THE FORCE OF NEWTONIAN COSMOLOGY: ACCELERATION IS RELATIVE*

JOHN D. NORTON†‡

Department of History and Philosophy of Science
University of Pittsburgh

1. Introduction. David Malament (1995) has described a natural and satisfying resolution of the traditional problems of Newtonian cosmology—natural in the sense that it effects the escape by altering Newtonian gravitation theory in a way that leaves its observational predictions completely unaffected. I am in full agreement with his approach. There is one part of his account, however, over which Malament has been excessively modest. The resolution requires a modification to Newtonian gravitation theory. Malament presents the modification as so straightforward as to be automatic. This trivializes the crucial postulate, which I shall call the "relativity of acceleration." It is a significant physical statement in its own right and requires careful justification. Moreover the postulate proved easy to overlook for decades of discussion of the paradox.[1] It really only becomes natural from the perspective of the newer geometric methods Malament exploits. There the postulate has become a commonplace. My purpose here is to develop the following:

- While Newtonian cosmology can be repaired satisfactorily, in its traditional form it remains deeply troubled. These troubles can be expressed most vividly as the paradoxical contradictions indicated below. They persist in both the integral and differential formulations of Newtonian gravitation theory. (Section 2)
- Malament's careful geometric treatment is necessarily dense. By taking some liberties with precision, his core result can be expressed in a far simpler form. (Section 4)
- Attempts to avoid the resolution Malament describes do lead to disaster. Therefore this episode can be inverted and used as the strongest extant argument for the relativity of acceleration in Newtonian gravitation theory. (Section 5)

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‡Send reprint requests to the author, Department of History and Philosophy of Science, 1017 Cathedral of Learning, University of Pittsburgh, Pittsburgh, PA 15260.

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[1]The woes and triumphs of Newtonian cosmology have had a rich history, some of which is recapitulated by Malament's paper and this note. This history will be examined in more detail in a study now in preparation.

The sketching of this last argument is the most important component of this note.

2. The Woes of Newtonian Cosmology. In its simplest form, a Newtonian cosmology consists of an infinite three dimensional Euclidean space with a uniform matter distribution. The problem arises when we ask after the net gravitational force on a test mass that results from Newton's inverse square law of gravitation. According to this law, the gravitational force of attraction between any two masses of size m_1 and m_2 separated by distance r is given as

$$F = Gm_1m_2/r^2 \qquad (1)$$

where G is the gravitational constant. To recover the net force, one needs to sum the forces between the test mass and all other masses in the universe. The infinite integral representing this sum proves not to be uniformly convergent: depending on how one approaches the limit, one can recover a force of any nominated magnitude and direction.

We can see this result informally following the approach of Norton (1993). We seek the net gravitational force on a test mass m with position vector **r** in three-space, as in Figure 1. Divide the homogeneous source mass distribution of density r into a sphere centered on some arbitrary point $\mathbf{r}_0$ and on whose surface the test mass sits. Locate the remaining masses in concentric spherical shells that contain the test mass. Using a well-known result in Newtonian gravitation theory, we conclude that the shells each exert no net force on the test mass. The net force on the test mass is simply the force

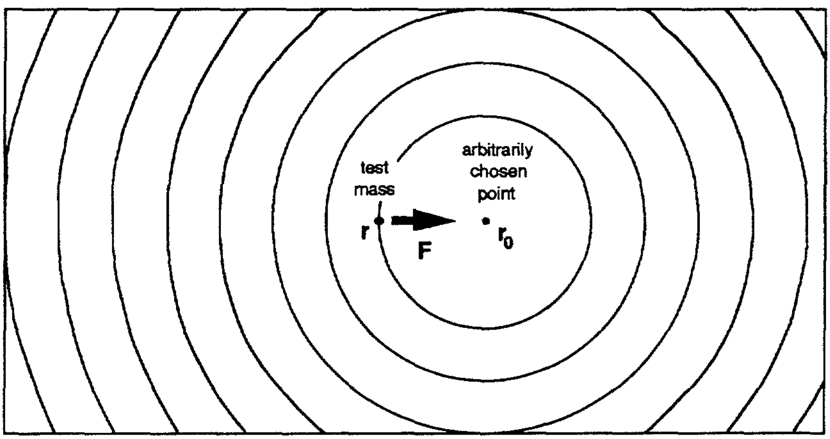

Figure 1. Arbitrariness of Gravitational force **F** on Test Mass in Newtonian Cosmology.

