

VOLUME AND TOPOLOGY OF BOUNDED AND CLOSED HYPERBOLIC 3-MANIFOLDS.

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ABSTRACT. Let N be a compact, orientable hyperbolic 3-manifold with ∂N a connected totally geodesic surface of genus 2. If N has Heegaard genus at least 5, then its volume is greater than 6.89. The proof of this result uses the following dichotomy: either the shortest *return path* (defined by Kojima-Miyamoto) of N is long, or N has an embedded codimension-0 submanifold X with incompressible boundary $T \sqcup \partial N$, where T is the frontier of X in N , which is not a book of I -bundles. As an application of this result, we show that if M is a closed, orientable hyperbolic 3-manifold with $\dim_{\mathbb{Z}_2} H_1(M; \mathbb{Z}_2) \geq 5$, and if the cup product map $H^1(M; \mathbb{Z}_2) \otimes H^1(M; \mathbb{Z}_2) \rightarrow H^2(M; \mathbb{Z}_2)$ has image of dimension at most one, then M has volume greater than 3.44.

1. INTRODUCTION

The results of this paper support an old theme in the study of hyperbolic 3-manifolds, that the volume of a hyperbolic 3-manifold increases with its topological complexity. The first main result, Theorem 1.1 below, reflects this theme in the context of hyperbolic manifolds with totally geodesic boundary. We denote the Heegaard genus of a 3-manifold N by $\text{Hg}(N)$.

Theorem 1.1. *Let N be a compact, orientable hyperbolic 3-manifold with ∂N a connected totally geodesic surface of genus 2. If $\text{Hg}(N) \geq 5$, then N has volume greater than 6.89.*

Our second main result concerns closed manifolds:

Theorem 1.2. *Let M be a closed, orientable hyperbolic 3-manifold with*

$$\dim_{\mathbb{Z}_2} H_1(M; \mathbb{Z}_2) \geq 5,$$

and suppose that the cup product map $H^1(M; \mathbb{Z}_2) \otimes H^1(M; \mathbb{Z}_2) \rightarrow H^2(M; \mathbb{Z}_2)$ has image of dimension at most one. Then M has volume greater than 3.44.

Theorem 1.1 builds on work by Kojima and Miyamoto. Miyamoto proved that the minimal-volume compact hyperbolic 3-manifolds with totally geodesic boundary of genus g decompose into g regular truncated tetrahedra, each with dihedral angle $\pi/3g$ [18, Theorem 5.4]. Their volumes increase with g , taking values 6.452... for $g = 2$ and 10.428... for $g = 3$. Miyamoto's theorem implies in particular that the minimal volume compact hyperbolic manifolds with totally geodesic boundary of genus g have Heegaard genus equal to $g + 1$.

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Prior to the work of Miyamoto, Kojima-Miyamoto established the universal lower bound of 6.452... for the volume of compact hyperbolic 3-manifolds with totally geodesic boundary and described the minimal-volume examples [17]. In fact their result is slightly stronger: if N is not “simple”, then $\text{vol}(N) > 6.47$. In the terminology of [17], a hyperbolic manifold with geodesic boundary is simple if it admits a decomposition into truncated polyhedra with one internal edge. Such manifolds are classified in [17, Lemma 2.2], and include those with minimal volume.

Theorem 1.1 can be regarded as an extension of Kojima-Miyamoto’s theorem. Experimental results of Frigerio-Martelli-Petronio [12] suggest that the next smallest manifolds with geodesic boundary after those of minimal volume have volume greater than 7.1, so it is likely that Theorem 1.1 is not close to sharp. Nonetheless it seems to be the only result of its kind in the literature.

Theorem 1.2 will be deduced by combining Theorem 1.1 with results from [6] and [8]. The transition from these results to Theorem 1.2 involves two other results, Theorems 1.3 and 1.4 below, which are of independent interest.

Theorem 1.3. *Let M be a closed, orientable hyperbolic 3-manifold containing a closed, connected incompressible surface of genus 2 or 3, and suppose that $\text{Hg}(M) \geq 8$. Then M has volume greater than 6.89.*

Theorem 1.3 is analogous to [6, Theorem 6.5], and follows in a similar way: we apply Theorem 1.1 and the results of Miyamoto and Kojima-Miyamoto discussed above, using work of Agol-Storm-Thurston [2], to the output of the topological theorem below. The notation in the statement is taken from [6]. In particular, below and in the remainder of this paper, we will use the term “simple” as it is defined in [6, Definitions 1.1], which differs from its usage in [17] mentioned above. We also recall the definitions of $M \setminus S$ from the first sentence of [6], and “kish” from Definition 1.1 there.

Theorem 1.4. *Suppose that M is a closed, simple 3-manifold which contains a connected closed incompressible surface of genus 2 or 3, and that $\text{Hg}(M) \geq 8$. Then M contains a connected closed incompressible surface S of genus at most 4, such that either $\bar{\chi}(\text{kish}(M \setminus S)) \geq 2$, or S is separating and $M \setminus S$ has an acylindrical component N with $\text{Hg}(N) \geq 7$.*

Theorem 1.4 follows by application of [6, Theorem 5.8] jointly with [6, Theorem 3.2]. It is the analog of [6, Corollary 5.9] for manifolds possessing an incompressible surface of genus 3.

The proof of Theorem 1.2 follows the outline of [6, Theorem 6.8]. In place of the results concerning 3-free groups used in [6], the proof uses results of [8] concerning 4-free groups, and Theorem 1.3 above replaces [6, Theorem 6.5].

All the theorems stated above are proved in Section 7. Sections 2–6 constitute preparation for the proof of Theorem 1.1. We introduce *return paths*, defined by Kojima [16], and (i, j, k) *hexagons* in Section 2. (The analysis of (i, j, k) hexagons is a crucial element of [17], but we have borrowed our notation for them from Gabai, Meyerhoff and Milley’s paper [13], which

is set in a different context.) Lemmas 2.7 and 2.8, which are due to Kojima-Miyamoto [17], respectively give an absolute lower bound on ℓ_1 , and a lower bound for ℓ_2 in terms of ℓ_1 . Lemma 2.9 refines Lemma 2.8, giving a bound for ℓ_2 which improves that of [17] when ℓ_1 is in a certain interval.

Section 3 describes Kojima-Miyamoto’s volume bounds. The main result rigorously establishes a lower bound which is apparent from inspection of [17, Graph 4.1].

Proposition 3.7. *Let N be a hyperbolic 3-manifold with ∂N connected, totally geodesic, and of genus 2, satisfying $\cosh \ell_1 \geq 1.215$. Then N has volume greater than 6.89.*

Also in Section 3, Proposition 3.9 shows that any manifold with no $(1, 1, 1)$ hexagon has a shortest return path satisfying $\cosh \ell_1 \geq 1.215$, thus has volume greater than 6.89 by the above. In the remaining sections, we explore the topological consequences of the presence of a $(1, 1, 1)$ hexagon in \tilde{N} , when $\cosh \ell_1 \leq 1.215$. In Section 6, we show that under these circumstances, N contains a submanifold X which is a *nondegenerate trimonic manifold relative to ∂N* (see Definition 5.7).

Section 4 develops results from the theory of books of I -bundles (which were introduced in [1]) that we use in Section 5. There we introduce trimonic manifolds and prove, in Proposition 5.14, that such a manifold X does not have the structure of a book of I -bundles. It follows that X has *kishkes* (or *guts*, cf. [2]) with negative Euler characteristic. Using volume bounds due to Agol-Storm-Thurston, we obtain the following result.

Theorem 7.4. *Let N be a compact, orientable hyperbolic 3-manifold with ∂N a connected totally geodesic surface of genus 2. If $\cosh \ell_1 \leq 1.215$ and there is a $(1, 1, 1)$ hexagon in \tilde{N} , then $\text{Hg}(N) \leq 4$ or $\text{vol}(N) > 7.32$.*

Together with the results of Section 3, this implies Theorem 1.1.

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2. GEOMETRIC PRELIMINARIES

Suppose N is a hyperbolic 3-manifold with totally geodesic boundary. Its universal cover \tilde{N} may be identified with a convex subset of \mathbb{H}^3 bounded by a collection of geodesic hyperplanes. The following terminology was introduced in [16] and used extensively in [17]; we will use it here as well.

Definition 2.1. Let N be a hyperbolic 3-manifold with totally geodesic boundary, and let $\tilde{N} \subset \mathbb{H}^3$ be its universal cover. A *short cut* in \tilde{N} is a geodesic arc joining the closest points of two distinct components of $\partial\tilde{N}$. A *return path* in N is the projection of a short cut under the universal covering map.

It is an easy consequence of the definitions that each return path is a homotopically nontrivial geodesic arc properly immersed in N , perpendicular to ∂N at each of its endpoints. Corollary

3.3 of [16] asserts that for a fixed hyperbolic manifold N with geodesic boundary and $K \in \mathbb{R}$, there are only finitely many return paths in N with length less than K . Thus the collection of return paths may be enumerated as $\{\lambda_1, \lambda_2, \dots\}$, where for each $i \in \mathbb{N}$, the length of λ_i is less than or equal to the length of λ_{i+1} . Fixing such an arrangement, we will denote by ℓ_i the length of λ_i .

It will prove important to understand the distance in ∂N , properly interpreted, between endpoints of return paths of N .

Definition 2.2. Let N be a compact hyperbolic 3-manifold with connected totally geodesic boundary, and suppose λ is a short cut in \widetilde{N} projecting to λ_i . Fix an endpoint x of λ , and let Π be the component of $\partial\widetilde{N}$ containing x . For $j \in \mathbb{N}$ define d_{ij} to be the minimum, taken over all short cuts λ' projecting to λ_j such that λ' has an endpoint $y \in \Pi$ and $\lambda' \neq \lambda$, of $d(x, y)$.

The requirement above that λ' be distinct from λ ensures that $d_{ii} > 0$. In general, d_{ij} is the length of the shortest geodesic arc in ∂N joining an endpoint of λ_i to an endpoint of λ_j .

A crucial tool for understanding the relationships between the lengths ℓ_i and distances d_{ij} is a class of totally geodesic hexagons in \widetilde{N} which have short cuts as edges. The two lemmas below describe the relevant hexagons.

Lemma 2.3. *Suppose that Π_1, Π_2 and Π_3 are mutually disjoint geodesic planes in \mathbb{H}^3 . For each two-element subset $\{i, j\}$ of $\{1, 2, 3\}$, let λ_{ij} denote the common perpendicular to Π_i and Π_j . Then $\lambda_{12}, \lambda_{13}$ and λ_{23} lie in a common plane Π .*

Proof. We may assume that the three lines $\lambda_{12}, \lambda_{13}$ and λ_{23} do not all coincide, and so by symmetry we may assume that $\lambda_{12} \neq \lambda_{13}$. For $i = 2, 3$ the line λ_{1i} meets Π_1 orthogonally at some point p_i . Let $L \subset \Pi_1$ denote the line joining p_2 and p_3 , and let Σ denote the plane which meets Π_1 perpendicularly along L . It is clear that Σ contains λ_{12} and λ_{13} . This implies that Σ meets the planes Π_2 and Π_3 perpendicularly. For $i = 2, 3$, let X_i denote the line $\Pi_i \cap \Sigma$. Since $\Pi_2 \cap \Pi_3 = \emptyset$, the lines $X_2, X_3 \subset \Sigma$ are disjoint. Hence the common perpendicular to X_2 and X_3 is a line $Y \subset \Sigma$. For $i = 2, 3$, the line Y meets the line $X_i \subset \Pi_i$ perpendicularly, and Y is contained in the plane Σ which is perpendicular to Π_i ; hence Y is itself perpendicular to Π_i . It follows that $Y = \lambda_{23}$. Thus the plane Σ contains $\lambda_{12}, \lambda_{13}$ and λ_{23} . \square

Lemma 2.4. *Let N be a compact hyperbolic 3-manifold with totally geodesic boundary, and suppose $\Pi_1, \Pi_2,$ and Π_3 are distinct components of $\partial\widetilde{N}$. Let Π be the plane containing the short cuts $\lambda_{12}, \lambda_{23}$ and λ_{13} , which exists by Lemma 2.3. Let C be the right-angled hexagon in Π with edges λ_{ij} and the geodesic arcs in the Π_i joining their endpoints. Then $C \subset \widetilde{N}$, and $C \cap \partial\widetilde{N} = \cup_i(C \cap \Pi_i)$.*

Proof. $\Pi \cap \widetilde{N}$ is a convex subset of Π bounded by the family of disjoint geodesics $\Pi \cap \partial\widetilde{N}$, which includes $\Pi \cap \Pi_i, i \in \{1, 2, 3\}$. If γ is another component of $\Pi \cap \partial\widetilde{N}$, then by definition $\Pi \cap \Pi_1, \Pi \cap \Pi_2,$ and $\Pi \cap \Pi_3$ are all contained in the component of $\Pi - \gamma$ intersecting \widetilde{N} . Thus there is a single component of $\partial_\infty \Pi - (\cup_{i=1}^3 \partial_\infty \Pi_i)$ containing both endpoints of γ .

Since γ is a component of $\Pi \cap \partial \widetilde{N}$, its endpoints are between those of two different geodesics $\Pi \cap \Pi_i$, say $\Pi \cap \Pi_1$ and $\Pi \cap \Pi_2$. Then since the geodesic containing λ_{12} intersects Π_1 and Π_2 perpendicularly, it is disjoint from γ and contained in the component of $\Pi - \gamma$ intersecting \widetilde{N} . The remainder of C is on the other side of λ_{12} from γ . Since γ was arbitrary, the lemma follows. \square

Definition 2.5. Let N be a compact hyperbolic 3-manifold with totally geodesic boundary, and let C be a right-angled hexagon supplied by Lemma 2.4. We call the edges of C which are short cuts *internal*, and the remaining edges *external*. If the internal edges project to λ_i , λ_j , and λ_k , we call C an (i, j, k) *hexagon*.

This terminology matches that defined in [13] in the context of horospheres and cusped hyperbolic manifolds. As we will see below, (i, j, k) hexagons were used extensively in the analysis of [17], although not by name.

The isometry class of a right-angled hexagon is determined by the lengths of three of its pairwise nonadjacent sides. If $\{\ell, \ell', \ell''\}$ is a collection of such lengths, and d is the length of a side abutting those with lengths ℓ and ℓ' , the *right-angled hexagon rule* (cf. eg. [19, Theorem 3.5.13]) describes d in terms of the other lengths.

$$(2.5.1) \quad \cosh d = \frac{\cosh \ell \cosh \ell' + \cosh \ell''}{\sinh \ell \sinh \ell'}$$

A prototypical application of the right-angled hexagon rule is the following initial lemma, proved in [17] during the proof of Lemma 3.2.

