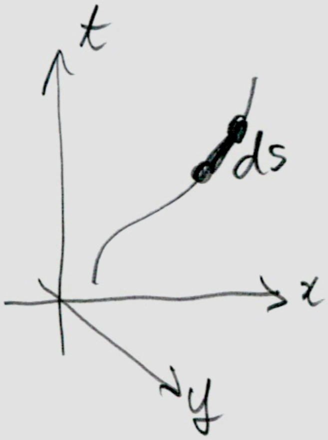
minkowski spacetime of special relativity is a metrical space



$$ds^2 = -c^2 dt^2 + dx^2 + dy^2 + dz^2$$

"indefinite"
time t different from space x

s = proper time along curves represent motion less than c

s = proper distance along curves within a hypersurface of simultaneity

s = 0 along a lightlike curve since then

$$ds^2 = -c^2 dt^2 + dx^2 + dy^2 + dz^2 = 0$$

$$\therefore \underbrace{\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}}_{\text{speed}} = \pm c$$

Lorentz transformation

$X = \beta(x - vt)$

$T = \beta(t - \frac{v}{c^2}x)$

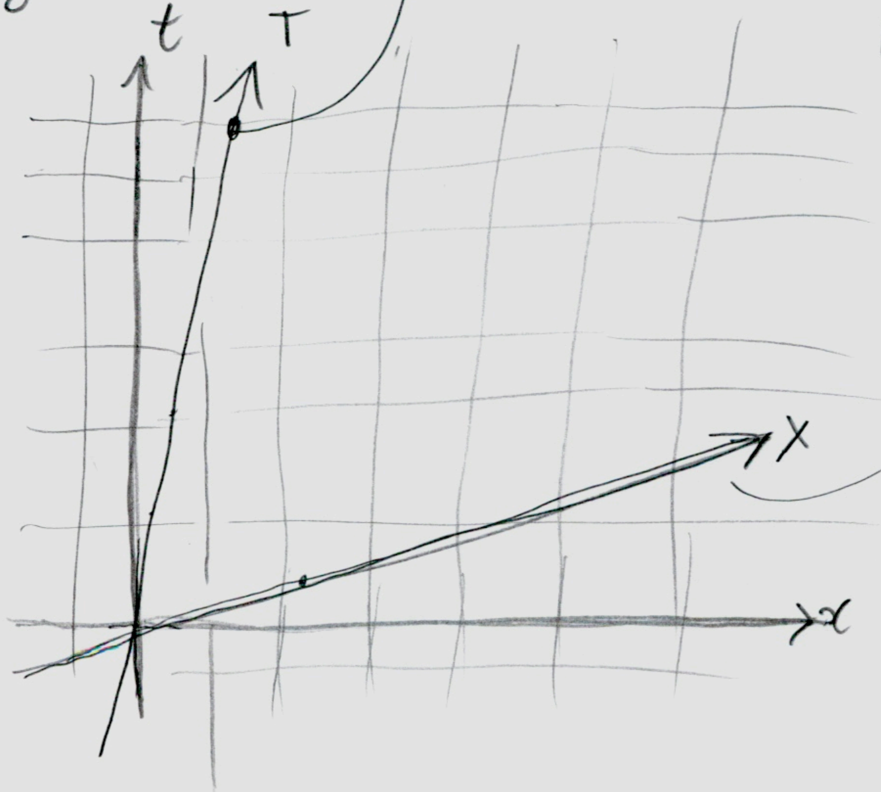
$Y = y$

$Z = z$

T axis is $X = 0$

i.e. $\beta(x - vt) = 0$ $\boxed{x = vt}$

$\boxed{t = \frac{1}{v}x}$



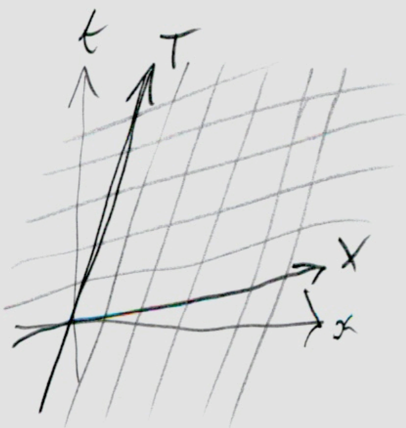
X axis is

$T = 0$

i.e.