$$\mathbf{F} = -m(4/3)\pi G\rho(\mathbf{r} - \mathbf{r}_o)$$

exerted by the sphere of matter centered on $\mathbf{r}_o$ and on whose surface the test mass sits. Since the position $\mathbf{r}_o$ of the sphere's center is arbitrarily chosen, the force F is arbitrary in both magnitude and direction.

Thus we have:

Contradiction 1: Indeterminacy of gravitational force
The following cannot all be true:
(a) Newtonian mechanics: Newtonian kinematics plus Newton's three laws of motion.
(b) Newton's inverse square law of gravitation (1).
(c) Matter is distributed homogeneously and isotropically in an infinite Euclidean space.
(d) There is a unique gravitational force on a test mass fixed by (b) and (c).

Malament's first response to this indeterminacy problem is to notice that most treatments of Newtonian gravitation theory do not simply use Newton's inverse square law (1). Rather they require that the vector force **f** on a unit test mass be given by the gradient $-\nabla\phi$ of a new intermediate quantity, the gravitational potential ϕ, which is governed by Poisson's equation

$$\nabla^2\phi = 4\pi G\rho \qquad (2)$$

where ρ is the mass density. For the case of a homogenous, isotropic matter distribution (i.e., ρ is a constant in space but not necessarily in time), the closest we can get to a homogeneous and isotropic field that satisfies (2) are the canonical solutions in which the field $\phi(\mathbf{r})$ at position **r** is

$$\phi(\mathbf{r}) = (2/3)\pi G\rho(t)\,|\mathbf{r} - \mathbf{r}_0|^2 \qquad (3)$$

where $\mathbf{r}_0$ is some arbitrarily selected point in space.[2] The resulting force field is given as

$$\mathbf{f} = -\nabla\phi = -(4/3)\pi G\rho(t)(\mathbf{r} - \mathbf{r}_o). \qquad (4)$$

It represents a force field everywhere directed towards the central point $\mathbf{r}_o$. The matter density $\rho(t)$ is some function of t whose form is fixed by further assumption. We shall assume that the matter is in the form of pressureless dust in free fall, so that the free fall motion of the dust particles will determine how the dust cloud expands or contracts and thus the functional form of $\rho(t)$.

[2]Poisson's equation (2) leaves ϕ underdetermined up to a arbitrary, additive harmonic function ψ, which satisfies $\nabla^2\psi = 0$. If one requires isotropy of the field about $\mathbf{r}_0$ this underdetermination disappears.

The addition of the potential ϕ and Poisson equation (2) does not materially affect the indeterminacy contradiction of Newtonian cosmology. There are as many canonical solutions (3) as there are choices for $\mathbf{r}_0$. Each distinct choice of $\mathbf{r}_0$ leads to a different force

$$\mathbf{F} = -m\nabla\phi = -m(4/3)\pi G\rho(\mathbf{r} - \mathbf{r}_o)$$

on the test body. This force $\mathbf{F}$ is precisely the same as the one computed through Figure 1 for a test mass by means of the sphere and shells construction. What has changed, however, is that once the point $\mathbf{r}_0$ is selected, the forces on test masses at spatial positions other than $\mathbf{r}$ become fixed if we also assume that the solution is canonical.