Lemma 2.6 (Kojima-Miyamoto). *Suppose N is a compact hyperbolic 3-manifold with connected totally geodesic boundary, and let R be the function of ℓ_1 defined by the following formula.*

$$(2.6.1) \quad \cosh R = \sqrt{1 + \frac{1}{2 \cosh \ell_1 - 2}}$$

Then $d_{11} \geq 2R$.

Proof. Let λ and λ' be short cuts in \widetilde{N} with length ℓ_1 whose feet are at distance d_{11} on some boundary component. The short cut λ'' joining the boundary components containing the other feet of λ and λ' has length ℓ_k for some $k \geq 1$. Applying the right-angled hexagon rule to the $(1, 1, k)$ hexagon containing λ , λ' , and λ'' yields the following inequality.

$$(2.6.2) \quad \cosh d_{11} = \frac{\cosh^2 \ell_1 + \cosh \ell_k}{\sinh^2 \ell_1} \geq \frac{\cosh^2 \ell_1 + \cosh \ell_1}{\sinh^2 \ell_1} = 1 + \frac{1}{\cosh \ell_1 - 1}$$

The lemma now follows upon applying the “half-angle formula” for hyperbolic cosine, $\cosh R = \sqrt{(\cosh(2R) + 1)/2}$. \square

Note that R is decreasing as a function of ℓ_1 . Hence using Lemma 2.6, an upper bound on d_{11} implies a lower bound on ℓ_1 . An upper bound for d_{11} follows from area considerations.

Lemma 2.7 ([17], Corollary 3.5). *Let N be a compact hyperbolic 3-manifold with ∂N connected, totally geodesic, and of genus 2. Then the following bounds hold for d_{11} and ℓ_1 .*

$$\cosh d_{11} \leq 3 + 2\sqrt{3} \qquad \cosh \ell_1 \geq \frac{3 + \sqrt{3}}{4}$$

Proof. By Lemma 2.6, there is a disk of radius R embedded on ∂N around each endpoint of λ_1 , and these two disks do not overlap. They lift to a radius R disk packing on a component Π of $\partial\tilde{N}$, invariant under the action of $\pi_1\partial N$. Boröczky’s Theorem [4] gives an upper bound $d(R)$ on the local density of a radius R disk packing of \mathbb{H}^2 . Since the packing in question is invariant under the action by covering transformations for the compact surface ∂N , $d(R)$ bounds the global density of the packing there, yielding the following inequality.

$$\frac{4\pi(\cosh R - 1)}{4\pi} \leq d(R)$$

The numerator on the left hand side of the inequality above is twice the area of a hyperbolic disk of radius R , and the denominator is the area of ∂N . (This follows from the Gauss-Bonnet theorem and the fact that ∂N has genus 2.)

Let α be the angle at a vertex of a hyperbolic equilateral triangle $T(R)$ with side length $2R$. $T(R)$ has area $\pi - 3\alpha$, and the intersection with $T(R)$ of disks of radius R centered at its vertices occupies a total area of $3\left(\frac{\alpha}{2\pi}\right)2\pi(\cosh R - 1)$. Boröczky’s bound $d(R)$ is defined as the ratio of these areas; thus after simplifying the inequality above we obtain the one below.

$$\cosh R - 1 \leq \frac{3\alpha(\cosh R - 1)}{\pi - 3\alpha}$$

Solving for α yields $\alpha \geq \pi/6$. The hyperbolic law of cosines describes the relationship between α and the side length of $T(R)$:

$$\cos \alpha = \frac{\cosh^2(2R) - \cosh(2R)}{\sinh^2(2R)} = \frac{\cosh(2R)}{\cosh(2R) + 1} = \frac{2 \cosh^2 R - 1}{2 \cosh^2 R}$$

Using the fact that $\cos \alpha \leq \sqrt{3}/2$ and solving for $\cosh R$ yields $\cosh R \leq (1 + \sqrt{3})/\sqrt{2}$. The inequality for $\cosh d_{11}$ follows using the “hyperbolic double angle formula”, and the inequality for $\cosh \ell_1$ follows upon solving the formula of Lemma 2.6. \square

The following lemma combines Lemmas 4.2 and 4.3 of [17] with the discussion below them.

Lemma 2.8 (Kojima-Miyamoto). *Let N be a compact hyperbolic 3-manifold with ∂N totally geodesic, connected, and of genus 2. Define quantities R' , E , and F , depending on ℓ_1 , by the following equations.*

$$(2.8.1) \qquad \cosh R' = 3 - \cosh R$$

$$(2.8.2) \qquad \cosh E = \frac{2}{\cosh^2(R + R') \cdot \tanh^2 \ell_1 - 1} + 1$$

$$(2.8.3) \qquad \cosh F = \sqrt{\frac{\cosh \ell_1 + 1}{\cosh 2R' - 1} + 1}$$

Then $\ell_2 \geq \max\{\ell_1, \min\{E, F\}\}$.

Proof. The right-angled hexagon rule can be used to obtain lower bounds on ℓ_2 depending on values for ℓ_1 and d_{12} or d_{22} , respectively.

$$(2.8.4) \quad \cosh \ell_2 \geq \frac{2}{\cosh^2 d_{12} \tanh^2 \ell_1 - 1} + 1$$

$$(2.8.5) \quad \cosh \ell_2 \geq \sqrt{\frac{\cosh \ell_1 + 1}{\cosh d_{22} - 1}} + 1$$

This is recorded in Lemma 4.2 of [17]. Therefore an upper bound for d_{12} or d_{22} gives a lower bound for ℓ_2 in terms of ℓ_1 .

Recall from above that an upper bound on ℓ_1 gives a lower bound of $2R(\ell_1)$, on d_{11} , the shortest distance between feet of two different shortest short cuts on (any) component of $\partial\tilde{N}$. Hence disks U and U' of radius R in ∂N , each centered at a foot of the shortest return path, are embedded and nonoverlapping. Area considerations imply that R' is an upper bound for the radii of two equal-size nonoverlapping disks in $\partial N - \text{int}(U \cup U')$ [17, Lemma 4.3]. It follows that at least one of $d_{12} \leq R + R'$ or $d_{22} \leq 2R'$ holds, since otherwise there would be disks of radius R' embedded around the feet of the second-shortest return path without overlapping U and U' . The inequalities (2.8.4) and (2.8.5) thus imply that $\ell_2 \geq \min\{E, F\}$. Note that by definition $\ell_2 \geq \ell_1$, which gives the lemma. \square

The following lemma contains a new observation improving on the bound of Lemma 2.8 for values of ℓ_1 with hyperbolic cosine near 1.4.

Lemma 2.9. *Let N be a compact hyperbolic 3-manifold with ∂N totally geodesic, connected, and of genus 2. Let R'' be determined by the following equation.*

$$(2.9.1) \quad \cosh R'' = \frac{1}{\sqrt{2(1 - \cos(2\pi/9))}} = 1.4619\dots$$

Define quantities L and M depending on ℓ_1 by

$$(2.9.2) \quad \cosh L = \frac{2}{\cosh^2(2R'') \tanh^2 \ell_1 - 1} + 1$$

$$(2.9.3) \quad \cosh M = \sqrt{\frac{\cosh \ell_1 + 1}{\cosh(2R'') - 1}} + 1$$

For any value of ℓ_1 with

$$(2.9.4) \quad \cosh \ell_1 \leq \frac{\cos(2\pi/9)}{2 \cos(2\pi/9) - 1} = 1.439\dots,$$

ℓ_2 is bounded below by $\max\{\ell_1, \min\{E, F\}, \min\{L, M\}\}$.

Proof. Applying Boröczky's theorem as in Lemma 2.7, but this time to four disks of equal radius packed on ∂N , we find that the radius is bounded above by the quantity R'' specified by the formula above. For ℓ_1 satisfying the bound of (2.9.4), the inequality (2.6.2) implies that d_{11} is at least $2R''$. If each of d_{12} and d_{22} were also larger than $2R''$, then disks of radius R'' around the feet of both the shortest and second-shortest return paths would be

embedded and nonoverlapping, a contradiction. Thus $\min\{d_{12}, d_{22}\} \leq 2R''$. Plugging into inequalities (2.8.4) and (2.8.5) gives the result. \square

The quantities L and M defined above offer a better lower bound on ℓ_2 than E and F , for values of ℓ_1 with $1.367 \leq \cosh \ell_1 \leq 1.439$. This is because E and F use the quantity R' of equation (2.8.1), and by definition twice the area of a hyperbolic disk of radius R' plus twice the area of a hyperbolic disk of radius R is 4π . However it is impossible to entirely cover a compact surface of genus 2 with 4 nonoverlapping embedded disks. Boröczky's theorem bounds the proportion of the area which can be covered by four disks of equal radius, and this supplies R'' . This is less than R' for R determined below.

$$\cosh R \leq 3 - \frac{1}{\sqrt{2(1 - \cos(2\pi/9))}}$$

Solving equation (2.6.1) for $\cosh \ell_1$, we find that this occurs when $\cosh \ell_1 \geq 1.366\dots$. It is at this point that the bound above for d_{22} given by $\cosh(K)$ is better than the bound above given by $2R'$. The bound on d_{12} given by $\cosh(K)$ becomes better than that given by $R + R'$ somewhat earlier, at least for values of ℓ_1 with $\cosh \ell_1 \geq 1.25$.

3. VOLUME WITH A LONG RETURN PATH

By definition, in a hyperbolic manifold N with totally geodesic boundary, $\ell_1/2$ is the height of a maximal embedded collar of ∂N . The volume of such a collar bounds the volume of N below, but leaves a lot out. The volume bounds of [17] are obtained by taking a larger collar of ∂N and using separate means to understand the region where it overlaps itself.

Definition 3.1. The *muffin of height ℓ* , here denoted Muf_ℓ , is the hyperbolic solid obtained by rotating a hyperbolic pentagon with base of length ℓ , opposite angle $2\pi/3$, and all other angles $\pi/2$, about its base (see [17, Figure 3.1]).

It is a standard fact of hyperbolic trigonometry (see [19, Theorem 3.5.14]) that positive real numbers a , b , and c determine a right-angled hexagon in \mathbb{H}^2 , unique up to isometry, with alternate sides of lengths a , b , and c . For $\ell > 0$, the hexagon specified by $a = b = c = \ell$ has an orientation preserving symmetry group of order three which cyclically permutes the sides of length ℓ . The hyperbolic pentagon mentioned in the definition above is a fundamental domain for this symmetry group.

For a compact hyperbolic 3-manifold N , a copy of Muf_{ℓ_1} is embedded in \widetilde{N} around each short cut with length ℓ_1 . Lemma 3.2 of [17] asserts that Muf_{ℓ_1} is embedded in N by the universal covering. Let A be the length of a side joining a vertex with angle $\pi/2$ to the vertex with angle $2\pi/3$ of the pentagon rotated to construct the muffin. In terms of ℓ_1 , A is given by the formula below.

$$(3.1.1) \quad \cosh A = \sqrt{\frac{2}{3}(\cosh \ell_1 + 1)}$$

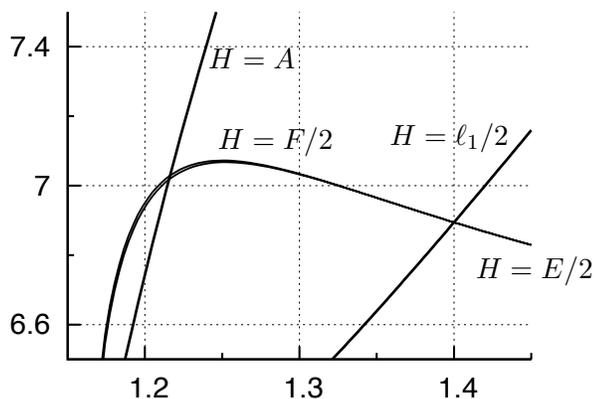


FIGURE 3.1. The volume estimates of Kojima–Miyamoto

A collar of ∂N with height less than both A and $\ell_2/2$ has its region of self-overlap entirely contained in Muf_{ℓ_1} . This yields the fundamental volume inequality of [17], stated there in the proof of Proposition 4.1, which we formulate in the following lemma.

Lemma 3.2 (Kojima–Miyamoto). *Let N be a compact hyperbolic 3-manifold with totally geodesic boundary, and let $H = \min\{A, \ell_2/2\}$. With Muf_{ℓ_1} as defined above and R as in Lemma 2.6, we have the following bound.*

$$(3.2.1) \quad \text{vol}(N) \geq \text{vol}(Muf_{\ell_1}) + \pi(2 - \cosh R)(2H + \sinh(2H))$$

Proof. The intersection of Muf_{ℓ_1} with ∂N is the union of disks U and U' of radius R , the quantity defined in Lemma 2.6. This is because the pentagon rotated to construct the muffin is a fundamental domain for the orientation-preserving symmetry group of a $(1, 1, 1)$ hexagon, thus its sides adjacent to the base each have length R (again see [17, Figure 3.1]). It follows from Lemma 2.6 that U and U' are embedded in ∂N without overlapping.

The area of $\partial N - (U \cup U')$ is $4\pi - 4\pi(\cosh R - 1)$. A collar of $\partial N - (U \cup U')$ of height H is embedded in N without overlapping Muf_{ℓ_1} , and the bound of the lemma is obtained by adding their volumes. This uses the following well known formula for the volume V of a collar of height H in \mathbb{H}^3 of a set in a plane with area A : $V = A \cdot (2H + \sinh 2H)/4$. \square

A formula for the volume of Muf_{ℓ_1} is recorded in [17, Lemma 3.3]. The output of inequality (3.2.1) is recorded as a function of $x = \cosh \ell_1$ in [17, Graph 4.1], which we have reproduced above in Figure 3.1. Lower bounds for H are determined at various points on the graph by A , the quantities E and F of Lemma 2.8, and ℓ_1 itself. The point on Graph 4.1 of [17] above $\cosh \ell_1 = 1.215$ is just to the left of the intersection of the curves labeled $H = A$ and $H = F/2$. A computation gives a volume bound of $7.007\dots$ here. To the right of $\cosh \ell_1 = 1.215$, inspection reveals a single local minimum at about $\cosh \ell_1 = 1.4$. Numerical experimentation indicates that the volume bound at this minimum is just larger than 6.89 . In this section, our main task is to prove this rigorously.

