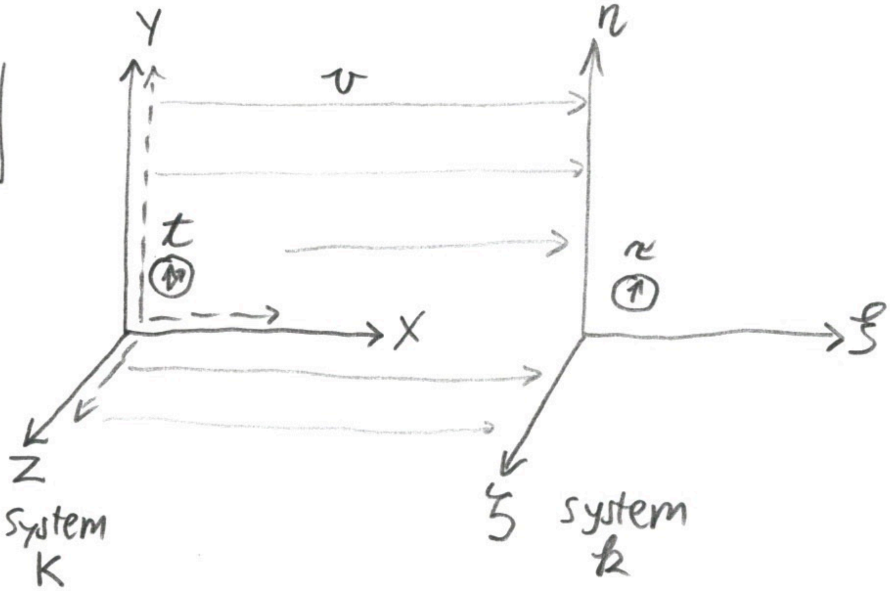
§3 Einstein's (unbelievably cumbersome) derivation of the Lorentz transformation

