§7 Theory of Doppler's Principle and Aberration

Overall project.

Plane wave in K

$X = X_0 \sin \Phi$

$\vdots$

$L = L_0 \sin \Phi$

$\Phi = \omega\left(t - \frac{1}{c}(lx + my + nz)\right)$

$2\pi \times$ frequency

speed wave c

Directional cosines

$\boxed{l^2 + m^2 + n^2 = 1}$

Transform

Plane wave in k

$X' = X_0 \sin \Phi'$

$\vdots$

$L' = L_0 \sin \Phi$

$\vdots$

$\Phi' = \omega'\left(\tau - \frac{1}{c}(l'\xi + m'\eta + n'\zeta)\right)$

Read off transformation equations

$\omega' = \omega\beta(1 - lv/c)$] Frequency shift

$l' = \dfrac{l - v/c}{1 - lv/c}$

$m' = \dfrac{m}{\beta(1 - lv/c)}$

$n' = \dfrac{n}{\beta(1 - lv/c)}$

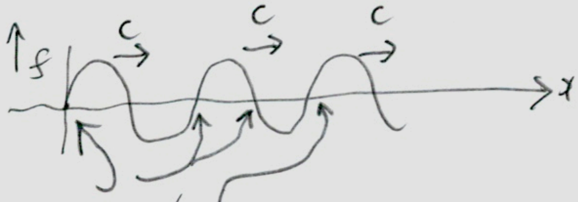
Direction shift

deduce

Simplified formula for special cases

Representing wave motion

Simplest case



sine wave propogates at c along x axis

$$f(x,t) = f_0 \sin \underbrace{\omega\left(t - \tfrac{1}{c}x\right)}_{\text{“}\Phi\text{”}}$$

To interpret: Track motion of points where

$f = 0$

$\Phi = 0, \pi, 2\pi, 3\pi, \ldots$

$= \text{constant}$

i.e. $\text{constant} = t - \frac{1}{c}x$

$\therefore \boxed{x = ct + \text{another constant}}$

Plane wave in three dimensional space

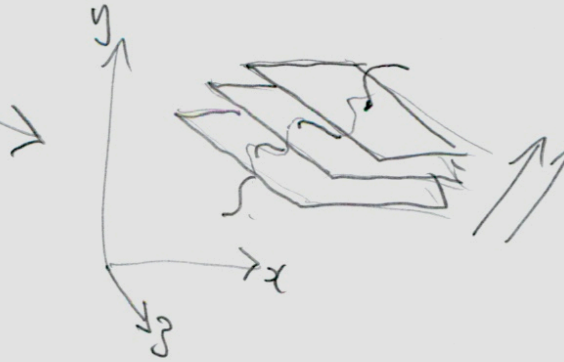
$$\Phi \longrightarrow \omega\left(t - \tfrac{1}{c}\underbrace{(lx + my + nz)}\right)$$

$$\underbrace{lx + my + nz}_{\text{describes a flat plane}} = \underbrace{\text{constant}}_{\text{distance to plane from origin }(0,0,0)}$$

see over

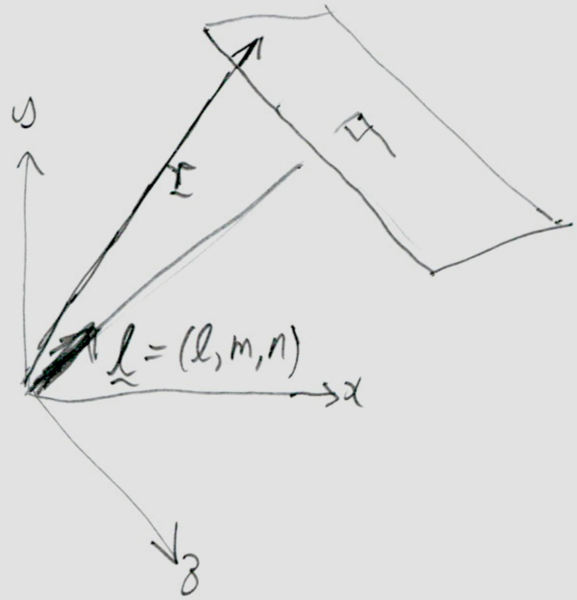
$$\Phi = \text{constant} \Rightarrow \underbrace{\text{distance to plane} = ct + \text{constant}}_{\text{plane moves at } c}$$

$\boxed{f(x,y,z,t) = f_0 \sin \Phi}$



surfaces of constant phase Φ propogate at c

$$\boxed{lx + my + nz = \text{constant} = \text{distance to plane}}$$



$\underset{\sim}{l} = (l, m, n)$ is unit norm vector along line perpendicular to plane