Proposition 3.7. *Let N be a hyperbolic 3-manifold with ∂N connected, totally geodesic, and of genus 2, satisfying $\cosh \ell_1 \geq 1.215$. Then N has volume greater than 6.89.*

Remark 3.3. Although Proposition 3.7 probably follows solely from results of [17], our proof uses Lemma 2.9. In fact, numerical experimentation indicates that the volume bound above could be improved to about 6.94 with this result. Since it is easier and suffices for applications, we prove only the bound of 6.89 here.

The lemma below proves useful for several results in this section.

Lemma 3.4. *For $\frac{3+\sqrt{3}}{4} \leq \cosh \ell_1 \leq 1.4$, the quantity E of (2.8.2) is decreasing in ℓ_1 .*

Proof. The formula (2.8.1) defines R' in terms of R by

$$R' = \cosh^{-1}(3 - \cosh R) = \log \left[(3 - \cosh R) + \sqrt{(3 - \cosh R)^2 - 1} \right]$$

Taking a derivative with respect to R , one finds that $R+R'$ increases with R for $1 < \cosh R < 3/2$, reaching a maximum at $\cosh R = 3/2$, and decreases when $\cosh R > 3/2$. This implies that $\cosh(R + R')$ is an increasing function of ℓ_1 on the interval $\frac{3+\sqrt{3}}{4} \leq \cosh \ell_1 \leq 1.4$, since values of ℓ_1 in this interval give R -values between $\frac{3}{2}$ and $\frac{1+\sqrt{3}}{\sqrt{2}}$, and R is a decreasing function of ℓ_1 . Since the hyperbolic tangent is an increasing function, the lemma follows from the definition of E in Lemma 2.8. \square

In the proof of Proposition 3.7, we divide the interval $1.215 \leq \cosh \ell_1 < \infty$ into subintervals:

$$[1.215, \infty) = [1.215, 1.367] \cup [1.367, 1.439] \cup [1.439, \infty)$$

We address each subinterval separately. The first is below.

Lemma 3.5. *A compact hyperbolic 3-manifold N with geodesic boundary satisfying $1.215 \leq \cosh \ell_1 \leq 1.367$ has volume greater than 6.89.*

Proof. The strategy of proof is to break the interval in question into subintervals, bound the constituent quantities in the formula (3.2.1) on each subinterval, and from this obtain a coarse lower bound for the right hand side of the inequality. Table 1 records this computation. We explain its entries below.

The leftmost column specifies the subinterval of values of $\cosh \ell_1$. The second column records a lower bound on this interval for the volume of the muffin. This is attained at the right endpoint, according to [17, Lemma 3.3]. The third column records a lower bound for the area on ∂N of the complement of the base of the muffin. This is attained at the left endpoint of the subinterval, since R is decreasing in ℓ_1 . The fourth column records a lower bound for H . This is obtained by computing minima for each of A , $E/2$, and $F/2$ on the subinterval and taking the minimum (in each case this lower bound is greater than $\ell_1/2$).

From its definition (3.1.1) one easily finds that A is increasing in ℓ_1 , hence a minimum for A is obtained at the left endpoint of each subinterval. By Lemma 3.4 above, E is decreasing in ℓ_1 on each subinterval in question, so its minimum is obtained at the right endpoint.

$\cosh \ell_1$	muffin volume	$\text{area}(\partial N - (U \cup U'))$	H	volume
[1.215, 1.220]	5.304	2.216	.629 (<i>E</i>)	6.899
[1.220, 1.226]	5.236	2.399	.611 (<i>E</i>)	6.899
[1.226, 1.233]	5.159	2.609	.592 (<i>E</i>)	6.900
[1.233, 1.241]	5.076	2.844	.574 (<i>E</i>)	6.901
[1.241, 1.250]	4.988	3.097	.556 (<i>E</i>)	6.901
[1.250, 1.260]	4.895	3.367	.539 (<i>F</i>)	6.898
[1.260, 1.270]	4.808	3.648	.524 (<i>F</i>)	6.908
[1.270, 1.281]	4.717	3.911	.510 (<i>F</i>)	6.898
[1.281, 1.292]	4.632	4.182	.498 (<i>F</i>)	6.900
[1.292, 1.303]	4.551	4.436	.488 (<i>F</i>)	6.898
[1.303, 1.314]	4.475	4.675	.479 (<i>F</i>)	6.894
[1.314, 1.324]	4.409	4.899	.471 (<i>F</i>)	6.899
[1.324, 1.334]	4.346	5.092	.464 (<i>F</i>)	6.891
[1.334, 1.343]	4.292	5.275	.459 (<i>F</i>)	6.893
[1.343, 1.351]	4.245	5.432	.454 (<i>F</i>)	6.894
[1.351, 1.358]	4.206	5.565	.451 (<i>F</i>)	6.895
[1.358, 1.364]	4.173	5.678	.448 (<i>F</i>)	6.896
[1.364, 1.367]	4.157	5.772	.447 (<i>F</i>)	6.917

TABLE 1.

The monotonicity of F is not apparent from its definition, so to find a lower bound on each subinterval, we plug the value of $\cosh \ell_1$ at the left endpoint and the value of $\cosh 2R'$ at the right endpoint to the formula (2.8.3). In addition to the resulting lower bound, we record which of A , E , and F supplies it in the fourth column of Table 1.

The final column assembles these bounds to give a volume bound. In each column after the first, the decimal approximation has been truncated after three places. \square

Using the bounds of Lemma 2.9 for ℓ_2 in the inequality (3.2.1) yields the lemma below.

Lemma 3.6. *A compact hyperbolic 3-manifold with geodesic boundary satisfying $1.367 \leq \cosh \ell_1 \leq 1.439$ has volume at least 6.89.*

Proof. We assemble a table as in the proof of Lemma 3.5, using L and M to bound H below, instead of E and F . Since L is decreasing in ℓ_1 , its minimum occurs at the right endpoint of each subinterval, whereas the minimum of M occurs at the left endpoint. The results of the computation are recorded in Table 2. \square

We now prove Proposition 3.7. Below we recall its statement.

Proposition 3.7. *Let N be a hyperbolic 3-manifold with ∂N connected, totally geodesic, and of genus 2, satisfying $\cosh \ell_1 \geq 1.215$. Then N has volume greater than 6.89.*

$\cosh \ell_1$	muffin volume	$\text{area}(\partial N - (U \cup U'))$	H	volume
[1.367, 1.377]	4.105	5.818	.447 (M)	6.892
[1.377, 1.392]	4.031	5.966	.448 (M)	6.894
[1.392, 1.416]	3.920	6.176	.449 (M)	6.893
[1.416, 1.439]	3.823	6.485	.451 (M)	6.959

TABLE 2.

Proof. Given Lemmas 3.5 and 3.6, we need only concern ourselves with hyperbolic 3-manifolds N with totally geodesic boundary satisfying $1.439 \leq \cosh \ell_1$. The union of the muffin and a collar of height $H = \ell_1/2$ has volume given by the following formula.

$$V(\ell_1) = \pi \left[\ell_1 + 2 \sinh \ell_1 + \sqrt{\frac{2 \cosh \ell_1 - 1}{2 \cosh \ell_1 - 2}} \left(\cosh^{-1} \left(\frac{4 \cosh \ell_1 + 1}{3} \right) - \ell_1 - \sinh \ell_1 \right) \right]$$

When $\cosh \ell_1 = 1.439$, this yields a volume of 7.1... We claim that $V(\ell_1)$ is increasing with ℓ_1 (as is suggested by Graph 4.1 in [17]). Once established, this will complete the proof.

It is clear that $\sqrt{\frac{2 \cosh \ell_1 - 1}{2 \cosh \ell_1 - 2}} = \cosh R$ (recall equation (2.6.1)) is decreasing in ℓ_1 , asymptotically approaching 1 from above. The derivative of $\cosh^{-1}((4 \cosh \ell_1 + 1)/3)$ is $1/\cosh R$, bounded above by one. Hence the quantity in parentheses above,

$$Q = \cosh^{-1} \left(\frac{4 \cosh \ell_1 + 1}{3} \right) - \ell_1 - \sinh \ell_1,$$

is decreasing in ℓ_1 . Its value is negative when $\cosh \ell_1 = 1.439$, and so also on the entire interval in question. Taking the derivative of $V(\ell_1)$ yields the following.

$$V'(\ell_1) = \pi \left[1 + 2 \cosh \ell_1 + \cosh R \left(\frac{1}{\cosh R} - 1 - \cosh \ell_1 \right) + \frac{d}{d\ell_1}(\cosh R) \cdot Q \right]$$

Since $\frac{d}{d\ell_1}(\cosh R)$ and Q are both negative, the above is the sum of a positive number with the product of $1 + \cosh \ell_1$ and $2 - \cosh R$. When $\cosh \ell_1 = 1.439$, $\cosh R = 1.46... < 2$; since $\cosh R$ is decreasing as a function of ℓ_1 , the derivative of the volume formula is positive for $\cosh \ell_1 \geq 1.439$. \square

Lemma 3.8. *Let N be a compact hyperbolic 3-manifold with ∂N connected, totally geodesic, and of genus 2. If $\cosh \ell_1 \leq 1.215$, then ℓ_1 and A are each less than $\ell_2/2$.*

Proof. Lemma 2.8 implies that $\ell_2 \geq \min\{E, F\}$. The values of ℓ_1 in question here have hyperbolic cosine between $\frac{3+\sqrt{3}}{4} \cong 1.186$ and 1.215. By Lemma 3.4, E is monotone decreasing in ℓ_1 on this interval; thus a lower bound, obtained by plugging in 1.215, is 1.961... The quantity F is not necessarily monotone in ℓ_1 , since both $\cosh \ell_1$ and $\cosh 2R'$ are increasing in ℓ_1 . However, a coarse lower bound may be found by substituting $\cosh^{-1}(1.215)$ for ℓ_1 in $\cosh \ell_1$ and $\cosh^{-1}(\frac{3+\sqrt{3}}{4})$ for ℓ_1 in $\cosh 2R'$. The lower bound obtained in this way is 1.960...

When $\cosh \ell_1 \leq 1.215$, $\ell_1 \leq .645$. Taking the inverse hyperbolic cosine of 1.960, we find $\ell_2 \geq 1.293$. The quantity A is clearly increasing in ℓ_1 . When $\cosh \ell_1 = 1.215$, the value obtained is .644.... This establishes the lemma. \square

Proposition 3.9. *Let N be a hyperbolic 3-manifold with ∂N connected, totally geodesic, and of genus 2, such that there is no $(1, 1, 1)$ hexagon in \widetilde{N} . Then $\cosh \ell_1 \geq 1.215$.*

Proof. Suppose N is a hyperbolic 3-manifold with totally geodesic boundary and no $(1, 1, 1)$ hexagons. According to Lemma 3.8, if ℓ_1 is at most 1.215, then ℓ_2 is at least twice ℓ_1 . Since N has no $(1, 1, 1)$ hexagons, in the right-angled hexagon in \widetilde{N} realizing d_{11} , the opposite side has length at least ℓ_2 ; thus at least twice ℓ_1 . Using the right-angled hexagon rule as in inequality (2.6.2), we obtain the following inequality.

$$\cosh d_{11} \geq \frac{\cosh^2 \ell_1 + \cosh(2\ell_1)}{\sinh^2 \ell_1} = \frac{3 \cosh^2 \ell_1 - 1}{\cosh^2 \ell_1 - 1} = 3 + \frac{2}{\sinh^2 \ell_1}$$

We recall from Lemma 2.7 that $\cosh d_{11} \leq 3 + 2\sqrt{3}$. Putting this together with the inequality above implies

$$\sinh^2 \ell_1 \geq \frac{1}{\sqrt{3}}.$$

This gives $\cosh \ell_1 > 1.255$, contradicting the hypothesis that $\cosh \ell_1 \leq 1.215$. \square

4. NORMAL BOOKS OF I -BUNDLES

4.1. The results of this section and the next are topological. We work in the PL category in these sections, and follow the conventions of [6]. In particular, we say that a subset Y of a space X is π_1 -*injective* if for all path components A of X and B of Y such that $B \subset A$, the inclusion homomorphism $\pi_1(B) \rightarrow \pi_1(A)$ is injective. We denote the Euler characteristic of a finite polyhedron X by $\chi(X)$, and we write $\bar{\chi}(X) = -\chi(X)$.

This section collects topological results to be used in the sequel. After some preliminary results and definitions, we define a *normal book of I -bundles* in 4.6 below, and follow with the main results: Proposition 4.7 gives a criterion (equivalent to having empty “kishkes,” or “guts”) for recognizing a simple 3-manifold as the underlying topological space of a normal book of I -bundles, and Lemma 4.9 identifies a property satisfied by certain submanifolds of a normal book of I -bundles.

Proposition 4.2. *Let X be a compact, orientable 3-manifold, and let S and T be components of ∂X . Suppose that for some field F the inclusion homomorphism $H_1(T; F) \rightarrow H_1(X; F)$ is surjective. Then S is π_1 -injective in X .*

Proof. Assume that S is not π_1 -injective. Then there is a properly embedded disk $D \subset X$ such that ∂D is a non-trivial simple closed curve in S . Let N be a regular neighborhood of D in X , let A denote the component of $\overline{X - N}$ containing T , let \star be a base point in T , and let $G \leq \pi_1(X, \star)$ denote the image of $\pi_1(T, \star)$ under the inclusion homomorphism. Then $\pi_1(X, \star)$ is the free product of G with another subgroup K , where $K \cong \mathbb{Z}$ if D does not separate X ,

and $K \cong \pi_1(\overline{X - A})$ if D does separate X . In the latter case, ∂D is a separating, non-trivial simple closed curve in S , and hence $\overline{X - A}$ has a boundary component of strictly positive genus. Thus in either case we have $H_1(K; F) \neq 0$. Hence the inclusion homomorphism $H_1(A; F) \rightarrow H_1(X; F)$ is not surjective. Since $S \subset A$, this contradicts the hypothesis. \square

Lemma 4.3. *Let $\tau : F \rightarrow F$ be a free involution of a compact orientable surface F , and let $C \subset F$ be a simple closed curve. Suppose that $\tau(C)$ is isotopic to a curve which is disjoint from C . Then C is isotopic to a curve C_1 such that either (i) $\tau(C_1) \cap C_1 = \emptyset$ or (ii) $\tau(C_1) = C_1$. Furthermore, if τ reverses orientation we may always choose C_1 so that (i) holds.*

Proof. We fix a metric with convex boundary on $F/\langle \tau \rangle$. Then F inherits a metric such that $\tau : F \rightarrow F$ is an isometry. Since C is a homotopically non-trivial simple closed curve in F , it is homotopic to a curve C_1 with shortest length in its homotopy class, which is a simple closed geodesic by [11, Theorem 2.1]. Let C' be homotopic to $\tau(C)$ and disjoint from C . Then since the shortest closed geodesics C_1 and $\tau(C_1)$ are respectively homotopic to the disjoint curves C and C' , it follows from [11, Corollary 3.4] that they either are disjoint or coincide. (The results we have quoted from [11] are stated there for the case of a closed surface, but it is pointed out in the first paragraph of [11, §4] that they hold in the case of a compact surface with convex boundary.)

