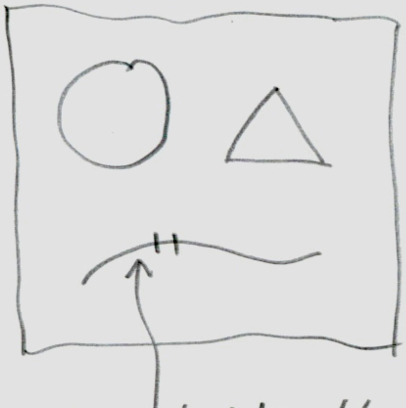
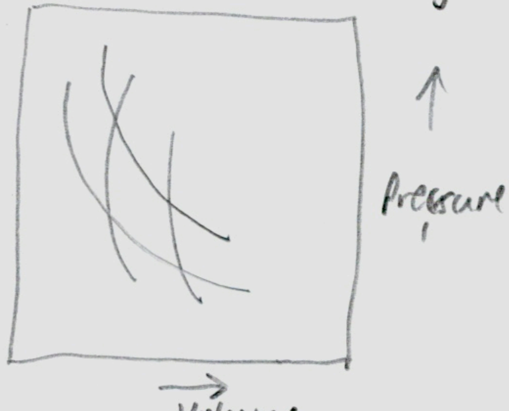
Metric Space: Euclidean Geometry



concept of length defined along a curve

vs

state space of thermodynamics



No natural notion of length of a curve

small interval

Pythagoras' theorem

cartesian coordinates

$$\boxed{dl^2 = dx^2 + dy^2}$$

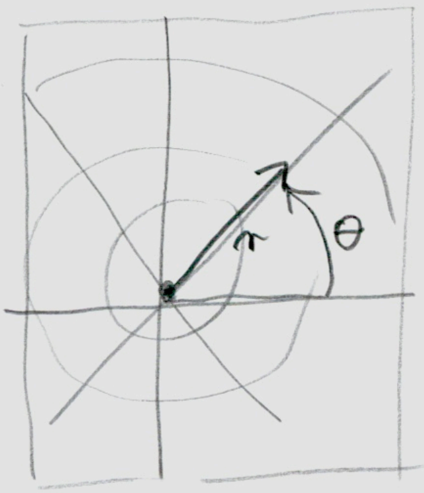
"line element"

"quadratic differential form"

More precisely

Tangent vector to curve $\left(\frac{dx}{dl}, \frac{dy}{dl}\right)$ has unit norm $\left(\frac{dx}{dl}\right)^2 + \left(\frac{dy}{dl}\right)^2 = 1$

Radial coordinates



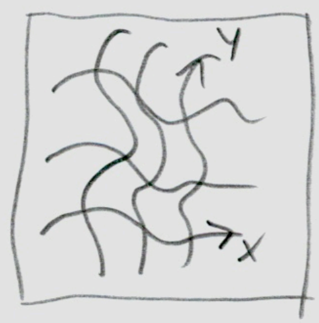
$r = \sqrt{x^2+y^2} \quad \theta = \tan^{-1} y/x$

$$\boxed{dl^2 = dr^2 + r^2 d\theta^2}$$

Arbitrary coordinates, X, Y

Write dx, dy as functions of dX, dY

$dx = \left(\frac{\partial x}{\partial X}\right) dX + \left(\frac{\partial x}{\partial Y}\right) dY \quad dy = \left(\frac{\partial y}{\partial X}\right) dX + \left(\frac{\partial y}{\partial Y}\right) dy$



$$\boxed{dl^2 = dx^2 + dy^2}$$

substitute for dx, dy

$$dl^2 = \underbrace{\left[\left(\frac{\partial x}{\partial X}\right)^2 + \left(\frac{\partial y}{\partial Y}\right)^2\right]}_{\text{“}g_{00}\text{”}} dX^2 + \underbrace{\left\{\left(\frac{\partial x}{\partial X}\right)\left(\frac{\partial x}{\partial Y}\right) + \left(\frac{\partial y}{\partial X}\right)\left(\frac{\partial y}{\partial Y}\right)\right\}}_{\text{“}2g_{01} = 2g_{10}\text{”}} dX dY + \underbrace{\left\{\left(\frac{\partial y}{\partial X}\right)^2 + \left(\frac{\partial y}{\partial Y}\right)^2\right\}}_{\text{“}g_{11}\text{”}} dY^2$$

The g's form the metric tensor $\begin{bmatrix} g_{00} & g_{01} \\ g_{10} & g_{11} \end{bmatrix}$ where $dl^2 = \begin{bmatrix} g_{00} & g_{01} \\ g_{10} & g_{11} \end{bmatrix} \begin{bmatrix} dX \\ dY \end{bmatrix}$

"$g_{\mu\nu}$"

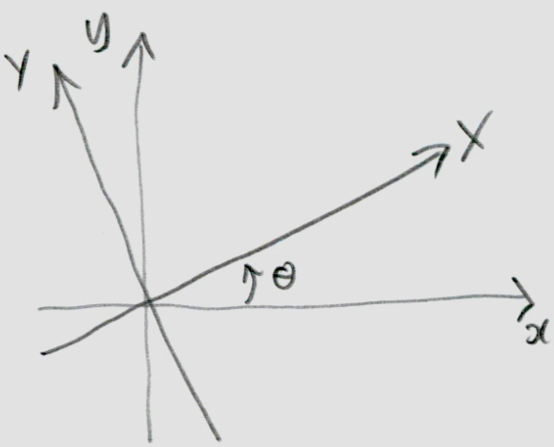
$\mu, \nu = 0, 1$

Symmetries of a Euclidean Space

Defined: Transform coordinates such that the space "looks the same" in the new coordinate system
i.e. $dl^2 = dx^2 + dy^2$ is preserved