$\underset{\sim}{r} = (x, y, z)$ is a vector from origin to any point on the plane

$$\underset{\sim}{r} = \underbrace{\underset{\sim}{r}_{\parallel}}_{\text{parallel to } \underset{\sim}{l}} + \underbrace{\underset{\sim}{r}_{\perp}}_{\text{orthogonal to } \underset{\sim}{l}}$$

$$(lx + my + nz) = \underset{\sim}{l} \cdot \underset{\sim}{r} = \underset{\sim}{l} \cdot (\underset{\sim}{r}_{\parallel} + \underset{\sim}{r}_{\perp})$$

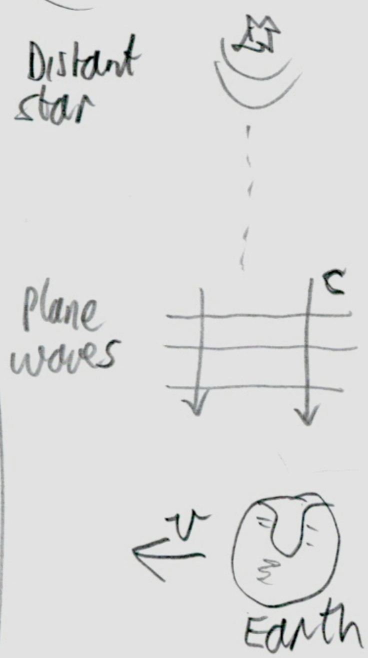
$$= \underset{\sim}{l} \cdot \underset{\sim}{r}_{\parallel} \quad \text{since } \underset{\sim}{l} \cdot \underset{\sim}{r}_{\perp} = 0$$

$$\underset{\sim}{r}_{\parallel} = \underbrace{|r_{\parallel}|}_{\substack{\text{Normal} \\ \text{distance} \\ \text{to plane}}} \underset{\sim}{l}$$

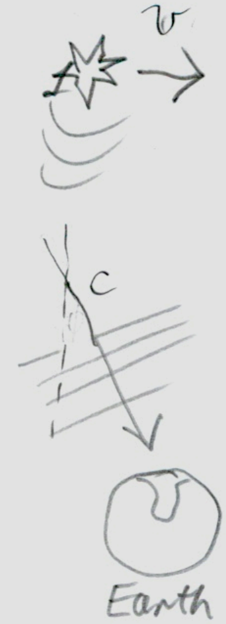
$$= (\underset{\sim}{l} \cdot \underset{\sim}{l}) |r_{\parallel}|$$

$$= |r_{\parallel}| = \text{Distance to plane}$$

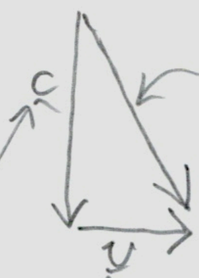
Stellar Aberration



Transform to earth frame of reference

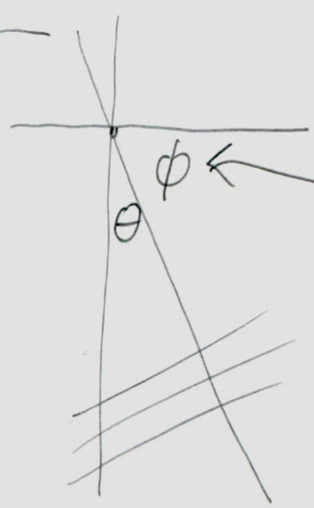


Angle of aberration given by emission theory of light



Angle of aberration θ

$$\frac{v}{c} = \tan\theta \underset{\text{small}}{\approx} \sin\theta$$



Einstein's "angle of aberration"

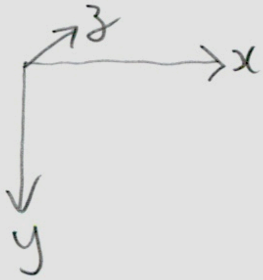
$$\phi = \pi/2 - \theta$$

$$\sin\theta = \cos\left(\tfrac{\pi}{2} - \theta\right) = \cos\phi$$

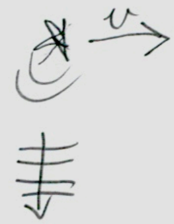
Problem: maxwell's ether-based electrodynamics is not an emission theory