Set up



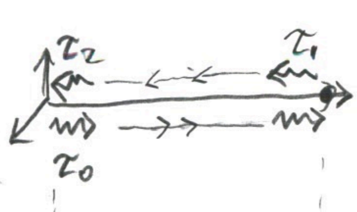
K and k coincide at $\tau = t = 0$

Find τ, ξ, η, ζ as function of t, x, y, z

Homogeneity of space & time
$\Downarrow$
Transformation must be linear

Reflected light signal

In k

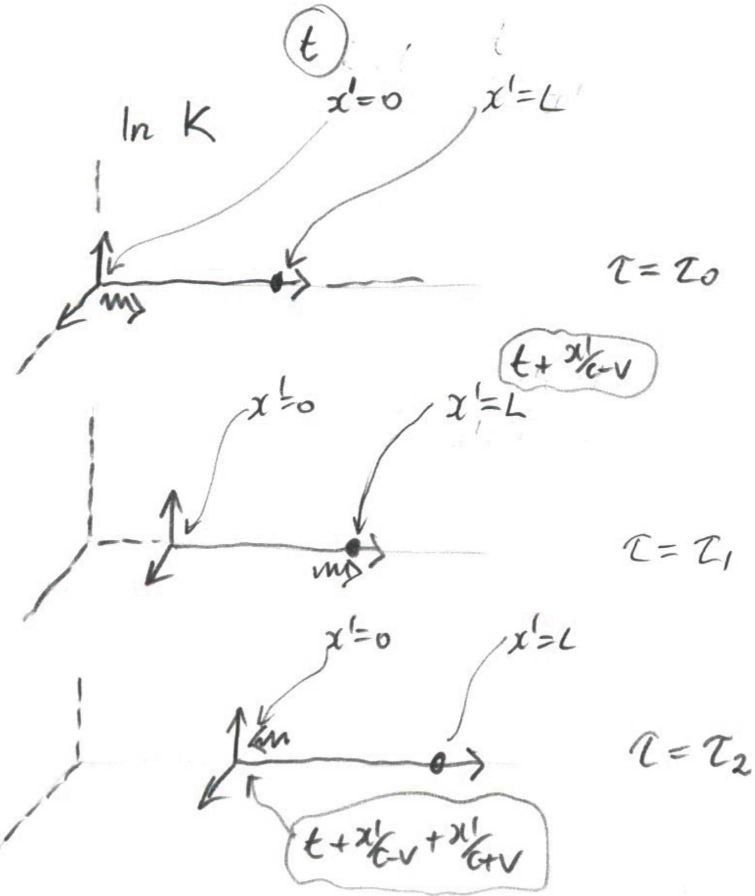


In K: $\leftarrow L \rightarrow$

$x = vt$ $\qquad$ $x = vt + L$

$x' = x - vt = 0$ $\qquad$ $x' = x - vt = L$

$\underbrace{\qquad\qquad\qquad}_{}$
x' values of both points indep. of t



Einstein's definition of clock synchrony for clocks in k:

$$\frac{1}{2}(\tau_0+\tau_2)=\tau_1$$

$$\tau=\tau(x',y,z,t)$$

x' used as coordinate!

K time for outward trip

K time for return

$$\frac{1}{2}\left[\tau(0,0,0,t)+\tau\left(0,0,0,\left\{t+\frac{x'}{c-v}+\frac{x'}{c+v}\right\}\right)\right]=\tau\left(x',0,0,t+\frac{x'}{c-v}\right)$$

Let x' become very small

$$\tau\left(x',0,0,t+\frac{x'}{c+v}\right)\approx\tau(0,0,0,t)+x'\frac{\partial\tau}{\partial x'}+\frac{x'}{c-v}\frac{\partial\tau}{\partial t}$$

$$\tau\left(0,0,0,t+\frac{x'}{c-v}+\frac{x'}{c+v}\right)\approx\tau(0,0,0,t)+x'\left(\frac{1}{c-v}+\frac{1}{c+v}\right)\frac{\partial\tau}{\partial t}$$

$$\frac{1}{2}\left(\frac{1}{c-v}+\frac{1}{c+v}\right)\frac{\partial\tau}{\partial t}=\frac{\partial\tau}{\partial x'}+\frac{1}{c-v}\cdot\frac{\partial\tau}{\partial t}$$

$$\frac{1}{c-v}-\frac{1}{2}\left(\frac{1}{c-v}+\frac{1}{c+v}\right)=\frac{c+v}{c^2-v^2}-\frac{1}{2}\,\frac{c+v-(c-v)}{c^2-v^2}=\frac{v}{c^2-v^2}$$

$$\boxed{\frac{\partial\tau}{\partial x'}+\frac{v}{c^2-v^2}\cdot\frac{\partial\tau}{\partial t}=0}$$

holds at all x', y, z

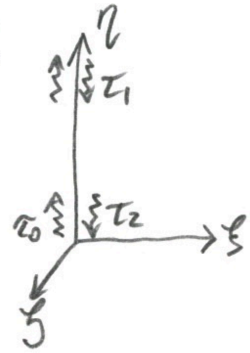
(Repeat argument with origin replaced by (x', y, z))

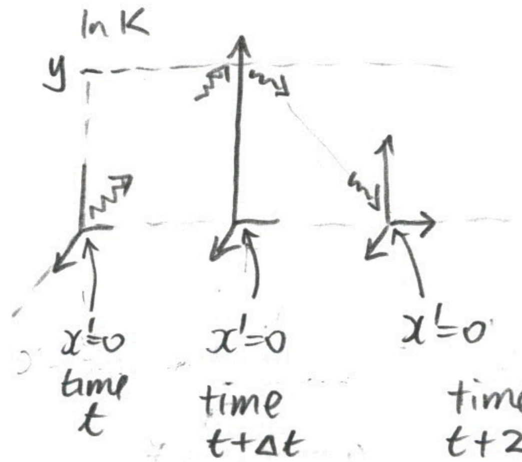
Analogous reasoning on y, z $\Longrightarrow$ $\dfrac{\partial \tau}{\partial z} = 0 \qquad \dfrac{\partial \tau}{\partial y} = 0$

(Not given)