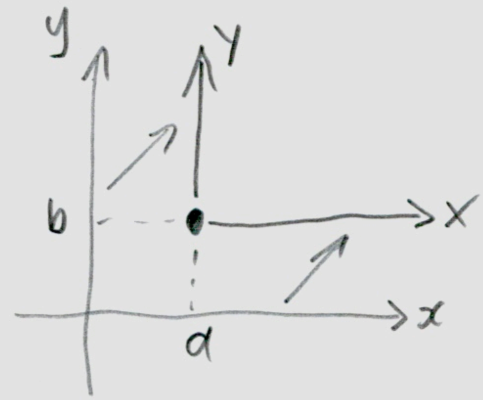
Rotation

$$X = x\cos\theta + y\sin\theta$$
$$Y = -x\sin\theta + y\cos\theta$$



Translation

$$X = x - a$$
$$Y = y - b$$

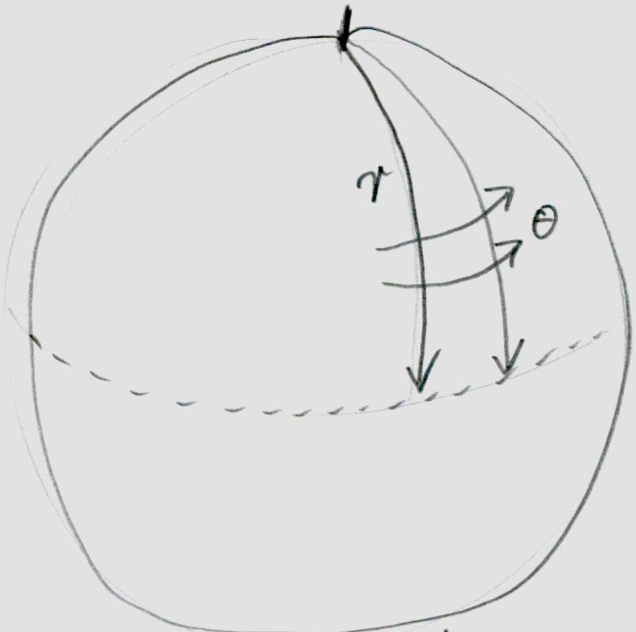


$$dl^2 = dx^2 + dy^2 \xrightarrow[\text{transform using } dl = dl']{} dl'^2 = dX^2 + dY^2$$

Non-Euclidean Geometry

$g_{\mu\nu}$ cannot be transformed to flat

$$g = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$



surface of a sphere
unit radius

$$dl^2 = dr^2 + \sin^2 r \, d\theta^2$$

scaled so that full circle is $r = 0$ to $r = 2\pi$

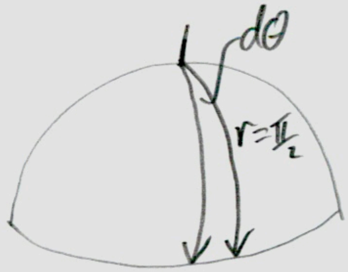
$$\underset{\sim}{g} = \begin{bmatrix} 1 & 0 \\ 0 & \sin^2 r \end{bmatrix}$$

[Small r near origin.

$\sin^2 r \approx r^2$

$$dl^2 \approx dr^2 + r^2 d\theta^2$$ } Euclidean line element]

[$r = \pi/2$



$$dl^2 = dr^2 + d\theta^2$$

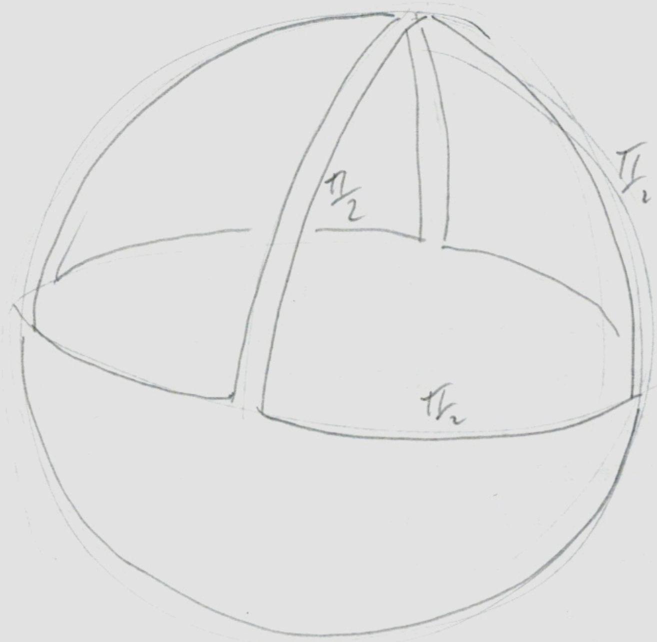
Integrate around equator

$$dl = d\theta \therefore \int dl = \int d\theta$$

Not Euclidean 2π

length equator $= 2\pi = 4 \times$ radius $(\pi/2)$]

Top half of sphere covered by four equilateral triangles



Flattened view from North pole