$y \neq 0\}$. Thus, if $w_1, w_2, \ldots, w_{n-k} \in$

$\frac{y^*\Lambda y}{y^*y}$

U^*w_{n-k}

$\sum_{i=1}^{n} \lambda_i |y_i|^2$

U^*w_{n-k}

$\sum_{i=1}^{n} \lambda_i |y_i|^2$

U^*w_{n-k}

$y_{k-1}=0$

up $\sum_{i=k}^{n} \lambda_i |y_i|^2 \geq \lambda_k$

$+\cdots+|y_n|^2=1$

$\ldots, U^*w_{n-k}$

ut (4.2.9) shows that equality namely $w_i = u_{n-i+1}$, where $U=$

ith "min" and "max" since the (4.2.13) is similar. □

of (4.2.13).

eigenvalues $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$.

$2, \ldots, n$

$k = 1, 2, \ldots, n$

where, in both cases, S_j denotes a subspace of dimension j and the outer optimization is over all subspaces of the indicated dimension.

2. If $A \in M_n$ is Hermitian, show that the following three optimization problems all have the same solution:

(a) $\max\limits_{x^*x=1} x^*Ax$

(b) $\max\limits_{x \neq 0} \frac{x^*Ax}{x^*x}$

(c) $\max\limits_{x^*Ax=1} \frac{1}{x^*x}$ if at least one eigenvalue of A is positive

3. If $A \in M_n$ is Hermitian and $x^*x = 1$, show that

$$\lambda_{\max} \geq x^*Ax \geq \lambda_{\min}$$

That $\max\limits_{\vec{x} \neq \vec{0}} \frac{(A\vec{x}|\vec{x})}{(\vec{x},\vec{x})} \leq \lambda_n$, for λ_n the largest eigenvalue of A.

4. Show that the assumption that A is Hermitian is essential in $\boxed{(4.2.2)}$ by considering $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$. What is $\max\{x^TAx/x^Tx : 0 \neq x \in \mathbf{R}^2\}$? What is $\max \operatorname{Re}\{x^*Ax/x^*x : 0 \neq x \in \mathbf{C}^2\}$?

5. Let $A \in M_n$ have eigenvalues $\{\lambda_i\}$. Show that, even if A is not Hermitian, one has the bounds

$$\min_{x \neq 0} \left| \frac{x^*Ax}{x^*x} \right| \leq |\lambda_i| \leq \max_{x \neq 0} \left| \frac{x^*Ax}{x^*x} \right|, \quad i = 1, 2, \ldots, n$$

Hint: Consider x = an eigenvector of A, and $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ to show that neither bound need be sharp.

4.3 Some applications of the variational characterizations

Among the many important applications of the Courant–Fischer theorem, one of the simplest is to the problem of comparing the eigenvalues of $A+B$ with those of A. We denote the eigenvalues of a matrix A by $\lambda_i(A)$.

4.3.1 Theorem (Weyl). Let $A, B \in M_n$ be Hermitian and let the eigenvalues $\lambda_i(A)$, $\lambda_i(B)$, and $\lambda_i(A+B)$ be arranged in increasing order (4.2.1). For each $k = 1, 2, \ldots, n$ we have

$$\lambda_k(A) + \lambda_1(B) \leq \lambda_k(A+B) \leq \lambda_k(A) + \lambda_n(B) \tag{4.3.2}$$

Proof: For any nonzero $x \in \mathbf{C}^n$ we have the bound

$$\lambda_1(B) \leq \frac{x^*Bx}{x^*x} \leq \lambda_n(B)$$

and hence for any $k = 1, 2, \ldots, n$ we have