On the basis of the naturalness of the canonical solutions, one may want to proceed by stipulating *in addition* that, in Newtonian cosmology, we proceed by picking just one canonical solution and ignoring the rest. The troubles still remain in the oddness of the result that each canonical solution has a preferred central point $\mathbf{r}_0$ even though the space and matter distribution that generated it are fully homogeneous and isotropic. The inhomogeneity cuts to the heart of the theory, for at that unique central point *alone* we see immediately from (4) that $\mathbf{f} = 0$, so that matter in free fall at that point alone is unaccelerated and all other matter accelerates towards that point with an acceleration proportional to the distance from it. This problem can be summarized as

Contradiction 2: Inhomogeneity of ϕ
The following cannot all be true:
(a) Newtonian mechanics: Newtonian kinematics plus Newton's three laws of motion.
(b′) The gravitational potential ϕ satisfies Poisson's equation (2).
(c) Matter is distributed homogeneously and isotropically in an infinite Euclidean space.
(d′) The gravitational potential ϕ is homogeneous.

3. The Relativity of Acceleration. One can only escape the contradictions by denying one or more of the component clauses. Historically, drastic assumptions have been invoked to effect the escape. Layzer (1954) urged that we give up (c) and consider only huge but finite distributions of matter. Einstein (1917), in his famous cosmology paper, preferred to entertain a slight modification to the inverse square law of gravitation (b)/Poisson's equation (b′).[3]

The escape Malament urges is less drastic. While the canonical solutions are inhomogeneous with a preferred central point $\mathbf{r}_0$ and distinguished

[3]He added a cosmological term to Poisson's equation which became $\nabla^2\phi - \lambda\phi = 4\pi G\rho$. For constant ρ, this modified field equation admits the homogeneous, isotropic solution $\phi = -4\pi G\rho/\lambda$.

from one another by the choice of $\mathbf{r}_0$, nothing *observable* depends on the choice of $\mathbf{r}_0$. If we take any two points $\mathbf{r}_1$ and $\mathbf{r}_2$ in space, then the relative acceleration **a** between any two masses in free fall at these points—the observable quantity—is derived directly from (4) as

$$\mathbf{a} = -(4/3)\pi G\rho(t)(\mathbf{r}_2 - \mathbf{r}_1) \tag{5}$$

That is, the arbitrarily chosen point $\mathbf{r}_0$ does not appear in the expression for the observable **a** so that the choice of $\mathbf{r}_0$ has no influence upon it. Again, while a particle at $\mathbf{r}_0$ is distinguished as being the only particle that moves inertially in the field, nothing observable reveals this special status. Indeed we could stipulate that *any* point, for example the $\mathbf{r}_1$ of equation (5), is the preferred center $\mathbf{r}_0$ and the observable relative accelerations would be compatible with it. Relabelling $\mathbf{r}_1$ as $\mathbf{r}_0$ in (5) reduces (5) to the canonical force field equation (4).[4]

In ordinary Newtonian gravitation theory, one decomposes a free fall motion into two parts: an inertial trajectory and a deflection due the gravitational field. The motion at $\mathbf{r}_0$ is distinct in so far as it is the only motion without gravitational deflection. Since the distinctness of $\mathbf{r}_0$ translates into nothing observable we may suspect that whatever distinguishes the special point $\mathbf{r}_0$ from all others cannot correspond to anything physical. It is merely a conventional element of the theory. That suggestion is Malament's proposal for the escape to both contradictions. It is a modification to Newtonian kinematics (a):

> *Relativity of Acceleration*
> The decomposition of gravitational free fall into an inertial trajectory and a gravitational deflection is conventional; we are free to divide free fall motion into any combination of inertial motion and gravitational deflection we please, as long as the latter corresponds to a gravitational potential satisfying Poisson's equation.

In the spacetime approach, the specification of which motions are inertial is made by the affine structure, represented by the derivative operator ∇_a, and the arbitrariness of the decomposition into inertial and gravitational parts appears in our freedom to adjust ∇_a by the term $C^a{}_{bc} = -t_{bc}\nabla^a\phi$ in Malament's "geometrization lemma." In the geometrized case, we choose to attribute all the free fall motion to inertial motion so that all gravitational deflections vanish. In this case, what is customarily thought of as gravitational effects arise as a geometric curvature in the affine structure; the gravitational field has been geometrized.