It follows from [10, Theorem 2.1] that C is isotopic to C_1 . This proves the first assertion.

To prove the second assertion, suppose that τ reverses orientation and that C is isotopic to a curve C_1 such that (ii) holds. Let A be an invariant annular neighborhood of C_1 . Since τ is a free involution it must preserve an orientation of the invariant curve C ; since it reverses an orientation of F , it must therefore interchange the components of ∂A . Hence (i) holds if C_1 is replaced by one of the components of ∂A . \square

If a 3-manifold X has the structure of an I -bundle over a surface T and $p : X \rightarrow T$ is the bundle projection, we will call $\partial_v X \doteq p^{-1}(\partial T)$ the *vertical* boundary of X and $\partial_h X \doteq \overline{\partial X - \partial_v X}$ the *horizontal* boundary of X . Note that $\partial_v X$ inherits the structure of an I -bundle over ∂T , and $\partial_h X$ the structure of a ∂I -bundle over T , from the original I -bundle structure on X . We call an annulus $A \subset X$ *vertical* if $A = p^{-1}(p(A))$.

Let M an orientable, irreducible 3-manifold M , and let F be a π_1 -injective 2-dimensional submanifold of ∂M . By an *essential annulus or torus in M relative to F* we mean a properly embedded annulus or torus in M which is π_1 -injective, has its boundary contained in F , and not parallel to a subsurface of F .

In the case where M is boundary-irreducible, we shall define an *essential annulus or torus in M* to be an essential annulus or torus in M relative to ∂M .

Proposition 4.4. *Let X be an I -bundle over a compact surface, and suppose that A is an essential annulus in X relative to $\partial_h X$. Then A is isotopic, by an ambient isotopy of X which is constant on $\partial_v X$, to a vertical annulus in X .*

Proof. Let $j : A \rightarrow X$ denote the inclusion. If j is homotopic to a map of A into $\partial_h X$ then by [21, Lemma 5.3], A is boundary parallel, contradicting essentiality relative to $\partial_h X$. Hence j is not homotopic to a map of A into $\partial_h X$.

It follows that if X is a trivial I -bundle then A has one boundary component on each component of $\partial_h X$. Lemma 3.4 of [21] then asserts that A is ambiently isotopic in X to a vertical annulus, by an isotopy fixing $\partial_v X$ and one component of $\partial_h X$. This gives the conclusion in the case where X is a trivial I -bundle.

We turn to the case where X is a twisted I -bundle. Then for some compact orientable surface F and some orientation-reversing free involution $\tau : F \rightarrow F$ we may write $X = (F \times I)/\hat{\tau}$, where $\hat{\tau} : F \times I \rightarrow F \times I$ is defined by $\hat{\tau}(x, t) = (\tau(x), 1-t)$. The quotient map $p : F \times I \rightarrow X$ is a two-sheeted covering map, and p maps $F \times \{i\}$ homeomorphically onto $\partial_h X$ for $i = 0, 1$. Hence $j : A \rightarrow X$ may be lifted to an embedding $\tilde{j} : A \rightarrow F \times I$. Since j is not homotopic to a map of A into $\partial_h X$, the annulus $\tilde{j}(A)$ is essential. By the case of the proposition already proved, \tilde{j} is ambiently isotopic in $F \times I$ to an embedding \tilde{j}' such that $\tilde{j}'(A)$ is vertical.

For $i = 0, 1$ let C_i denote the component of ∂A with $\tilde{j}'(C_i) \subset F \times \{i\}$. Define a simple closed curve $C \subset F$ by $C \times \{0\} = \tilde{j}'(C_0)$. Then $C \times \{0\}$ is isotopic to $\tilde{j}'(C_0)$. Since $\tilde{j}'(A)$ is vertical, $\tilde{j}'(C_1) = C \times \{1\}$; hence by definition we have $\tau(C) \times \{0\} = \hat{\tau}(\tilde{j}'(C_1))$, and so $\tau(C) \times \{0\}$ is isotopic to $\hat{\tau}(\tilde{j}'(C_1))$. But since j is an embedding, $\tilde{j}'(C_0)$ and $\hat{\tau}(\tilde{j}'(C_1))$ are disjoint curves in $F \times \{0\}$. Thus C and $\tau(C)$ are isotopic in F to disjoint curves, and it follows from Lemma 4.3 that C is isotopic in F to a curve C_1 such that $\tau(C_1) \cap C_1 = \emptyset$. This implies that \tilde{j}' is ambiently isotopic to an embedding \tilde{j}'_1 such that $\tilde{j}'_1(A)$ is a vertical annulus and $\tau(\tilde{j}'_1(A)) \cap \tilde{j}'_1(A) = \emptyset$.

Hence $p \circ \tilde{j}'_1 : A \rightarrow X$ is an embedding, and $A_1 \doteq p \circ \tilde{j}'_1(A)$ is a vertical annulus in X . Since \tilde{j}'_1 is ambiently isotopic in $F \times I$ to \tilde{j} , the homeomorphism $p \circ \tilde{j}'_1 : A \rightarrow A_1$ is homotopic to j by a boundary-preserving homotopy in X . It then follows from [21, Corollary 5.5] that A is isotopic to A_1 by an ambient isotopy fixing $\partial_v X$. \square

4.5. Let W be a compact, orientable, irreducible and boundary-irreducible 3-manifold with $\partial W \neq \emptyset$. We recall the definition of the characteristic submanifold Σ of W relative to ∂W . Up to ambient isotopy, Σ is the unique compact submanifold of W with the following properties.

- (1) Every component of Σ is either an I -bundle P over a surface such that $P \cap \partial W = \partial_h P$, or a Seifert fibered space S such that $S \cap \partial W$ is a saturated 2-manifold in ∂S .
- (2) Every component of the frontier of Σ is an essential annulus or torus in W .
- (3) No component of Σ is ambiently isotopic in W to a submanifold of another component of Σ .
- (4) If Σ_1 is a compact submanifold of W such that (1) and (2) hold with Σ_1 in place of Σ , then Σ_1 is ambiently isotopic in W to a submanifold of Σ .

Recall that a submanifold of a Seifert fibered space M is *saturated* if it is a union of fibers. For further details on the characteristic submanifold, see [14] and [15].

It follows from Property (2) of W that Σ is π_1 -injective. Hence if W is simple, the fundamental group of a component of Σ cannot have a rank-2 free abelian subgroup. It follows that if W is simple then every Seifert-fibered component of Σ is a solid torus. In particular, all the components of the frontier of Σ are essential annuli in this case.

The rest of this section is concerned with books of I -bundles. We shall follow the conventions of [6], and we refer the reader to [6, 5.1] for the definition of a book of I -bundles. As in [6] we shall denote the union of the pages of a book of I -bundles by $\mathcal{P}_{\mathcal{W}}$ and the union of its bindings by $\mathcal{B}_{\mathcal{W}}$, and we shall set $|\mathcal{W}| = \mathcal{P}_{\mathcal{W}} \cup \mathcal{B}_{\mathcal{W}}$ and $\mathcal{A}_{\mathcal{W}} = \mathcal{P}_{\mathcal{W}} \cap \mathcal{B}_{\mathcal{W}}$.

Definition 4.6. Let \mathcal{W} be a book of I -bundles, and set $W = |\mathcal{W}|$, and let C denote a regular neighborhood of $\mathcal{A}_{\mathcal{W}}$ in W . We shall say that \mathcal{W} is *normal* if (i) W is a simple 3-manifold, and (ii) $\overline{W - C}$ is ambiently isotopic in W to the characteristic submanifold of the pair $(W, \partial W)$.

Proposition 4.7. *Let W be a simple 3-manifold. Let Σ denote the characteristic submanifold of the pair $(W, \partial W)$, and suppose that $\chi(\overline{W - \Sigma}) \geq 0$. Then $\overline{W - \Sigma}$ is a regular neighborhood of a properly embedded submanifold \mathcal{A} of W , each component of which is an annulus. Furthermore, there is a normal book of I -bundles \mathcal{W} such that $|\mathcal{W}| = W$ and $\mathcal{A}_{\mathcal{W}} = \mathcal{A}$.*

Proof. Set $\mathcal{C} = \overline{W - \Sigma}$. As we observed in 4.5, the components of the frontier of Σ are π_1 -injective annuli. In particular, no component of $\partial\mathcal{C}$ is a 2-sphere. Since $\chi(\mathcal{C}) \geq 0$, it follows that every component of $\partial\mathcal{C}$ is a torus. On the other hand, the property (2) of Σ stated in 4.5 implies that \mathcal{C} is π_1 -injective in W . Since W is simple, it follows that the fundamental group of a component of \mathcal{C} cannot have a rank-2 free abelian subgroup. Hence the components of \mathcal{C} are boundary-reducible, and in view of the irreducibility of W they must be solid tori.

Let C be any component of \mathcal{C} . The frontier components of C are among the frontier components of Σ , and we observed in 4.5 that these are essential annuli in W . In particular it follows that the components of $C \cap \partial W$ are π_1 -injective annuli on the solid torus C . Hence C may be given the structure of a Seifert fibered space in such a way that $C \cap \partial W$ is saturated. It now follows from Property (4) of Σ that C is ambiently isotopic in W to a submanifold of Σ . Since C is also ambiently isotopic to a submanifold of $W - \Sigma$, it must be ambiently isotopic to the regular neighborhood of a frontier component of Σ . This proves:

4.7.1. *\mathcal{C} is a regular neighborhood of a properly embedded 2-manifold $\mathcal{A} \subset W$ whose components are annuli. In particular, each component of \mathcal{C} has two frontier annuli.*

In particular this gives the first assertion of the proposition.

Let Σ_0 denote the union of all components of Σ that are solid tori, and set $\Sigma_- = \Sigma - \Sigma_0$. Since every component of Σ is a solid torus or an I -bundle P with $P \cap \partial W = \partial_h P$, each component of Σ_- is an I -bundle P over a surface of negative Euler characteristic with $P \cap \partial W = \partial_h P$.

We now claim:

4.7.2. *Each component of \mathcal{C} has one frontier annulus contained in Σ_0 and one contained in Σ_- .*

Let C be any component of \mathcal{C} . According to 4.7.1, C has two frontier annuli A_1 and A_2 . Let Q_i denote the component of Σ containing A_i (where a priori we might have $Q_1 = Q_2$). To prove 4.7.2 we must show that Q_1 and Q_2 cannot both be contained in Σ_- or both be contained in Σ_0 .

First suppose that the Q_i are both contained in Σ_- . Then $Q \doteq Q_1 \cup C \cup Q_2$ may be given the structure of an I -bundle over a surface in such a way that $\partial_h Q = Q \cap \partial W$. It therefore follows from the property (4) of Σ stated in 4.5 that Q is ambiently isotopic to a submanifold of Σ . But since $\chi(Q_i) < 0$ for $i = 1, 2$ we have $\bar{\chi}(Q) > \bar{\chi}(Q_i)$ for $i = 1, 2$. Hence Q cannot be ambiently isotopic to a submanifold of Q_i for $i = 1, 2$. Furthermore, if Q' is a component of Σ distinct from Q_1 and Q_2 , then since $Q \cap Q' = \emptyset$ and $\chi(Q) < 0$, the I -bundle Q cannot be ambiently isotopic to a submanifold of Q' . This is a contradiction.

Now suppose that the Q_i are both contained in Σ_0 . Then the Q_i are solid tori, and the image of the inclusion homomorphism $\pi_1(A_i) \rightarrow \pi_1(Q_i)$ has some finite index m_i in $\pi_1(Q_i)$ for $i = 1, 2$. The fundamental group of $Q \doteq Q_1 \cup C \cup Q_2$ has presentation $\langle x_1, x_2 : x_1^{m_1} = x_2^{m_2} \rangle$. But since W is simple and the frontier annuli of Q are essential, $\pi_1(Q)$ has no rank-2 free abelian subgroup. Hence at least one of the m_i must be equal to 1, and we may assume that $m_2 = 1$. But this implies that Q_2 is ambiently isotopic to a submanifold of Q_1 , which contradicts the property (3) of Σ stated in 4.5. This completes the proof of 4.7.2.

It follows from 4.7.1 and 4.7.2 that $\mathcal{B} \doteq \Sigma_0 \cup \mathcal{C}$ is a regular neighborhood of Σ_0 in W , and that the frontier \mathcal{A}' of \mathcal{B} is ambiently isotopic to \mathcal{A} . If we set $\mathcal{P} = \Sigma_-$, it follows from the definition given in [6, 5.1] that $\mathcal{W}' = (W, \mathcal{B}, \mathcal{P})$ is a book of I -bundles. Normality is immediate from the construction. Since $\mathcal{A}_{\mathcal{W}'} = \mathcal{A}'$ is ambiently isotopic to \mathcal{A} , there is a normal book of I -bundles \mathcal{W} with $|\mathcal{W}| = |\mathcal{W}'| = W$ and $\mathcal{A}_{\mathcal{W}} = \mathcal{A}$. \square

Requiring that a book of I -bundles structure be normal rules out certain degeneracies. For example, if W^b is an I -bundle over a closed surface of negative Euler characteristic, we may write $W^b = |\mathcal{W}^b|$ for some book of I -bundles \mathcal{W}^b with $\mathcal{B}_{\mathcal{W}^b} \neq \emptyset$. Such a book of I -bundles \mathcal{W}^b is not normal, and the following proposition would become false if the normal book of I -bundles \mathcal{W} were replaced by \mathcal{W}^b .