JDN: For y-axis

In k





Never actually need to compute it!

$$\frac{1}{2}\left(\tau_0 + \tau_2\right) = \tau_1$$

$$\therefore \underbrace{\frac{1}{2}\Big[\tau(0,0,0,t) + \tau(0,0,0,t+2\Delta t)\Big] = \tau(0,y,0,t+\Delta t)}$$

small y

$$\tau(0,0,0,t+2\Delta t) \approx \tau(0,0,0,t) + 2\Delta t \frac{\partial \tau}{\partial t}$$

$$\tau(0,y,0,t+\Delta t) \approx \tau(0,0,0,t) + y\frac{\partial \tau}{\partial y} + \Delta t \frac{\partial \tau}{\partial t}$$

$$\Downarrow$$

$$\frac{1}{2}\left(\tau + \tau + 2\Delta t \frac{\partial \tau}{\partial t}\right) = \tau + y\frac{\partial \tau}{\partial y} + \Delta t \frac{\partial \tau}{\partial t}$$

$$0 = y \frac{\partial \tau}{\partial y}$$

$$\boxed{\frac{\partial \tau}{\partial y} = 0}$$

$$\boxed{\tau = a t + B x' + D y + E z}$$

since linear in t, x, y, z

$a = \frac{\partial \tau}{\partial t}$

$D = 0$ since $\frac{\partial \tau}{\partial y} = 0$

$E = 0$ since $\frac{\partial \tau}{\partial z} = 0$

$$B = \frac{\partial \tau}{\partial x'} = -\frac{v}{c^2 - v^2}\frac{\partial \tau}{\partial t} = -a\frac{v}{c^2 - v^2}$$

$$\boxed{\tau = \underbrace{a}_{\text{"}\varphi(v)\text{"}}\left(t - \frac{v}{c^2 - v^2}x'\right)}$$

Find ξ as function t, x', y, z

Light signal $\xi = c\tau$ in k $\longrightarrow$ $\overbrace{x' = (c - v)t}^{\text{same as } x - vt = (c-v)t,\ x = ct}$ in K

$$\tau = a\left(t - \frac{v}{c^2 - v^2}x'\right) = a\left(\underbrace{\frac{x'}{c - v} - \frac{v}{c^2 - v^2}x'}\right)$$

$$= a\frac{c^2}{c^2 - v^2}x' \quad \text{since } \frac{1}{c - v} - \frac{v}{c^2 - v^2} = \frac{c + v}{c^2 - v^2} - \frac{v}{c^2 - v^2} = \frac{c}{c^2 - v^2}$$

$$\underbrace{\boxed{\xi = c\tau = a\frac{c^2}{c^2 - v^2}x'}}_{\text{True for all } t, x, y, z \text{ since transformation is linear}}$$

Find η as function of t, x, y, z

Light signal $\eta = c\tau \longrightarrow y = c_{\text{in } y \text{ dirn}}\, t$

$$c_{\text{in } y \text{ dirn}} = c\sqrt{1-v^2/c^2}$$

$$x' = 0$$

$$\therefore \eta = c\tau = a\left(t - \frac{v}{c^2-v^2}x'\right) = a\,\frac{y}{c\sqrt{1-v^2/c^2}} = a\,\frac{c}{\sqrt{c^2-v^2}}\cdot y$$

True for all x', y, z, t due to linearity

Analogously $\zeta = a\,\frac{c}{\sqrt{c^2-y^2}}\,z$

$$\boxed{\tau = a\left(t - \frac{v}{c^2-v^2}x'\right) \quad \xi = a\,\frac{c^2}{c^2-v^2}x' \quad \eta = a\,\frac{c}{\sqrt{c^2-v^2}}\,y \quad \zeta = a\,\frac{c}{c^2-v^2}\,z}$$

$$\tau = a\left(t - \frac{v}{c^2-v^2}(x-vt)\right) \qquad \xi = a\,\frac{c^2}{c^2-v^2}(x-vt)$$