What makes this escape more attractive than the others is that it leaves

[4]What is essential in deriving this result is that the masses of the bodies at $\mathbf{r}_1$ and $\mathbf{r}_2$ do not appear in equation (4). This follows from the celebrated equality of inertial and gravitational mass in Newtonian gravitation theory.

the observational consequences of Newtonian gravitation theory untouched. What remains to be shown is that this escape does succeed.

4. Malament's Core Proposition. Malament's core proposition is stated in his Section 4. Its principal burden is to show that the relativity of acceleration is indeed sufficient to eliminate both indeterminacy and inhomogeneity contradictions. Loosely it states that a canonical solution is homogeneous in its physically significant properties if we accept the relativity of acceleration. This immediately resolves the inhomogeneity contradiction, since whatever inhomogeneity is present in a canonical solution is merely an artifact of our conventional choice of decomposition of free fall. It also follows from the homogeneity that all canonical solutions (based on the same ρ) are the same physically. Thus the indeterminacy contradiction is resolved: the different gravitational forces assigned by different canonical solutions only reflect differences in our conventional choice of decomposition of free fall motion.

The essence of Malament's core proposition can be captured in nongeometric terms as a simple covariance property of canonical solutions. To see it, take some particular canonical solution and consider an inertial frame of reference I in which the cosmic matter at the central point $\mathbf{r}_0$ is

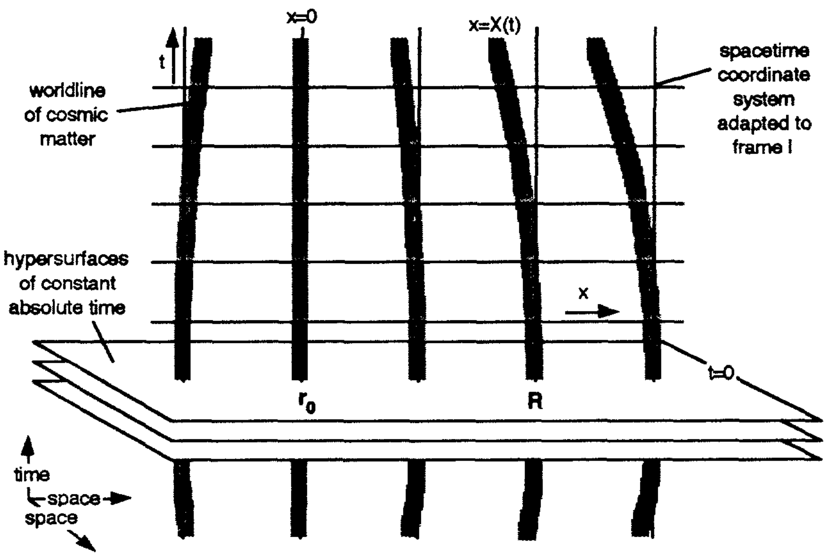

Figure 2. Canonical solution in Frame of Reference I.

at rest.[5] This is depicted in a spacetime diagram in Figure 2. Let (x,y,z,t) be a coordinate system adapted to this inertial frame in the usual sense.[6] The central point $\mathbf{r}_0$ has coordinates $x=y=z=0$ so we have $\mathbf{r}_0 = \mathbf{0}$. In this coordinate system, the gravitational potential is given by

$$\phi(r,t) = (2/3)\pi G\rho(t)r^2 \tag{6}$$

where $r^2 = x^2 + y^2 + z^2$. The free fall trajectories of matter are governed by

$$d^2\mathbf{r}/dt^2 = -(4/3)\pi G\rho(t)\mathbf{r} \tag{7}$$

As before, the functional form of $\rho(t)$ will follow from the free fall motion of the cosmic dust. Select some arbitrary point $\mathbf{R}$ in space other than $\mathbf{r}_0$ at the instant $t = 0$. The trajectory of free fall of the cosmic matter through this point will be described by some functions $\mathbf{R}(t) = (X(t), Y(t), Z(t))$ governed by (7) so that

$$d^2\mathbf{R}(t)/dt^2 = -(4/3)\pi G\rho(t)\mathbf{R}(t) \tag{8}$$

We now transform to an accelerating frame of reference I′ with adapted coordinates (x′,y′,z′,t′) in which the free fall trajectory through $\mathbf{R}$ is at rest. See Figure 3. The transformation is given by

$$x' = x - X(t) \qquad y' = y - Y(t) \qquad z' = z - Z(t) \qquad t' = t. \tag{9}$$

