approx 2\frac{A^2}{8\pi}\cos^2\phi$$

main result

Energy density $= \dfrac{E^2 + H^2}{8\pi} = \dfrac{A^2}{8\pi}$

propagates at c in direction ϕ

Incident energy flux is

$$\frac{A^2}{8\pi}\underbrace{(c\cos\phi)}_{\text{component propagation in } x \text{ direction}}$$

But mirror recedes at v

$\therefore$ Net incident flux is

$$\frac{A^2}{8\pi}(c\cos\phi - v)$$