Proposition 4.8. *Let \mathcal{W} be a normal book of I -bundles, and suppose that A is an essential annulus in $|\mathcal{W}|$. Then A is ambiently isotopic in $|\mathcal{W}|$ to either a vertical annulus in a page of \mathcal{W} , or an annulus contained in a binding.*

Proof. Set $W = |\mathcal{W}|$. Let \mathcal{C} be a regular neighborhood of $\mathcal{A}_{\mathcal{W}}$ in W . The definition of normality implies that (up to isotopy) $\Sigma \doteq \overline{W - \mathcal{C}}$ is the characteristic submanifold of W relative to ∂W .

Let V be a regular neighborhood of A in W . Then V may be given the structure of a Seifert fibered space in such a way that $V \cap \partial W$ is a saturated 2-manifold in ∂V . The components

of the frontier of V in W are essential annuli in W . Hence the property (4) of Σ stated in 4.5 implies that V is ambiently isotopic in W to a submanifold of Σ . In particular, A is ambiently isotopic in W to an annulus $A' \subset W - \mathcal{A}_W$. Thus A' is contained in either a page or a binding of W .

If A' is contained in a page P , then P is an I -bundle over a surface. Since A is essential in W (relative to ∂W), it is in particular essential in P relative to $\partial_h P$. It therefore follows from Proposition 4.4 that A' is isotopic in P , by an ambient isotopy of P which fixes $\partial_v P$, to a vertical annulus. The conclusion of the proposition follows. \square

In the lemma below, recall that if K is a polyhedron then by definition $\bar{\chi}(K) = -\chi(K)$.

Lemma 4.9. *Let \mathcal{W} be a normal book of I -bundles. Let Y be a compact 3-dimensional submanifold of $|\mathcal{W}|$. Suppose that the following conditions hold.*

- (1) *Each component of the frontier of Y in $|\mathcal{W}|$ is an essential properly embedded annulus in $|\mathcal{W}|$.*
- (2) *The 2-manifold $Y \cap \partial|\mathcal{W}|$ has two components Z_0 and Z_1 , with $\bar{\chi}(Z_0) = \bar{\chi}(Z_1) = 1$.*
- (3) *The inclusion homomorphism $H_1(Z_0; \mathbb{Z}_2) \rightarrow H_1(Y; \mathbb{Z}_2)$ is injective.*
- (4) *For every solid torus $L \subset Y$, such that $L \cap Z_0$ is an annulus which is homotopically non-trivial in $|\mathcal{W}|$, the inclusion homomorphism $H_1(L \cap Z_0; \mathbb{Z}) \rightarrow H_1(L, \mathbb{Z})$ is surjective.*

Then the inclusion homomorphism $\pi_1(Z_0) \rightarrow \pi_1(Y)$ is surjective.

Proof. We set $W = |\mathcal{W}|$, $\mathcal{P} = \mathcal{P}_W$ and $\mathcal{B} = \mathcal{B}_W$.

Since the frontier components of Y relative to W are annuli, we have

$$\bar{\chi}(Y) = \frac{1}{2}\bar{\chi}(\partial Y) = \frac{1}{2}(\bar{\chi}(Z_0) + \bar{\chi}(Z_1)) = 1.$$

By Proposition 4.8 we may assume that each frontier component of Y is either a vertical annulus contained in \mathcal{P} and disjoint from \mathcal{A} , or an annulus in \mathcal{B} . Then each component of $Y \cap \mathcal{P}$ is an I -sub-bundle of a page of \mathcal{W} , whose frontier is a disjoint union of vertical annuli in the page; each of these vertical annuli is either contained in, or disjoint from, the vertical boundary of the page. Furthermore, each component of $Y \cap \mathcal{B}$ is a solid torus in a binding, whose frontier is a disjoint union of π_1 -injective annuli in the binding. In particular we have $\bar{\chi}(C) \geq 0$ for every component C of $Y \cap \mathcal{P}$, and $\bar{\chi}(C) = 0$ for every component C of $Y \cap \mathcal{B}$.

Since the frontier components of $Y \cap \mathcal{B}$ and $Y \cap \mathcal{P}$ relative to Y are annuli, we have

$$1 = \bar{\chi}(Y) = \bar{\chi}(Y \cap \mathcal{B}) + \bar{\chi}(Y \cap \mathcal{P}) = \bar{\chi}(Y \cap \mathcal{P}) = \sum_U \bar{\chi}(U),$$

where U ranges over the components of $Y \cap \mathcal{P}$. Hence there is a component U_1 of $Y \cap \mathcal{P}$ with $\bar{\chi}(U_1) = 1$, and $\bar{\chi}(U) = 0$ for every component $U \neq U_1$ of $Y \cap \mathcal{P}$.

Since U_1 is a sub-bundle of a page of \mathcal{W} , it is an I -bundle over a surface S with $\bar{\chi}(S) = 1$, and the frontier of U_1 relative to W is its vertical boundary.

We set $\mathcal{K} = \overline{Y - U_1}$.

Each component of the frontier of U_1 or \mathcal{K} relative to W is a component of either the frontier \mathcal{A}_W of \mathcal{P}_W relative to W , or the frontier of Y relative to W . According to [6, Lemma 5.2], the components of \mathcal{A}_W are π_1 -injective annuli in W . The components of the frontier of Y are π_1 -injective by hypothesis. Hence:

4.9.1. *Each component of the frontier of U_1 or \mathcal{K} relative to W is a π_1 -injective annulus in W .*

The horizontal boundary of U_1 is a two-sheeted covering space of S , which is connected if and only if U_1 is a twisted I -bundle. In particular we have $\bar{\chi}(\partial_h U_1) = 2\bar{\chi}(S) = 2$. On the other hand, we have $\partial_h U_1 = U_1 \cap \partial W \subset Y \cap \partial W = Z_0 \cup Z_1$. Hence $\partial_h U_1 \subset Z_j$ for some $j \in \{0, 1\}$. It follows from 4.9.1 that $\partial_h U_1$ is π_1 -injective in W and hence in $Z_0 \cup Z_1$. Since $\bar{\chi}(Z_0) = \bar{\chi}(Z_1) = 1$, the surface $\partial_h U_1$ cannot be contained in Z_0 or in Z_1 . Hence:

4.9.2. *U_1 is a trivial I -bundle over S , and its horizontal boundary has one component contained in Z_0 and one in Z_1 .*

For $i = 1, 2$, let us denote by Δ_i the component of $\partial_h U_1$ contained in Z_i .

Since W is simple by the definition of a normal book of I -bundles, it follows from 4.9.1 that each component of \mathcal{K} is simple. If V is any component of \mathcal{K} , the components of $V \cap \mathcal{P}$ are components of $Y \cap \mathcal{P}$ distinct from U_1 , and the components of $V \cap \mathcal{B}$ are components of $Y \cap \mathcal{B}$. Hence all components of $V \cap \mathcal{P}$ and $V \cap \mathcal{B}$ have Euler characteristic 0. As the components of $(V \cap \mathcal{P}) \cap (V \cap \mathcal{B})$ are annuli it follows that $\chi(V) = 0$. But the only simple 3-manifold with non-empty boundary having Euler characteristic 0 is a solid torus. This shows:

4.9.3. *Every component of \mathcal{K} is a solid torus.*

Let us consider any component A of $U_1 \cap \mathcal{K}$, and let V denote the component of \mathcal{K} containing A . According to 4.9.3, V is a solid torus. Since A is a component of the frontier of U_1 in W , it is a π_1 -injective annulus in W by 4.9.1, and hence in V . On the other hand, it follows from 4.9.2 that some component c of ∂A is contained in Z_0 . A small non-ambient isotopy of V gives a solid torus L such that

$L \cap Z_0$ is a regular neighborhood R of c . Since A is π_1 -injective in W , the annulus R is homotopically non-trivial in W . Hypothesis (4) then implies that the inclusion homomorphism $H_1(L \cap Z_0; \mathbb{Z}) \rightarrow H_1(L, \mathbb{Z})$ is an isomorphism, and hence that the inclusion homomorphism $\pi_1(A) \rightarrow \pi_1(V)$ is an isomorphism. This shows:

4.9.4. *For every component A of $U_1 \cap \mathcal{K}$, if V denotes the component of \mathcal{K} containing A , the inclusion homomorphism $\pi_1(A) \rightarrow \pi_1(V)$ is an isomorphism.*

Now suppose that A_1 and A_2 are two components of $U_1 \cap \mathcal{K}$ contained in a single component V of \mathcal{K} . For $i = 1, 2$ it follows from 4.9.2 that some component c_i of ∂A_i is contained in Z_0 . Since c_1 and c_2 are disjoint, non-trivial simple closed curves on the torus $\partial V \subset$

W , they represent the same element of $H_1(W; \mathbb{Z}_2)$. Since by hypothesis (3) the inclusion homomorphism $H_1(Z_0; \mathbb{Z}_2) \rightarrow H_1(Y; \mathbb{Z}_2)$ is injective, c_1 and c_2 cobound a subsurface Q of Z_0 . Since c_1 and c_2 are homotopically non-trivial, Q is π_1 -injective in Z_0 , and hence $\bar{\chi}(Q) \leq \bar{\chi}(Z_0) = 1$. As Q is orientable and has exactly two boundary curves, it must be an annulus. On the other hand, Q and Δ_0 are subsurfaces of Z_0 , and we have $\partial Q \subset \partial \Delta_0$. Hence either $\Delta_0 \subset A$ or A is a component of $\overline{Z_0 - \Delta_0}$. But it follows from 4.9.1 that Δ_0 is π_1 -injective in W and hence in Z_0 ; since $\bar{\chi}(\Delta) = 1$, we cannot have $\Delta_0 \subset A$. Thus we have proved:

4.9.5. *If $A^{(0)}$ and $A^{(1)}$ are two components of $U_1 \cap \mathcal{K}$ contained in the same component of \mathcal{K} , then some component of $\overline{Z_0 - \Delta_0}$ is an annulus Q having one boundary component contained in $\partial A^{(0)}$ and one contained in $\partial A^{(1)}$.*

In particular, if Q is an annulus having the properties stated in 4.9.5, then $\partial Q \subset \partial \Delta_0$. Since $\bar{\chi}(\Delta_0) = 1$, the surface Δ_0 has at most three boundary curves. Hence there is at most one unordered pair $\{A^{(0)}, A^{(1)}\}$ of components of $U_1 \cap \mathcal{K}$ such that $A^{(0)}$ and $A^{(1)}$ are contained in the same component of \mathcal{K} . Equivalently:

4.9.6. *There is no component of \mathcal{K} whose boundary contains more than two components of $U_1 \cap \mathcal{K}$, and there is at most one component of \mathcal{K} whose boundary contains two components of $U_1 \cap \mathcal{K}$.*

In view of 4.9.6, the argument now divides into two cases.

Case I: The boundary of each component of \mathcal{K} contains only one component of $U_1 \cap \mathcal{K}$. In this case, let A_1, \dots, A_m denote the components of $U_1 \cap \mathcal{K}$. (Since $\bar{\chi}(S) = 1$ we have $m \leq 3$.) Let V_i denote the component of \mathcal{K} containing A_i ; thus the solid tori V_1, \dots, V_m are all distinct. We have $Y = U_1 \cup V_1 \cup \dots \cup V_m$. By 4.9.4 the inclusion homomorphism $\pi_1(A_i) \rightarrow \pi_1(V_i)$ is an isomorphism for $i = 1, \dots, m$. Hence the inclusion homomorphism $\pi_1(U_1) \rightarrow \pi_1(Y)$ is an isomorphism. But by 4.9.2, the inclusion homomorphism $\pi_1(\Delta_0) \rightarrow \pi_1(U_1)$ is an isomorphism. Hence the inclusion homomorphism $\pi_1(\Delta_0) \rightarrow \pi_1(Y)$ is an isomorphism, and in particular the inclusion homomorphism $\pi_1(Z_0) \rightarrow \pi_1(Y)$ is surjective. This establishes the conclusion of the lemma in this case.

Case II: There is a component V_0 of \mathcal{K} whose boundary contains more than one component of $U_1 \cap \mathcal{K}$. By 4.9.6, this component V_0 is unique and contains exactly two components of $U_1 \cap \mathcal{K}$, which we denote $A_0^{(0)}$ and $A_0^{(1)}$.

According to 4.9.3, V_0 is a solid torus, and according to 4.9.4 the inclusion homomorphism $\pi_1(A_0^{(i)}) \rightarrow \pi_1(V_0)$ is an isomorphism for $i = 0, 1$. Hence there is a homeomorphism $h : V_0 \rightarrow S^1 \times [0, 1] \times [0, 1]$ such that $h(A_0^{(i)}) = S^1 \times \{i\} \times [0, 1]$ for $i = 0, 1$. This allows us to extend the I -bundle structure of U_1 to an I -bundle structure for $L \doteq U_1 \cup V_0$. (Note, however, that the horizontal boundary of L need not be contained in ∂W .)

According to 4.9.5, some component of $\overline{Z_0 - \Delta_0}$ is an annulus Q having one boundary component contained in $\partial A^{(0)}$ and one contained in $\partial A^{(1)}$. This implies that the I -bundle L is

trivial, and that one of its horizontal boundary components, which we shall denote by Θ , is contained in Z_0 .

Now let A_1, \dots, A_m denote the components of $U_1 \cap \mathcal{K}$ distinct from $A_0^{(0)}$ and $A_0^{(1)}$. (Since $\bar{\chi}(S) = 1$ one may show that $m \leq 1$, but this will not be used.) Let V_i denote the component of \mathcal{K} containing A_i ; thus the solid tori V_1, \dots, V_m are all distinct. We have $Y = L \cup V_1 \cup \dots \cup V_m$. By 4.9.4 the inclusion homomorphism $\pi_1(A_i) \rightarrow \pi_1(V_i)$ is an isomorphism for $i = 1, \dots, m$. Hence the inclusion homomorphism $\pi_1(L) \rightarrow \pi_1(Y)$ is an isomorphism. But since L is a trivial I -bundle and Θ is one component of its horizontal boundary, the inclusion homomorphism $\pi_1(\Theta) \rightarrow \pi_1(L)$ is an isomorphism. Hence the inclusion homomorphism $\pi_1(\Theta) \rightarrow \pi_1(Y)$ is an isomorphism, and in particular the inclusion homomorphism $\pi_1(Z_0) \rightarrow \pi_1(Y)$ is surjective. Thus the conclusion of the lemma is established in this case as well. \square

5. TRIMONIC MANIFOLDS

The kind of object discussed in this section, defined in 5.7 below, is obtained from a trivial I -bundle over a surface S by first attaching a single 1-handle α to one boundary component, then attaching a single 2-handle whose boundary runs thrice over α . Here we prove that such a *trimonic manifold* has incompressible boundary (in Lemma 5.13) and is not the underlying space of a normal book of I -bundles (Lemma 5.14), if it is *non-degenerate* (cf. 5.10).