It follows from (9) that $\mathbf{r} = \mathbf{r}' + \mathbf{R}(t)$, where $\mathbf{r}' = (x',y',z')$, so that (7) can be rewritten as

$$d^2(\mathbf{r}' + \mathbf{R}(t))/dt^2 = -(4/3)\pi G\rho(t)(\mathbf{r}' + \mathbf{R}(t)).$$

With substitution from (8) and setting $t' = t$ and $\rho'(t') = \rho(t)$, we recover

$$d^2\mathbf{r}'/dt'^2 = -(4/3)\pi G\rho'(t')\mathbf{r}' \tag{7′}$$

which derives from the gravitational potential

$$\phi'(r',t') = (2/3)\pi G\rho'(t')r'^2. \tag{6′}$$

Tacit in equation (6′) is that the gravitational potential ϕ has *not* transformed in the usual manner of Newtonian theory as a scalar. Rather it transformed as[7]

$$\phi \rightarrow \phi' = \phi + \mathbf{r}\cdot d^2\mathbf{R}/dt^2 + \phi(\mathbf{R}) \tag{10}$$

[5]Here a frame of reference is understood as a set of non-intersecting timelike worldlines that fill the spacetime. In an inertial frame, the trajectories are inertial. What motivates this definition is the picture of a frame of reference as a body (of possibly negligible material) filling the space and with some definite state of motion. The worldlines of the frame are the worldlines of the individual points in the body.

[6]The coordinates (x,y,z) are Cartesian coordinates of the space, t coincides with absolute time and the curves of constant (x,y,z) coincide with the worldlines of the inertial frame.

[7]Since $\phi + \mathbf{r}\cdot d^2\mathbf{R}/dt^2 + \phi(\mathbf{R}) = (2/3)\pi G\rho(r^2 - 2\mathbf{r}\cdot\mathbf{R} + R^2) = (2/3)\pi G\rho\,(\mathbf{r} - \mathbf{R})^2 = \phi'$.

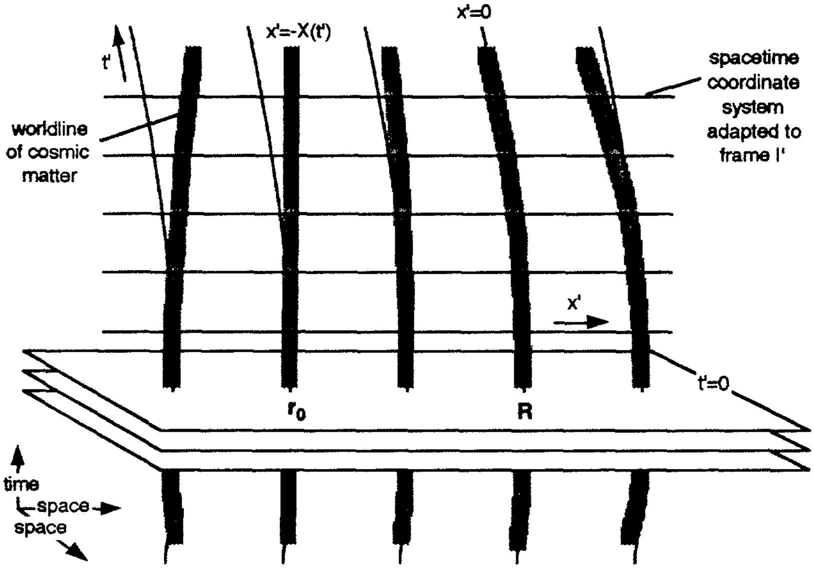

Figure 3. Canonical solution in Frame of Reference I′.