In star frame

$$f(x,y,z,t) = f_0 \sin \omega\left(t - \frac{1}{c} y\right)$$



In earth frame



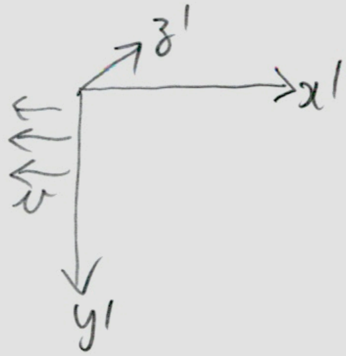
$$f(x,y,z,t) = f_0 \sin \omega\left(t' - \frac{1}{c} y'\right)$$

wave still propogates vertically i.e. along y' axis

Galilean transform

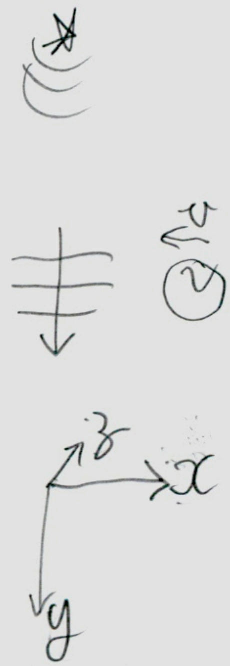
$t' = t$
$x' = x + vt$
$y' = y$
$z' = z$

$\boxed{\text{simplest case} \quad f' = f}$



Einstein recovers stellar aberration from the transformation of the directional cosines

Star frame



$l = n = 0$

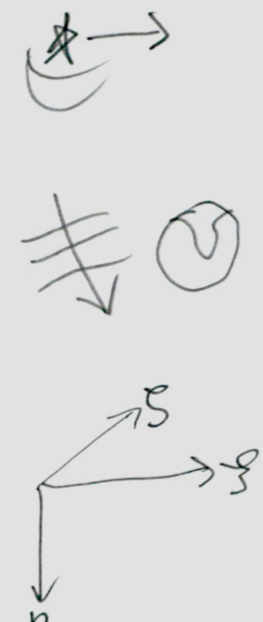
$m = 1$

Transform

$$l' = \frac{l + v/c}{1 + l v/c}$$

$$m' = \frac{m}{\beta(1 + l v/c)}$$

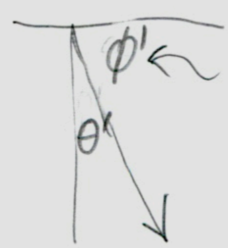
Earth frame



$$l' = \frac{0 + v/c}{1 + 0\, v/c} = v/c$$

$$\boxed{\cos\phi' = v/c}$$

$l = \cos\phi$
$= \cos\frac{\pi}{2} = 0$



$l' = \cos\phi'$

Einstein's "Angle of aberration" is $\phi' = \frac{\pi}{2} - \theta'$

Deriving the transformation for ω, l, m, n

$$\boxed{\Phi = \omega\left(t - \frac{1}{c}(lx + my + nz)\right)}$$

$\Phi' = \Phi$ since points of zero field remain zero fields

$t = \beta(\tau + \frac{v}{c^2}\xi)$ $y = \eta$

$x = \beta(\xi + v\tau)$ $z = \zeta$

$$\Phi = \omega\left[\beta(\tau + \tfrac{v}{c^2}\xi) - \tfrac{1}{c}\left(l\beta(\xi + v\tau) + m\eta + n\zeta\right)\right]$$

$$= \omega\left[\beta\tau + \beta\tfrac{v}{c^2}\xi - \tfrac{1}{c}l\beta\xi - \tfrac{1}{c}l\beta v\tau - \tfrac{1}{c}m\eta - \tfrac{1}{c}n\zeta\right]$$

$$= \omega\left[\beta(1 - \tfrac{lv}{c})\tau - \tfrac{1}{c}\beta(l - \tfrac{v}{c})\xi - \tfrac{1}{c}m\eta - \tfrac{1}{c}n\zeta\right]$$

$$= \underbrace{\omega\beta(1 - \tfrac{lv}{c})}_{\omega' = \omega\beta(1 - \frac{lv}{c})}\left[\tau - \frac{1}{c}\underbrace{\left(\frac{l - v/c}{1 - lv/c}\right)}_{l' = \frac{l - v/c}{1 - lv/c}}\xi - \frac{1}{c}\underbrace{\frac{m}{\beta(1 - \frac{lv}{c})}}_{m' = \frac{m}{\beta(1 - \frac{lv}{c})}}\eta - \frac{1}{c}\underbrace{\frac{n}{\beta(1 - \frac{lv}{c})}}_{n' = \frac{n}{\beta(1 - \frac{lv}{c})}}\zeta\right]$$

Read off transformation by matching terms

$$\boxed{\Phi' = \omega'\left[\tau - \frac{1}{c}l'\xi - \frac{1}{c}m'\eta - \frac{1}{c}n'\zeta\right]}$$