THE MEANING OF RELATIVITY

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LECTURE I

SPACE AND TIME IN PRE-RELATIVITY PHYSICS

THE theory of relativity is intimately connected with the theory of space and time. I shall therefore begin with a brief investigation of the origin of our ideas of space and time, although in doing so I know that I introduce a controversial subject. The object of all science, whether natural science or psychology, is to co-ordinate our experiences and to bring them into a logical system. How are our customary ideas of space and time related to the character of our experiences?

The experiences of an individual appear to us arranged in a series of events; in this series the single events which we remember appear to be ordered according to the criterion of "earlier" and "later," which cannot be analysed further. There exists, therefore, for the individual, an I-time, or subjective time. This in itself is not measurable. I can, indeed, associate numbers with the events, in such a way that a greater number is associated with the later event than with an earlier one; but the nature of this association may be quite arbitrary. This association I can define by means of a clock by comparing the order of events furnished by the clock with the order of the given series of events. We understand by a clock something which provides a series of events which can be counted, and which has other properties of which we shall speak later.

By the aid of speech different individuals can, to a certain extent, compare their experiences. In this way it is shown that certain sense perceptions of different individuals correspond to each other, while for other sense perceptions no such correspondence can be established. We are accustomed to regard as real those sense perceptions which are common to different individuals, and which therefore are, in a measure, impersonal. The natural sciences, and in particular, the most fundamental of them, physics, deal with such sense perceptions. The conception of physical bodies, in particular of rigid bodies, is a relatively constant complex of such sense perceptions. A clock is also a body, or a system, in the same sense, with the additional property that the series of events which it counts is formed of elements all of which can be regarded as equal.

The only justification for our concepts and system of concepts is that they serve to represent the complex of our experiences; beyond this they have no legitimacy. I am convinced that the philosophers have had a harmful effect upon the progress of scientific thinking in removing certain fundamental concepts from the domain of empiricism, where they are under our control, to the intangible heights of the *a priori*. For even if it should appear that the universe of ideas cannot be deduced from experience by logical means, but is, in a sense, a creation of the human mind, without which no science is possible, nevertheless this universe of ideas is just as little independent of the nature of our experiences as clothes are of the form of the human body. This is particularly true of our concepts of time and space, which physicists have been obliged by the facts to bring down from the Olympus of the *a priori* in order to adjust them and put them in a serviceable condition.

We now come to our concepts and judgments of space. It is essential here also to pay strict attention to the relation of experience to our concepts. It seems to me that Poincaré clearly recognized the truth in the account he gave in his book, "La Science et l'Hypothese." Among all the changes which we can perceive in a rigid body those are marked by their simplicity which can be made reversibly by an arbitrary motion of the body; Poincaré calls these, changes in position. By means of simple changes in position we can bring two bodies into contact. The theorems of congruence, fundamental in geometry, have to do with the laws that govern such changes in position. For the concept of space the following seems essential. We can form new bodies by bringing bodies B, C, . . . up to body A; we say that we *continue* body A. We can continue body A in such a way that it comes into contact with any other body, X. The ensemble of all continuations of body A we can designate as the "space of the body A." Then it is true that all bodies are in the "space of the (arbitrarily chosen) body A." In this sense we cannot speak of space in the abstract, but only of the "space belonging to a body A." The earth's crust plays such a dominant rôle in our daily life in judging the relative positions of bodies that it has led to an abstract conception of space which certainly cannot be defended. In order to free ourselves from this fatal error we shall speak only of "bodies of reference," or "space of reference." It was only through the theory of general relativity that refinement of these concepts became necessary, as we shall see later.

I shall not go into detail concerning those properties of the space of reference which lead to our conceiving points as elements of space, and space as a continuum. Nor shall I attempt to analyse further the properties of space which justify the conception of continuous series of points, or lines. If these concepts are assumed, together with their relation to the solid bodies of experience, then it is easy to say what we mean by the three-dimensionality of space; to each point three numbers, x_1, x_2, x_3 (co-ordinates), may be associated, in such a way that this association is uniquely reciprocal, and that x_1, x_2, and x_3 vary continuously when the point describes a continuous series of points (a line).

It is assumed in pre-relativity physics that the laws of the orientation of ideal rigid bodies are consistent with Euclidean geometry. What this means may be expressed as follows: Two points marked on a rigid body form an *interval*. Such an interval can be oriented at rest, relatively to our space of reference, in a multiplicity of ways. If, now, the points of this space can be referred to co-ordinates x_1, x_2, x_3, in such a way that the differences of the co-ordinates, Δx_1, Δx_2, Δx_3, of the two ends of the interval furnish the same sum of squares,

$$s^2 = \Delta x_1^2 + \Delta x_2^2 + \Delta x_3^2 \quad . \quad . \quad (1)$$

for every orientation of the interval, then the space of reference is called Euclidean, and the co-ordinates Cartesian.* It is sufficient, indeed, to make this assumption in the limit for an infinitely small interval. Involved in this assumption there are some which are rather less special, to which we must call attention on account of their fundamental significance. In the first place, it is assumed that one can move an ideal rigid body in an arbitrary manner. In the second place, it is assumed that the behaviour of ideal rigid bodies towards orientation is independent of the material of the bodies and their changes of position, in the sense that if two intervals can once be brought into coincidence, they can always and everywhere be brought into coincidence. Both of these assumptions, which are of fundamental importance for geometry and especially for physical measurements, naturally arise from experience; in the theory of general relativity their validity needs to be assumed only for bodies and spaces of reference which are infinitely small compared to astronomical dimensions.