The additional terms in the transformation law (10) exploit our freedom to adjust the gravitational potential in any way that does not disturb the observables, the relative accelerations of equation (5). They are unaffected by the addition of a constant field $\phi(\mathbf{R})$. They are also unaffected by the addition of the term $\mathbf{r}\cdot d^2\mathbf{R}/dt^2$. It merely corresponds to the addition of a homogeneous field $d^2\mathbf{R}/dt^2$ to the field strength, which becomes

$$-\nabla'\phi' = -\nabla\phi - d^2\mathbf{R}/dt^2$$

Since this homogeneous field accelerates all bodies alike in space, its presence is invisible to the observable relative accelerations of equation (5).

Comparing equations (6) and (7) with equations (6′) and (7′), we see that the free fall trajectories and cosmic matter density appear identical in the frame I and the frame I′. That is, select any trajectory of the cosmic matter in the canonical solution and align the rest frame with it according to the transformation (9). In any such frame, the canonical solution will look exactly the same; the cosmic matter will now be accelerated in the same manner towards the arbitrarily selected matter at $\mathbf{r} = \mathbf{R}$. In traditional approaches, one had to allow that some difference remained. At most one of the frames so selected—the frame I in this case—could be inertial, so the point at $\mathbf{r} = \mathbf{r}_0 = \mathbf{0}$ was the unique point at which the cosmic matter moved inertially. Any other I′ must be non-inertial and the cosmic matter at the corresponding center point $\mathbf{r} = \mathbf{R}$ accelerated. According to the relativity of acceleration, however, we are free to stipulate which of these frames is inertial. In particular, we may choose to nominate either I or I′ as inertial. Therefore mere conventional stipulation is all that accords preferred status to the central point in a given frame dependent description of the canonical solution.

This result gives us the sense in which the canonical solution is homogeneous. No point in space is distinguished as the preferred center. Any point can take on that role merely by a suitable stipulation as to which frame is inertial.

More formally Malament's core proposition is a symmetry principle.[8] It asserts that a canonical solution admits a rich class of symmetry transformations, under which the canonical solution remains unchanged. These symmetry transformations include that given by (9) which maps the frame I into the frame I′. This type of symmetry principle is familiar to us as a relativity principle. The most famous example is the relativity principle of special relativity, which expresses the equivalence of all inertial frames of reference. The relevant symmetry transformations comprise the Lorentz group. They map arbitrary inertial frames into arbitrary inertial frames. The accelerating frames of the canonical solutions are equivalent in precisely the same sense as the inertial frames of special relativity are equivalent. Thus we can think of Malament's core result as stating a relativity principle which extends the relativity of motion to acceleration in Newtonian gravitation theory—but only within the narrow confines of the canonical solutions of Newtonian cosmology.

5. The Case for the Relativity of Acceleration. The relativity of acceleration—that we are free to stipulate which are the inertial motions—is a standard assumption of the geometrized theory of Newtonian spacetime. But what justifies this assumption? Malament follows the standard approach in noting (Section I, footnote 8) that observation cannot pick out which are the true inertial motions. All the different decompositions of free fall into an inertial motion and a gravitational deflection are observationally equivalent. While this observational equivalence certainly inclines us toward accepting the relativity of acceleration, the mere fact of observational equivalence alone cannot decide the issue. In recent philosophy of space and time there have been two prominent conventionality claims and both have been supported by an observational equivalence of

[8]In developing Malament's core proposition, we have purchased simplicity of expression at the cost of some imprecision. In the formulation given above it is only implicit that the symmetry transformations also preserve absolute time and the Euclidean geometry of the hypersurfaces of simultaneity. Also one must exercise care in expressing true relativity principles as covariance principles for not all covariance principles express relativity principles. This has been a subject of decades of controversy. See Norton 1993a, especially Section 6.

the different choices. They are Reichenbach's celebrated claims for the conventionality of geometry and for the conventionality of simultaneity in special relativity. In spite of support from observational equivalence, the acceptability of the conventionality claims remains a matter of extended debate (see Norton 1992, 1994).