5.1. Let V be a point of an oriented surface S . By an *ordered triod* based at V we shall mean an ordered triple (A_0, A_1, A_2) of closed topological arcs in S , each having V as an endpoint, such that $A_i \cap A_j = \{V\}$ whenever $i \neq j$.

Suppose that (A_0, A_1, A_2) is an ordered triod based at V . For $i = 0, 1, 2$ let x_i denote the endpoint of A_i that is distinct from V . Then there is a disk $\delta \subset S$ such that $A_0 \cup A_1 \cup A_2 \subset \delta$ and $(A_0 \cup A_1 \cup A_2) \cap \partial\delta = \{x_0, x_1, x_2\}$. We shall express this by saying that the triod (A_0, A_1, A_2) is *properly embedded* in δ . The orientation of S restricts to an orientation of δ , which in turn induces an orientation of $\partial\delta$. We shall say that the ordered triod (A_0, A_1, A_2) is *positive* if the ordered triple (x_0, x_1, x_2) is in counterclockwise order on $\partial\delta$, and *negative* otherwise.

5.2. Suppose that θ is an oriented open arc. We denote by θ' the same arc with the opposite orientation. By a *terminal segment* of θ we mean a subset A of θ which has the form $h((t, 1))$ for some orientation-preserving homeomorphism $h : (0, 1) \rightarrow \theta$ and some point $t \in (0, 1)$. By an *initial segment* of θ we mean a terminal segment of θ' .

5.3. Now suppose that Γ is a graph (i.e. a 1-dimensional CW complex) contained in an oriented surface S . By an *oriented edge* of Γ we mean simply an (open) edge which is equipped with an orientation. Let V be a vertex of Γ , and let (e_0, e_1, e_2) be an ordered triple of oriented edges of Γ with terminal vertex V . Assume that the e_i are distinct as oriented edges (although two of them may be opposite orientations of the same underlying edge). We may choose terminal segments A_i of the e_i in such a way that $(\bar{A}_0, \bar{A}_1, \bar{A}_2)$ is an ordered triod in S . We shall say that the ordered triple (e_0, e_1, e_2) is *positive* if the ordered triod $(\bar{A}_0, \bar{A}_1, \bar{A}_2)$ is positive, and *negative* otherwise.

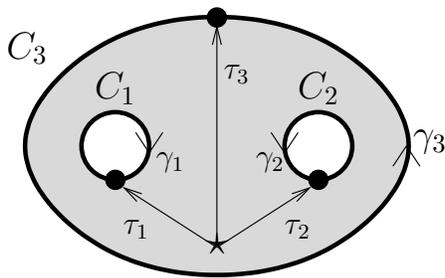


FIGURE 5.1. A geometric basis

5.4. Let Z be a planar surface with three boundary curves, and let $\star \in \text{int } Z$ be a base point. An ordered pair (z_1, z_2) of elements of $\pi_1(Z, \star)$ will be called a *geometric basis* for $\pi_1(Z, \star)$ if the boundary curves of Z may be indexed as $(C_i)_{1 \leq i \leq 3}$ in such a way that for $i = 1, 2, 3$ there exist a point $p_i \in C_i$, a closed path γ_i in C_i based at p_i and an oriented (embedded) arc τ_i from \star to p_i , such that

- the τ_i have pairwise disjoint interiors;
- $\tau_i \cap C_i = \{p_i\}$ for each i ;
- $[\gamma_i]$ generates $\pi_1(C_i, p_i)$ for each i ;
- $z_i = [\tau_i * \gamma_i * \bar{\tau}_i]$ for $i = 1, 2$; and
- $z_1^{-1} z_2 = [\tau_3 * \gamma_3 * \bar{\tau}_3]$.

The constituent parts of a geometric basis are pictured in Figure 5.1. Note that if (z_1, z_2) is a geometric basis for $\pi_1(Z, \star)$ then z_1 and z_2 freely generate $\pi_1(Z, \star)$.

5.5. Let Γ be a theta graph contained in an oriented surface S . Let W and V denote the vertices of Γ , and let β_0, β_1 and β_2 denote the oriented edges having initial vertex W and terminal vertex V . Suppose that the ordered triple $(\beta_0, \beta_1, \beta_2)$ of edges terminating at V is positive, and that the ordered triple $(\beta'_0, \beta'_1, \beta'_2)$ of oriented edges terminating at W is negative. Then a regular neighborhood Z of Γ in S is a planar surface with three boundary curves, and $([\beta'_0 * \beta_1], [\beta'_0 * \beta_2])$ is a geometric basis of $\pi_1(Z, V)$.

5.6. Let Γ be an eyeglass graph contained in an oriented surface S . Let W and V denote the vertices of Γ , let β_0 denote the oriented edge having initial vertex W and terminal vertex V , and let β_1 and β_2 be oriented loops based at W and V respectively. Suppose that the ordered triple $(\beta_0, \beta'_2, \beta_2)$ of edges terminating at V is positive, and that the ordered triple $(\beta'_0, \beta'_1, \beta_1)$ of edges terminating at W is negative. Then a regular neighborhood Z of Γ in S is a planar surface with three boundary curves, and $([\beta_2], [\beta'_0 * \beta_1 * \beta_0])$ is a geometric basis of $\pi_1(Z, V)$.

Definition 5.7. Let X be a compact orientable 3-manifold, and let S be a component of ∂X . We shall say that X is a *trimonic manifold relative to S* if there exists a properly embedded arc $\alpha \subset X$ and a PL map f of a PL 2-disk D into X , such that the following conditions hold:

- (1) $f^{-1}(\alpha)$ is a union of three disjoint arcs in ∂D ;

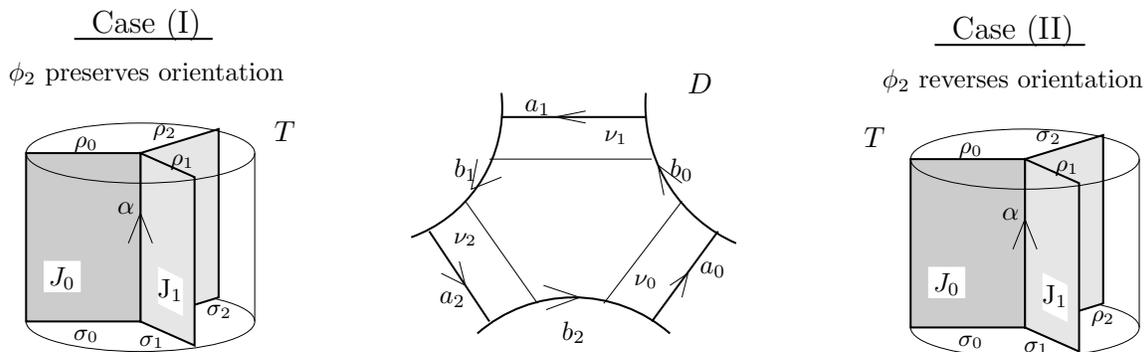


FIGURE 5.2. Some objects defined in the proof of Lemma 5.9

- (2) f maps each component of $f^{-1}(\alpha)$ homeomorphically onto α ;
- (3) $f|(\text{int } D \cup ((\partial D) - f^{-1}(\alpha)))$ is one-to-one;
- (4) $f(\text{int } D) \subset \text{int } X$;
- (5) $f((\partial D) - f^{-1}(\alpha)) \subset S$;
- (6) X is a semi-regular neighborhood of $S \cup f(D)$.

By a *semi-regular neighborhood* of K in M we mean a neighborhood of K which is a compact PL submanifold of M and admits a polyhedral collapse to K , but does not necessarily meet the boundary of M regularly.

Note that condition (5) implies that the endpoints of α lie in S .

A PL map f of a 2-disk D into X such that (1)–(6) hold for some properly embedded arc α in X will be called a *defining hexagon* for the trimonic manifold X relative to S .

Notation 5.8. Suppose that $f : D \rightarrow X$ is a defining hexagon for a trimonic manifold X relative to S . Note that the arc α appearing in Definition 5.7 is uniquely determined by D . We shall denote this arc by α_f . Furthermore, we shall denote by Γ_f the PL set $f((\partial D) - f^{-1}(\alpha)) = f(D) \cap \partial X \subset S$.

Lemma 5.9. *Suppose that X is a trimonic manifold relative to S . Let $f : D \rightarrow X$ be a defining hexagon for X , and set $\alpha = \alpha_f$ and $\Gamma = \Gamma_f$. Then Γ is homeomorphic to either a theta graph (the “theta case”) or an eyeglass graph (the “eyeglass case”), and a regular neighborhood Z of Γ in S is a planar surface with three boundary curves. Furthermore, for some (and hence for any) base point $\star \in \text{int } Z$, there is an ordered basis (t, u_1, u_2) of the rank-3 free group $\pi_1(Z \cup \alpha, \star)$ with the following properties:*

- the inclusion homomorphism $\pi_1(Z, \star) \rightarrow \pi_1(Z \cup \alpha, \star)$ maps some geometric basis of $\pi_1(Z, \star)$ to the pair (u_1, u_2) ; and
- ∂D may be oriented so that the conjugacy class in $\pi_1(Z \cup \alpha, \star)$ represented by $f|\partial D : \partial D \rightarrow Z \cup \alpha$ is $t^2 u_1 t u_2$ in the theta case, or $t^2 u_1 t^{-1} u_2$ in the eyeglass case.

Proof. We denote the components of $f^{-1}(\alpha)$ by a_0, a_1 and a_2 , and we denote the components of $(\partial D) - f^{-1}(\alpha)$ by b_0, b_1 and b_2 . We take the a_i and b_i to be labeled in such a way that

a_i and b_j share an endpoint if and only if i is congruent to either j or $j + 1$ modulo 3. For $i = 0, 1, 2$ we set $\beta_i = f(\text{int } b_i)$. Then β_i is an open arc in $S - \partial\alpha$, and $\overline{\beta_i} - \beta_i \subset \partial\alpha$. Hence $\Gamma = \overline{\beta_0 \cup \beta_1 \cup \beta_2}$ may be given the structure of a graph whose vertices are the endpoints of α , and whose edges are β_0 , β_1 and β_2 .

Since α is an embedded arc, f maps the terminal point of β_i and the initial point of β_{i+1} to distinct vertices of Γ . In particular, each vertex of Γ has valence 3. Since Γ has three edges and two vertices, it is either a theta graph or an eyeglass graph.

For $i = 0, 1, 2$ we denote by P_i the common endpoint of the arcs a_i and b_{i-1} in ∂D (where subtraction is interpreted modulo 3), and by Q_i the common endpoint of a_i and b_i .

If we fix an orientation of ∂D then each of the arcs a_i and b_i inherits an orientation. We choose the orientation of ∂D in such a way that P_i and Q_i are respectively the initial and terminal points of a_i , while Q_i and P_{i+1} are respectively the initial and terminal points of b_i . According to Definition 5.7, f restricts to a homeomorphism $\phi_i : a_i \rightarrow \alpha$ for $i = 0, 1, 2$.

Let T be a PL tubular neighborhood of α in X . We may choose T in such a way that f is transverse to the frontier of T and $f^{-1}(T)$ is a regular neighborhood of $f^{-1}(\alpha) = a_0 \cup a_1 \cup a_2$ in D . For each $i \in \{0, 1, 2\}$, let ν_i denote the component of $f^{-1}(T)$ containing a_i . Then $\nu_i \cap \partial D$ has the form $a_i \cup s_i \cup r_i$, where s_i is the closure of a terminal segment of b_{i-1} and r_i is the closure of an initial segment of b_i .

It follows from Definition 5.7 that f maps each ν_i homeomorphically on to a PL disk $J_i \subset T$, and that $J_i \cap J_j = \alpha$ when $i \neq j$. The intersection of T with ∂X consists of two PL disks; for each $i \in \{0, 1, 2\}$, one of these disks meets J_i in the arc $\sigma_i \doteq f(s_i)$, and the other meets J_i in the arc $\rho_i \doteq f(r_i)$. Hence we may identify T by a PL homomorphism with $\delta \times \alpha$, where δ is a PL disk, in such a way that $\alpha = \{o\} \times \alpha$ for some interior point o of δ ; and so that for $i = 0, 1, 2$ we have $J_i = t_i \times \alpha$ for some arc $t_i \subset \delta$. Each t_i has one endpoint in $\partial\delta$ and one at o , and we have $t_i \cap t_j = o$ for $i \neq j$. The components of $t_i \times \partial\alpha$ are σ_i and ρ_i .

We may orient α in such a way that at least two of the homeomorphisms $\phi_i : a_i \rightarrow \alpha$ are orientation-preserving. Hence, after a possible cyclic relabeling of the a_i (and the b_i), we may assume that ϕ_0 and ϕ_1 are orientation-preserving.

We denote by V and W , respectively, the initial and terminal endpoints of α with respect to the orientation that we have chosen. Thus for $i = 0, 1$ we have $f(P_i) = V$ and $f(Q_i) = W$.

We shall distinguish two cases according to whether the homeomorphism $\phi_2 : a_2 \rightarrow \alpha$ (I) preserves or (II) reverses orientation. In case I we have $f(P_2) = V$ and $f(Q_2) = W$, while in case II we have $f(P_2) = W$ and $f(Q_2) = V$. Hence we may define ordered triods (see 5.1) \mathcal{T}_V and \mathcal{T}_W based at V and W respectively by setting $\mathcal{T}_V = (\sigma_0, \sigma_1, \sigma_2)$ and $\mathcal{T}_W = (\rho_0, \rho_1, \rho_2)$ in case I, and $\mathcal{T}_V = (\sigma_0, \sigma_1, \rho_2)$ and $\mathcal{T}_W = (\rho_0, \rho_1, \sigma_2)$ in case II. In each case, if we denote by δ_V and δ_W the components of $T \cap \partial X$ containing V and W respectively, the triods \mathcal{T}_V and \mathcal{T}_W are properly embedded (see 5.1) in δ_V and δ_W .