$$= a\left(t + \frac{v^2}{c^2-v^2}t - \frac{v}{c^2-v^2}x\right)$$

$$= a\cdot\frac{c^2}{c^2-v^2}\left(t - \frac{v}{c^2}x\right)$$

$$a\,\frac{c^2}{c^2-v^2} = \underbrace{\frac{a}{\sqrt{1-v^2/c^2}}}_{\text{“}\varphi(v)\text{”}}\cdot\underbrace{\frac{1}{\sqrt{1-v^2/c^2}}}_{\text{“}\beta\text{”}}$$

$$\boxed{\tau = \varphi(v)\,\beta\left(t - \frac{v}{c^2}x\right) \quad \xi = \varphi(v)\,\beta(x - vt) \quad \eta = \varphi(v)\,y \quad \zeta = \varphi(v)\,z}$$

$\boxed{\text{Compatability with light postulate with } \varphi(v) \text{ undetermined}}$

Light shell expanding at c in K

$x^2+y^2+z^2=c^2t^2$

$\Longrightarrow$

Light shell expanding at c in k

$\xi^2+\eta^2+\zeta^2=c^2\tau^2$

... by direct substitution

$\boxed{\text{Determine } \varphi(v)=1}$

since

$$K(x,y,z,t) \xrightarrow[\text{in x-direction}]{+v} k(\xi,\eta,\zeta,\tau) \xrightarrow[\text{in x direction}]{-v} K'(x',y',t',z')$$

must return original K so that $t'=t$, $x'=x$, ...

By direct substitution, find $t'=\varphi(v)\varphi(-v)t$ $\quad y'=\varphi(v)\varphi(-v)y$

$x'=\varphi(v)\varphi(-v)x$ $\quad z'=\varphi(v)\varphi(-v)z$

e.g.

$$t'=\varphi(-v)\beta(-v)\left(\tau-\tfrac{v}{c^2}\xi\right)$$

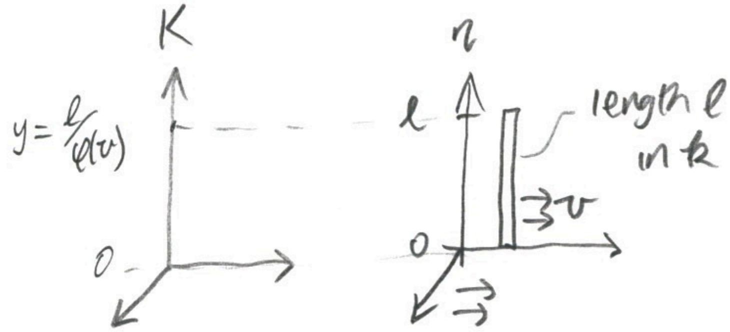
$$=\varphi(-v)\varphi(v)\underbrace{\beta(-v)\beta(v)}\left(t-\cancel{\tfrac{v}{c^2}x}-\tfrac{v}{c^2}(\cancel{x}-vt)\right)$$

$$=\varphi(v)\varphi(-v)\frac{1}{1-v^2/c^2}\left[t\left(1-v^2/c^2\right)\right]$$

$$=\varphi(v)\varphi(-v)\,t$$

$$\boxed{\varphi(v)\varphi(-v)=1}$$

and | $\varphi(v) \sim$ length change of rod perpendicular to direction of motion



$$\underbrace{\underset{v}{\text{motion}} : \; l \longrightarrow \frac{l}{\varphi(v)}}_{\text{same effect for } v \text{ or } -v}$$

$$\boxed{\varphi(v) = \varphi(-v)}$$

so | $\varphi(v)\,\varphi(-v) = 1 \xrightarrow[\varphi(-v)=\varphi(v)]{} [\varphi(v)]^2 = 1 \quad \varphi(v) = \pm 1$

choose $\varphi(v) = +1$ since ξ, x axes point in same direction

Final Result |

$$\tau = \beta\left(t - \frac{v}{c^2}x\right) \qquad \eta = y$$

$$x = \beta\,(x - vt) \qquad \zeta = z$$

$$\beta = \frac{1}{\sqrt{1 - v^2/c^2}}$$