$\beta(t - \frac{v}{c^2}x) = 0$

$\boxed{t = \frac{v}{c^2}x}$



Lorentz transformation is a symmetry of the Minkowski spacetime
(= principle of relativity satisfied)

$$\boxed{ds^2 = -c^2 dt^2 + dx^2 + dy^2 + dz^2}$$

$$\Big\Downarrow \quad \begin{aligned} S &= S' \\ T &= \beta(t - \tfrac{v}{c^2}x) \\ X &= \beta(x - vt) \\ Y &= y \\ Z &= z \end{aligned}$$

$$\boxed{ds'^2 = -c^2 dT^2 + dX^2 + dY^2 + dZ^2}$$

Introduce general coordinates

$$ds^2 = -c^2dt^2 + dx^2 + dy^2 + dz^2$$

$\Downarrow$ Replace (t, x, y, z) with $x^\mu = (x^0, x^1, x^2, x^3)$
$\mu = 0, 1, 2, 3$

$$ds^2 = \sum_{\substack{\mu=0,3 \\ \nu=0,3}} g_{\mu\nu}\, dx^\mu dx^\nu$$

$x^\mu \leftarrow$ index "upstairs"
"contravariant

contravariant vector transforms as

$$\frac{dy^\mu}{ds} = \frac{\partial y^\mu}{\partial x^\nu} \cdot \frac{dx^\nu}{ds}$$

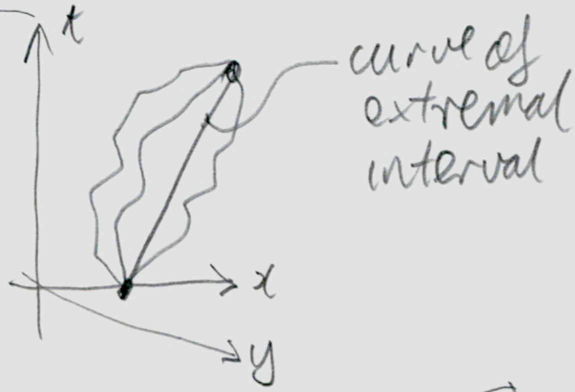
$\Downarrow$ "Einstein summation convention"

$$ds^2 = g_{\mu\nu}\, dx^\mu dx^\nu$$

Index "downstairs"
covariant tensor transforms as

$$g_{\alpha\beta} = \frac{\partial x^\mu}{\partial y^\alpha} \cdot \frac{\partial x^\nu}{\partial y^\beta}\, g_{\mu\nu}$$

$$\begin{array}{l}\text{Inertial} \\ \text{motion}\end{array} \equiv \begin{array}{l}\text{Geodesic} \\ \text{timelike curve}\end{array}$$



In normal coordinates $x^\mu = (t, x, y, z)$

$$\frac{d^2 x^\mu}{ds^2} = 0$$

Transform to arbitrary coordinats

$$\frac{d^2 x^\mu}{ds^2} + \Gamma^\mu_{\rho\sigma} \frac{dx^\rho}{ds} \frac{dx^\sigma}{ds} = 0$$

where $\Gamma^\mu_{\rho\sigma} = \frac{1}{2} g^{\lambda\sigma} \left(\frac{\partial g_{\nu\sigma}}{\partial x^\mu} + \frac{\partial g_{\sigma\mu}}{\partial x^\nu} - \frac{\partial g_{\mu\nu}}{\partial x^\sigma} \right)$

"christoffel symbols"

source for equations and conventions:

Sean Carroll, *Spacetime and Geometry*

$$\boxed{\text{Minkowski Spacetime} \quad ds^2 = g_{\mu\nu}\,dx^\mu dx^\nu \text{ AND the spacetime is flat}}$$

$\Downarrow$ There is a coordinate transformation

$$\boxed{ds^2 = -c^2dt^2 + dx^2 + dy^2 + dz^2}$$

Which special $g_{\mu\nu}$ satisfy this condition?
Exactly those for which:

$$\boxed{\text{Riemann curvature tensor} \quad R^\rho{}_{\sigma\mu\nu} = 0}$$

$$R^\rho{}_{\sigma\mu\nu} = \frac{\partial \Gamma^\rho_{\nu\sigma}}{\partial x^\mu} - \frac{\partial \Gamma^\rho_{\mu\sigma}}{\partial x^\nu} + \Gamma^\rho_{\mu\lambda}\Gamma^\lambda_{\nu\sigma} - \Gamma^\rho_{\nu\lambda}\Gamma^\lambda_{\mu\sigma}$$