If we identify T as above with $\delta \times \alpha$, we have $\delta_V = \delta \times \{V\}$ and $\delta_W = \delta \times \{W\}$. We may then define a homeomorphism $\psi : \delta_V \rightarrow \delta_W$ by $\psi(x, V) = (x, W)$.

We orient S in such a way that \mathcal{T}_V is a positive ordered triod (see 5.1). Each of the disks δ_V and δ_W inherits an orientation from S . The orientability of the 3-manifold X implies that $\psi : \delta_V \rightarrow \delta_W$ is an orientation-reversing homeomorphism.

By the properties of the identification stated above, we have $\psi(\sigma_i) = \rho_i$ for $i = 0, 1$; and in Case I we have $\psi(\sigma_2) = \rho_2$, while in Case II we have $\psi(\rho_2) = \sigma_2$. Since \mathcal{T}_V is a positive ordered triod, and since ψ reverses orientation, the ordered triod \mathcal{T}_W is negative.

We orient each open arc β_i in such a way that $f|_{\text{int } b_i} : \text{int } b_i \rightarrow \beta_i$ is orientation-preserving. Since α , β_0 , β_1 and β_2 are now equipped with orientations, their closures define elements of the fundamental groupoid $\Pi(f(D))$ which we denote by $[\alpha]$, $[\beta_0]$, $[\beta_1]$ and $[\beta_2]$. We let c denote the closed path $a_0 * b_0 * a_1 * b_1 * a_2 * b_2$ in ∂D , based at P_0 . Then $[c]$ generates $\pi_1(\partial D, P_0)$. We set $\gamma = f \circ c$.

We have

$$[\gamma] = [\alpha][\beta_0][\alpha][\beta_1][\alpha]^\epsilon[\beta_2] \in \pi_1(f(D), V) \subset \Pi(f(D)),$$

where $\epsilon = 1$ in Case I, and $\epsilon = -1$ in Case II.

To prove the conclusions of the lemma in Case I, we note that since $f(P_i) = V$ and $f(Q_i) = W$ for each i , each β_i has W as initial vertex and V as terminal vertex. Thus Γ is a theta graph, and Case I is the ‘‘theta case’’ referred to in the statement of the Lemma.

Since $\mathcal{T}_V = (\sigma_0, \sigma_1, \sigma_2)$ is a positive ordered triod based at V , and the interior of σ_i is a terminal segment of β_{i-1} , the triple $(\beta_2, \beta_0, \beta_1)$ of edges terminating at V is positive in the sense of 5.3. Hence the triple $(\beta_0, \beta_1, \beta_2)$ is positive. Likewise, since $\mathcal{T}_W = (\rho_0, \rho_1, \rho_2)$ is a negative ordered triod based at W , and since the interior of ρ_i is an initial segment of β_i —and therefore a terminal segment of β'_i —the triple $(\beta'_0, \beta'_1, \beta'_2)$ of edges terminating at W is negative. It now follows from 5.5 that a regular neighborhood Z of Γ in S is a planar surface with three boundary curves, and that if we set $z_1 = [\beta'_0 * \beta_1]$ and $z_2 = [\beta'_0 * \beta_2]$, then (z_1, z_2) is a geometric basis of $\pi_1(Z, V)$.

In particular $\pi_1(Z, V)$ is freely generated by z_1 and z_2 , and hence $\pi_1(Z \cup \alpha, V)$ is generated by t , u_1 and u_2 , where u_i denotes the image of z_i under the inclusion homomorphism $\pi_1(Z, V) \rightarrow \pi_1(Z \cup \alpha, V)$, and $t = [\alpha * \beta_0]$.

We have

$$\begin{aligned} [\gamma] &= [\alpha * \beta_0 * \alpha * \beta_1 * \alpha * \beta_2] \\ &= [\alpha * \beta_0 * \alpha * \beta_0 * \beta'_0 * \beta_1 * \alpha * \beta_0 * \beta'_0 * \beta_2] \\ &= t^2 u_1 t u_2. \end{aligned}$$

This establishes the conclusions of the Lemma in the theta case.

In Case II we have $f(P_i) = V$ and $f(Q_i) = W$ for $i = 0, 1$, while $f(P_2) = W$ and $f(Q_2) = V$. It follows that the closures of β_1 and β_2 are loops based at W and V respectively, whereas β_0 has initial point W and terminal point V . Thus Γ is an eyeglass graph, and Case II is the ‘‘eyeglass case’’ referred to above.

Since $\mathcal{T}_V = (\sigma_0, \sigma_1, \rho_2)$ is a positive ordered triod based at V , and since the interiors of σ_0 , σ_1 and ρ_2 are respectively terminal segments of β_2 , β_0 and β'_2 , the triple $(\beta_2, \beta_0, \beta'_2)$ of

edges terminating at V is positive. Hence the triple $(\beta_0, \beta'_2, \beta_2)$ is positive. Likewise, since $\mathcal{T}_W = (\rho_0, \rho_1, \sigma_2)$ is a negative ordered triod based at W , and since the interiors of ρ_0 , ρ_1 and σ_2 are respectively terminal segments of β'_0 , β'_1 and β_1 , the triple $(\beta'_0, \beta'_1, \beta_1)$ of edges terminating at W is negative. It now follows from 5.6 that a regular neighborhood Z of Γ in S is a planar surface with three boundary curves, and that if we set $z_1 = [\beta'_0 * \beta_1 * \beta_0]$ and $z_2 = [\beta_2]$, then (z_1, z_2) is a geometric basis of $\pi_1(Z, V)$.

In particular $\pi_1(Z, V)$ is freely generated by z_1 and z_2 , and hence $\pi_1(Z \cup \alpha, V)$ is generated by t , u_1 and u_2 , where u_i denotes the image of z_i under the inclusion homomorphism $\pi_1(Z, V) \rightarrow \pi_1(Z \cup \alpha, V)$, and $t = [\alpha * \beta_0]$.

We have

$$\begin{aligned} [\gamma] &= [\alpha * \beta_0 * \alpha * \beta_1 * \alpha' * \beta_2] \\ &= [\alpha * \beta_0 * \alpha * \beta_0 * \beta'_0 * \beta_1 * \beta_0 * \beta'_0 * \alpha' * \beta_2] \\ &= (t)^2 u_1 (t)^{-1} u_2. \end{aligned}$$

This establishes the conclusions of the lemma in the eyeglass case. \square

Definition 5.10. Let X be a trimonic manifold relative to S . We shall say that X is *non-degenerate* if there is a defining hexagon $f : D \rightarrow X$ such that no component of $S - \Gamma_f$ is an open disk.

Lemma 5.11. *Suppose that X is a non-degenerate trimonic manifold relative to a component S of ∂X . Then $Hg(X) \leq 1 + \text{genus}(S)$. Furthermore, X contains a compact connected 3-dimensional submanifold Y with the following properties.*

- (1) *Each component of the frontier of Y in X is an essential annulus in X , joining distinct components of ∂X .*
- (2) *The 2-manifold $Y \cap \partial X$ has two components Z_0 and Z_1 , where $Z_0 \subset S$ and $Z_1 \not\subset S$, and each Z_i is a planar surface with three boundary curves.*
- (3) *The group $\pi_1(Y)$ is free of rank 2.*
- (4) *For any base point $\star \in \text{int } Z_0$, there exists an ordered pair of generators (x, y) of $\pi_1(Y, \star)$, such that either the pair $(x, yx^{-1}y^2)$ or the pair $(x, y^{-1}x^{-1}y^2)$ is the image of some geometric basis of $\pi_1(Z_0, \star)$ under the inclusion homomorphism $\pi_1(Z_0, \star) \rightarrow \pi_1(Y, \star)$.*
- (5) *Each component Q of $\overline{X - Y}$ may be given the structure of a trivial I -bundle over a 2-manifold, in such a way that $Y \cap Q$ is the vertical boundary of Q .*

Proof. We fix a defining hexagon $f : D \rightarrow X$, and set $\alpha = \alpha_f$ and $\Gamma = \Gamma_f$. Since X is non-degenerate, we may choose f in such a way that:

5.11.1. *No component of $S - \Gamma$ is an open disk.*

Let T be a PL tubular neighborhood of α in X , and let δ be a properly embedded disk in $T \cap \text{int } X$ which crosses α transversally in one point. We may choose T in such a way that $f^{-1}(T)$ is a regular neighborhood of $f^{-1}(\alpha) = a_0 \cup a_1 \cup a_2$ in D . Let D' denote the disk

$\overline{D - f^{-1}(T)}$. Then the frontier of D' is the union of three arcs a'_0, a'_1 and a'_2 , where a'_i is the frontier in D of the component of $f^{-1}(T)$ containing a_i .

According to Definition 5.7, f maps each a_i homeomorphically onto α . Hence we may choose δ so that for each i , the arc $f(a'_i)$ meets $\partial\delta$ transversally in exactly one point. In particular:

5.11.2. *The simple closed curve $f(\partial D')$ meets $\partial\delta$ in exactly three points, and these are transversal points of intersection within the frontier annulus of T .*

Let η denote a regular neighborhood in $\overline{X - T}$ of the properly embedded disk $f(D') \subset \overline{X - T}$. Then $K \doteq T \cup \eta$ is a regular neighborhood of $f(D)$ in X .

Let $\mathcal{C} = \Sigma \times I$, where Σ is a surface homeomorphic to S , and attach \mathcal{C} to X by using such a homeomorphism to identify $\Sigma \times \{1\}$ with S . We note that the pair (X, S) is homeomorphic to $(X \cup \mathcal{C}, S_0)$, where $S_0 \doteq \Sigma \times \{0\} \subset \partial(X \cup \mathcal{C})$. We set $X_0 = \mathcal{C} \cup K \subset X \cup \mathcal{C}$. Since K is a regular neighborhood of $f(D)$ in X , the manifold $X \cup \mathcal{C}$ is a semi-regular neighborhood of its 3-dimensional submanifold X_0 . It follows that the pair (X_0, S_0) is homeomorphic to $(X \cup \mathcal{C}, S_0)$, and hence to (X, S) . It therefore suffices to show that the conclusions of the lemma are true when X and S are replaced by X_0 and S_0 .

We first note that the manifold $V \doteq \mathcal{C} \cup T$ is a compression body with $\partial_- V = S_0$. If we set $g = \text{genus}(S)$, the genus of $\partial_+ V$ is $g + 1$. In particular V admits a Heegaard splitting of genus $g + 1$. We have $X_0 = V \cup \eta$, so that X_0 is obtained from V by adding a 2-handle. It therefore follows from [6, Lemma 2.1] that $\text{Hg}(X_0) \leq g + 1$. This gives the first assertion of the lemma.

We must now construct a compact connected 3-dimensional submanifold Y of X_0 such that Properties (1)–(5) hold for Y when X and S are replaced by X_0 and S_0 .

We have depicted the one-handle T attached to S in Figure 5.3, along with some of the objects defined above. We fix a regular neighborhood R_1 of $K \cap S$ in S , and observe that R_1 is a regular neighborhood of $f(D) \cap S = \Gamma$ in S . It therefore follows from Lemma 5.9 that R_1 is a planar surface with three boundary curves. Since S is identified with $\Sigma \times \{1\}$, we have $R_1 = R \times \{1\}$ for some $R \subset \Sigma$. We set $Y = (R \times I) \cup K \subset (\Sigma \times I) \cup K = X_0$.

By construction we have $\overline{X - Y} = (\overline{\Sigma - R}) \times I$ and $(\overline{X - Y}) \cap S_0 = (\overline{\Sigma - R}) \times \{0\}$. This implies Property (5) for Y .

To show that Y has Property (2), we first observe that $Y = (R \times I) \cup K = (R \times I) \cup T \cup \eta$, and that T and η are disjoint from $S_0 = \Sigma \times \{0\}$ (since their intersection with $\mathcal{C} = \Sigma \times I$ is contained in $S = \Sigma \times \{1\}$). Hence $Y \cap \partial X_0$ is the disjoint union of $Z_0 \doteq R \times \{0\}$ and $Z_1 \doteq (R_1 \cup T \cup \eta) \cap \partial X_0 \not\subset S_0$. The 2-manifold Z_0 is homeomorphic to R_1 , and is therefore a planar surface with three boundary curves.

To determine the topological type of Z_1 , we first consider the 2-manifold $N \doteq (R_1 \cup T) \cap \partial(\mathcal{C} \cup T)$. We may obtain N from R_1 by removing the interior of $T \cap R_1$, which is a union of two disjoint disks, and attaching the annulus $\overline{(\partial T) - (T \cap R_1)}$. Since R_1 is connected and $\bar{\chi}(R_1) = 1$, it follows that N is connected and that $\bar{\chi}(N) = 3$. The surface N contains

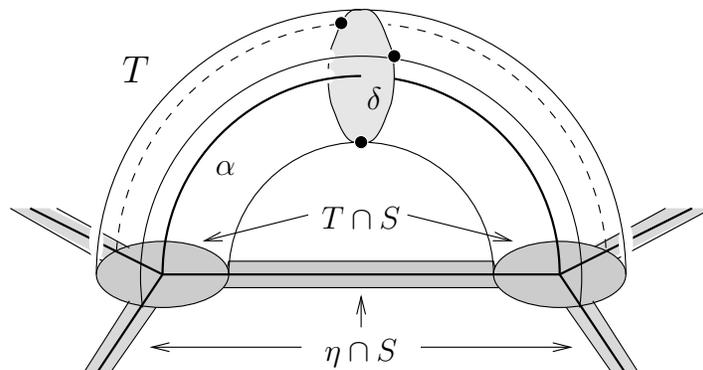


FIGURE 5.3. The shaded neighborhood is $K \cap S = (T \cap S) \cup (\eta \cap S)$. The graph $\Gamma \subset K \cap S$ is indicated in bold, as are the three points of $f(\partial D') \cap \partial \delta$.

the simple closed curves $f(\partial D')$ and $\partial \delta$, which by 5.11.2 meet transversally in exactly three points. In particular, their mod 2 homological intersection number in N is equal to 1. Hence $f(\partial D')$ does not separate the connected surface N . The 2-manifold Z_1 is obtained from N by removing the interior of the annulus $N \cap \eta$ and attaching the two components of $\overline{(\partial \eta) - (N \cap \eta)}$, which are disks. Since η is a regular neighborhood of $f(D')$ in $\overline{X - T}$, the annulus $N \cap \eta$ is a regular neighborhood of the non-separating curve $f(\partial D')$ in N . Hence Z_1 is connected, and so Z_0 and Z_1 are the components of $Y \cap \partial X_0$. Furthermore, we have $\partial Z_1 = \partial R_1$, and since R_1 is a planar surface with three boundary curves, Z_1 has three boundary curves. Since in addition we have $\bar{\chi}(Z_1) = \bar{\chi}(N) - 2 = 1$, the surface Z_1 must be planar.