The quantity s we call the length of the interval. In order that this may be uniquely determined it is necessary to fix arbitrarily the length of a definite interval; for example, we can put it equal to 1 (unit of length). Then the lengths of all other intervals may be determined. If we make the x_ν linearly dependent upon a parameter λ,

$$x_\nu = a_\nu + \lambda b_\nu,$$

* This relation must hold for an arbitrary choice of the origin and of the direction (ratios $\Delta x_1 : \Delta x_2 : \Delta x_3$) of the interval.

we obtain a line which has all the properties of the straight lines of the Euclidean geometry. In particular, it easily follows that by laying off n times the interval s upon a straight line, an interval of length $n \cdot s$ is obtained. A length, therefore, means the result of a measurement carried out along a straight line by means of a unit measuring rod. It has a significance which is as independent of the system of co-ordinates as that of a straight line, as will appear in the sequel.

We come now to a train of thought which plays an analogous rôle in the theories of special and general relativity. We ask the question: besides the Cartesian co-ordinates which we have used are there other equivalent co-ordinates? An interval has a physical meaning which is independent of the choice of co-ordinates; and so has the spherical surface which we obtain as the locus of the end points of all equal intervals that we lay off from an arbitrary point of our space of reference. If x_ν as well as x'_ν (ν from 1 to 3) are Cartesian co-ordinates of our space of reference, then the spherical surface will be expressed in our two systems of co-ordinates by the equations

$$\sum \Delta x_\nu^2 = \text{const.} \qquad (2)$$

$$\sum \Delta x'^2_\nu = \text{const.} \qquad (2a)$$

How must the x'_ν be expressed in terms of the x_ν in order that equations (2) and (2a) may be equivalent to each other? Regarding the x'_ν expressed as functions of the x_ν, we can write, by Taylor's theorem, for small values of the Δx_ν,

$$\Delta x'_\nu = \sum_a \frac{\partial x'_\nu}{\partial x_a}\Delta x_a + \frac{1}{2}\sum_{a\beta}\frac{\partial^2 x'_\nu}{\partial x_a \partial x_\beta}\Delta x_a \Delta x_\beta \ldots$$

If we substitute (2a) in this equation and compare with (1), we see that the x'_ν must be linear functions of the x_ν. If we therefore put

$$x'_\nu = a_\nu + \sum_a b_{\nu a} x_a \quad . \quad . \quad . \quad (3)$$

or

$$\Delta x'_\nu = \sum_a b_{\nu a} \Delta x_a \quad . \quad . \quad . \quad . \quad (3a)$$

then the equivalence of equations (2) and (2a) is expressed in the form

$$\sum \Delta x'^2_\nu = \lambda \sum \Delta x_\nu^2 \ (\lambda \text{ independent of } \Delta x_\nu) \quad . \ (2b)$$

It therefore follows that λ must be a constant. If we put $\lambda = 1$, (2b) and (3a) furnish the conditions

$$\sum_\nu b_{\nu a} b_{\nu \beta} = \delta_{a\beta} \quad . \quad . \quad . \quad (4)$$

in which $\delta_{a\beta} = 1$, or $\delta_{a\beta} = 0$, according as $a = \beta$ or $a \neq \beta$. The conditions (4) are called the conditions of orthogonality, and the transformations (3), (4), linear orthogonal transformations. If we stipulate that $s^2 = \sum \Delta x_\nu^2$ shall be equal to the square of the length in every system of co-ordinates, and if we always measure with the same unit scale, then λ must be equal to 1. Therefore the linear orthogonal transformations are the only ones by means of which we can pass from one Cartesian system of co-ordinates in our space of reference to another. We see that in applying such transformations the equations of a straight line become equations of a straight line. Reversing equations (3a) by multiplying both sides by $b_{\nu\beta}$ and summing for all the ν's, we obtain

$$\sum b_{\nu\beta}\Delta x'_{\nu} = \sum_{\nu\alpha} b_{\nu\alpha}b_{\nu\beta}\Delta x_{\alpha} = \sum_{\alpha} \delta_{\alpha\beta}\Delta x_{\alpha} = \Delta x_{\beta} \quad . \qquad (5)$$

The same coefficients, b, also determine the inverse substitution of Δx_{ν}. Geometrically, $b_{\nu\alpha}$ is the cosine of the angle between the x'_{ν} axis and the x_{α} axis.

To sum up, we can say that in the Euclidean geometry there are (in a given space of reference) preferred systems of co-ordinates, the Cartesian systems, which transform into each other by linear orthogonal transformations. The distance s between two points of our space of reference, measured by a measuring rod, is expressed in such co-ordinates in a particularly simple manner. The whole of geometry may be founded upon this conception of distance. In the present treatment, geometry is related to actual things (rigid bodies), and its theorems are statements concerning the behaviour of these things, which may prove to be true or false.

One is ordinarily accustomed to study geometry divorced from any relation between its concepts and experience. There are advantages in isolating that which is purely logical and independent of what is, in principle, incomplete empiricism. This is satisfactory to the pure mathematician. He is satisfied if he can deduce his theorems from axioms correctly, that is, without errors of logic. The question as to whether Euclidean