Thus, even though two theoretical descriptions may be observationally equivalent, it would seem that there may remain other grounds for deciding between the two descriptions and that observational equivalence does not automatically license a conventionality claim. The various choices of simultaneity relation in special relativity may be observationally equivalent, for example. Yet Malament (1977) has argued that there are compelling reasons for preferring just one of them, the so-called standard simultaneity relation.[9] Such distinguishing grounds seem readily available in Newtonian gravitation theory also—at least in the examples most commonly considered. The most familiar application of the theory is the gravitational field of a body such as the sun. In the geometrized spacetime view, one routinely assumes that the free fall trajectories of the field can be decomposed conventionally into inertial trajectories and gravitational deflection. However just one decomposition stands out. In regions of spacetime remote from the source mass, one naturally assumes that there is no gravitational deflection and it follows that the inertial structure is affinely flat. It seems natural to require that this flatness persists as we proceed into regions close to the source mass and also that the deflecting gravitational field be distributed isotropically in the space surrounding the source mass, reflecting the isotropy of the matter distribution. These two requirements—affine flatness of the inertial structure and isotropy of the gravitational field—pick out a unique decomposition.[10] Is this decomposition just the simplest of a range of conventional choices? Or does it reflect a deeper aspect of physical reality? Are we in fact free to choose the decomposition of free fall by convention? If we had to decide this question on the basis of the familiar example of the gravitational field of the sun, then we may well expect a somewhat tortured debate, revolving at least in part on a determination of how closely the conventionality claimed in the relativity of acceleration is like that of the conventionality of geometry or simultaneity.

The whole question looks quite different when we turn to the example of Newtonian cosmology. The challenges to the conventionality of a

[9]Glymour (1977) and Malament (1977a) describe another striking case. There are relativistic spacetimes which are observationally indistinguishable but which have distinct global structure.

[10]Affine flatness alone is not strong enough for uniqueness of decomposition. It follows from Malament's Geometrization Lemma (G3) that one can always decompose a flat affine structure into a second flat affine structure plus the gravitational deflection of a unidirectional gravitational field that satisfies $\nabla_b \nabla^a \phi = 0$.

choice all depend on there being some basis for picking between the choices. If observation cannot pick between simultaneity relations, then Malament (1977) showed that the requirement of definability in terms of causal structure does force a single choice. And the requirements of flatness and field isotropy forces a particular decomposition of free fall motion in the field of the sun. In both cases some condition stated in terms of theoretical structure was used to pick out a unique choice. In the case of Newtonian cosmology this cannot be done. Our choice is to decide which in the class of frames related by the group of transformations (9) is to be designated as inertial. *Prior to the decision of which are the inertial motions, no condition stated in either observational or theoretical terms can pick between the frames of reference in question.* Every frame in the class relates to the canonical solution in an identical way, on both the observational and theoretical level—this is the force of Malament's core proposition. Any condition that points to a frame I on the basis the observational or theoretical structure of the canonical solution must, therefore, also point to all other frames I′ related to I by the symmetry transformation (9). Because the transformation (9) is a symmetry of the canonical solution, the situation is akin to that in special relativity. No observational or theoretical condition within the theory can pick between the inertial frames and lead to the selection of one as a preferred rest frame.

With the example of Newtonian cosmology in hand, let us return to the strongest claim:

> *Absoluteness of Acceleration*
> In Newtonian gravitation theory, it is always possible to decompose gravitational free fall non-conventionally into a unique inertial trajectory and gravitational deflection.

The case of Newtonian cosmology provides a clear counterexample to this absoluteness claim. At best we might hope to salvage the absoluteness by requiring that, in *some* cases, there is a non-conventional decomposition. But we must surely find our confidence shaken in an absolute that is only absolute some of the time.

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