This establishes Property (2) for Y .

We now turn to the verification of Properties (3) and (4). By construction the pair (Y, Z_0) is homotopy equivalent to $(R_1 \cup f(D), R_1)$. Hence it suffices to show that if $\star \in \text{int } R_1$ is a base point, then $\pi_1(R_1 \cup f(D), \star)$ is free of rank 2 and has an ordered basis (x, y) , such that the image of some geometric basis of $\pi_1(R_1, \star)$ under the inclusion homomorphism is either (x, y^2xy) or (x, y^2xy^{-1}) .

We fix an ordered basis (t, u_1, u_2) of the rank-3 free group $\pi_1(R_1 \cup \alpha, \star)$ having the properties stated in the conclusion of Lemma 5.9. In particular, if $\star \in \text{int } R_1$ is any base point, $\pi_1(R_1 \cup \alpha, \star)$ is free on the generators t, u_1 and u_2 . Furthermore, $R_1 \cup f(D)$ is obtained from $R_1 \cup \alpha$ by attaching a 2-cell, and the attaching map realizes either the conjugacy class of $t^2u_1tu_2$ or that of $t^2u_1t^{-1}u_2$ in $\pi_1(R_1 \cup \alpha, \star)$. Hence $\pi_1(R_1 \cup f(D), \star)$ is given by either the presentation

$$(5.11.3) \quad |t, u_1, u_2 : t^2u_1tu_2 = 1|,$$

in the theta case, or the presentation

$$(5.11.4) \quad |t, u_1, u_2 : t^2u_1t^{-1}u_2 = 1|,$$

in the eyeglass case.

Let \bar{t} and \bar{u}_i denote the respective images in $\pi_1(R_1 \cup f(D), \star)$ of the generators t and u_i of the free group on t, u_1 and u_2 . From the properties of the basis (t, u_1, u_2) stated in the conclusion of Lemma 5.9, it follows that u_1 and u_2 are the images of the elements of a geometric basis of $\pi_1(R_1, \star)$ under the inclusion homomorphism $\pi_1(R_1, \star) \rightarrow \pi_1(R_1 \cup f(D), \star)$. On the other hand, it is clear from the presentation (5.11.3) or (5.11.4) that $\pi_1(R_1 \cup f(D), \star)$ is free on the generators $x \doteq \bar{u}_1$ and $y \doteq \bar{t}^{-1}$. Furthermore, in the theta case, we have $\bar{u}_1 = x$ and $\bar{u}_2 = yx^{-1}y^2$; while in the eyeglass case, we have $\bar{u}_1 = x$ and $\bar{u}_2 = y^{-1}x^{-1}y^2$. Thus properties (3) and (4) of Y are established.

To prove that Y has Property (1), we first observe that by construction the frontier of Y is $\partial R \times I$. Hence each component A of the frontier has the form $c \times I$, where c is a component of ∂R . Thus A is an annulus having one boundary curve in the component $\Sigma \times \{0\}$ of ∂X_0 . The other boundary curve of A is contained in $\Sigma \times \{1\}$ and is therefore disjoint from $S_0 = \Sigma \times \{0\}$. In particular, A has its boundary curves in distinct components of ∂X_0 . To prove that A is essential, it therefore suffices to prove that it is π_1 -injective in X_0 . This in turn reduces to showing that A is π_1 -injective (4.1) in Y and in $\overline{X_0 - Y}$.

To prove π_1 -injectivity of A in Y , we observe that since the surface R is homeomorphic to a regular neighborhood of Γ in S , we have $\bar{\chi}(R) = 1$. Hence the component c of ∂R is π_1 -injective in R . This implies that $c \times \{0\}$ is π_1 -injective in $Z_0 = R \times \{0\}$. Since Z_0 is π_1 -injective in Y by Property (4), it follows that $c \times \{0\}$ is π_1 -injective in Y , and hence that A is π_1 -injective in Y .

To prove π_1 -injectivity of A in $\overline{X_0 - Y}$, we observe that by construction we have $\overline{X_0 - Y} = \overline{\Sigma - R} \times I \subset (\Sigma \times I) \cup K = X_0$. Since $A = c \times I$, it suffices to prove that the boundary component c of R is π_1 -injective in $\overline{\Sigma - R}$, or equivalently that no component of $\overline{\Sigma - R}$ is a disk. But this follows immediately from 5.11.1. Thus Property (1) of Y is established. \square

Remark 5.12. It follows from Property (3) and (4) of Y , as stated in the conclusion of Lemma 5.11, that the inclusion homomorphism $H_1(Z_0; \mathbb{Z}_2) \rightarrow H_1(Y; \mathbb{Z}_2)$ is an isomorphism.

Lemma 5.13. *Suppose that X is a trimonic manifold relative to S . Then ∂X has exactly one component $T \neq S$, whose genus is equal to that of S , and T is π_1 -injective in X .*

Proof. Let us fix a compact, connected 3-dimensional submanifold Y of X having Properties (1)–(5) stated in the conclusion of Lemma 5.11. Define Z_0 and Z_1 as in the statement of Property (2) of that lemma. According to Property (2), we have $Z_0 \subset S$, and Z_1 is contained in a component $T \neq S$ of ∂X . If Q is a component of $\overline{X - Y}$, then by Property (5) of Y , Q is a trivial I -bundle. By Property (1), each component of the frontier of Q is an essential annulus with one boundary component in Z_0 and one in Z_1 . Thus the components of $\partial_h Q$ may be labeled F_0 and F_1 in such a way that F_0 meets $Z_0 \subset S$ and F_1 meets $Z_1 \subset T$. Note that F_0 and F_1 are homeomorphic. Since this holds for each component of $\overline{X - Y}$, and since Z_0 and Z_1 are homeomorphic, it follows that S and T have the same Euler characteristic, and furthermore that they are the only components of ∂X .

It remains to show that T is π_1 -injective in X . According to Proposition 4.2, it suffices to show that the inclusion homomorphism $H_1(S; \mathbb{Z}_2) \rightarrow H_1(M; \mathbb{Z}_2)$ is surjective. For this

purpose we set $\mathcal{Q} = \overline{X - Y}$ and $\partial_0 \mathcal{Q} = \mathcal{Q} \cap S$, we let F denote the frontier of Y in X , and we consider the commutative diagram

$$\begin{array}{ccccccccc}
 H_1(F \cap S) & \longrightarrow & H_1(Z_0) \oplus H_1(\partial_0 \mathcal{Q}) & \longrightarrow & H_1(S) & \longrightarrow & H_0(F \cap S) & \longrightarrow & H_0(Z_0) \oplus H_0(\partial_0 \mathcal{Q}) \\
 \downarrow \alpha_1 & & \downarrow \beta_1 \oplus \gamma_1 & & \downarrow \delta & & \downarrow \alpha_0 & & \downarrow \beta_0 \oplus \gamma_0 \\
 H_1(F) & \longrightarrow & H_1(Y) \oplus H_1(\mathcal{Q}) & \longrightarrow & H_1(M) & \longrightarrow & H_0(F) & \longrightarrow & H_0(Y) \oplus H_0(\mathcal{Q}),
 \end{array}$$

where all homology groups are defined with \mathbb{Z}_2 -coefficients, the rows are segments of Mayer-Vietoris exact sequences, and the homomorphisms $\alpha_i, \beta_i, \gamma_i$ and δ are induced by inclusion. Since $Z_0 \subset S$ and $Z_1 \subset T$, each component of F is an annulus with exactly one boundary curve in S . Hence the maps α_0 and α_1 are isomorphisms.

If Q is any component of \mathcal{Q} , then Property (5) of Y , as stated in Lemma 5.11, implies that Q may be given the structure of a trivial I -bundle over a 2-manifold in such a way that $Q \cap \partial X$ is the horizontal boundary of Q . Since Q must contain at least one component of the frontier of Y , exactly one component of the horizontal boundary of Q lies in S . It follows that γ_0 and γ_1 are isomorphisms. The map β_0 is an isomorphism because Z_0 and Y are both connected, while β_1 is an isomorphism by Remark 5.12.

Since the α_i, β_i and γ_i are isomorphisms, it follows from the Five Lemma that δ is an isomorphism. In particular it is surjective, as required. \square

Lemma 5.14. *Suppose that X is a non-degenerate trimonic manifold relative to a component S of ∂X . Then there is no normal book of I -bundles \mathcal{W} with $|\mathcal{W}| = X$.*

Proof. Let us fix a compact, connected 3-dimensional submanifold Y of X having Properties (1)–(5) of the conclusion of Lemma 5.11. Define Z_0 as in the statement of Property (2) there.

Suppose that $X = |\mathcal{W}|$ for some normal book of I -bundles \mathcal{W} . Then Properties (1) and (2) of Y , as stated in 5.11, give hypotheses (1) and (2) of Lemma 4.9. According to Remark 5.12, the inclusion homomorphism $H_1(Z_0; \mathbb{Z}_2) \rightarrow H_1(Y; \mathbb{Z}_2)$ is an isomorphism. This is Hypothesis (3) of Lemma 4.9.

Let us fix a base point $\star \in Z_0$. According to the properties (2) (3) and (4) of Y stated in the conclusion of Lemma 5.11, Z_0 is a planar surface with three boundary curves, and the group $\pi_1(Y)$ is free of rank 2, and there exists a pair of generators (x, y) of $\pi_1(Y, \star)$, such that either the pair $(x, yx^{-1}y^2)$ or the pair $(x, y^{-1}x^{-1}y^2)$ is the image of some geometric basis of $\pi_1(Z_0, \star)$ under the inclusion homomorphism $\pi_1(Z_0, \star) \rightarrow \pi_1(Y, \star)$. Since neither of the pairs $(x, yx^{-1}y^2)$ or $(x, y^{-1}x^{-1}y^2)$ generates the free group on x and y , the inclusion homomorphism $\pi_1(Z_0, \star) \rightarrow \pi_1(Y, \star)$ is not surjective. Hence Lemma 4.9 implies that there is a solid torus $L \subset Y$ such that $A \doteq L \cap Z_0$ is an annulus which is homotopically non-trivial in $|\mathcal{W}|$, and the inclusion homomorphism $H_1(A; \mathbb{Z}) \rightarrow H_1(L, \mathbb{Z})$ is not surjective.

Let c denote a core curve of the annulus A . Since c is in particular homotopically non-trivial in Z_0 , which is a planar surface with three boundary curves, c is parallel to one of the boundary curves of Z_0 . In view of the definition of a geometric basis, it follows that the conjugacy class in $\pi_1(Y, \star)$ defined by a suitably chosen orientation of c is represented by one

of the elements x , $yx^{-1}y^2$, $x^{-1}yx^{-1}y^2$, $y^{-1}x^{-1}y^2$ or $x^{-1}y^{-1}x^{-1}y^2$. On the other hand, since the inclusion homomorphism $H_1(A; \mathbb{Z}) \rightarrow H_1(L, \mathbb{Z})$ is not surjective, a representative of a conjugacy class in $\pi_1(Y, \star)$ defined by c must be an n -th power in $\pi_1(Y, \star)$ for some $n \neq \pm 1$. But none of the elements x , $yx^{-1}y^2$, $x^{-1}yx^{-1}y^2$, $y^{-1}x^{-1}y^2$ or $x^{-1}y^{-1}x^{-1}y^2$ is a proper power in the free group on x and y . This contradiction completes the proof. \square

6. WITH A (1, 1, 1) HEXAGON.

In this section and the next we will be working with hyperbolic 3-manifolds with totally geodesic boundary. As is customary in low-dimensional topology, we shall implicitly carry over the PL results proved in Sections 5 and 4 to the smooth category. For example, to say that a smooth manifold X is a trimonic manifold relative to a component S of ∂X means that the pair (X, S) is topologically homeomorphic to a PL pair (X', S') such that X' is a trimonic manifold relative to S' . Likewise, to say that a smooth manifold X has the form $|\mathcal{W}|$ for some normal book of I -bundles \mathcal{W} means that X is topologically homeomorphic to a PL manifold which has the form $|\mathcal{W}|$ for some normal book of I -bundles \mathcal{W} .

Here we will describe some topological consequences of the presence of a (1, 1, 1) hexagon in \widetilde{N} , when the shortest return path of N is not too long. The main result of the section, Proposition 6.8, asserts the existence of a trimonic manifold in N under these circumstances. Below we prove a series of separate lemmas concerning the geometry of (1, 1, 1) hexagons, from which the proposition follows quickly. The first follows from [17, Lemma 3.2], but for self-containedness we prove it here.

Lemma 6.1. *Let N be a compact hyperbolic 3-manifold with totally geodesic boundary, and $\lambda, \lambda' \subset \widetilde{N}$ short cuts with length ℓ_1 . Then $\lambda = \lambda'$ or $\lambda \cap \lambda' = \emptyset$.*

Proof. Suppose λ intersects λ' at a single point $y \in \lambda$. Let Π be the component of $\partial \widetilde{N}$ containing the endpoint x of λ closest to y , and let Π' be the component containing the endpoint x' of λ' closest to y . The subarcs $[x, y]$ and $[x', y]$ of λ and λ' , respectively, each have length at most $\ell_1/2$. They meet at y at an angle properly less than π , since $\lambda \neq \lambda'$ and each is geodesic. But then Π and Π' are components of $\partial \widetilde{N}$ at distance less than ℓ_1 , so they are equal. But since λ and λ' are geodesic arcs, each perpendicular to Π , in \widetilde{N} they are disjoint or equal, a contradiction. \square

Remark 6.2. Suppose N is a compact hyperbolic 3-manifold with totally geodesic boundary and $g : \widetilde{N} \rightarrow \widetilde{N}$ is a covering transformation. If λ is a short cut with length ℓ_1 , then according to Lemma 6.1, either $g(\lambda) = \lambda$ or $g(\lambda) \cap \lambda = \emptyset$. But in the first case, g would fix a point in λ , a contradiction. It follows that every short cut of length ℓ_1 is embedded in N by the universal cover.

Lemma 6.3. *Let N be a compact hyperbolic 3-manifold with ∂N connected, totally geodesic, and of genus 2, and suppose ℓ_1 satisfies the bound below.*

$$\cosh \ell_1 < \frac{\cos(2\pi/9)}{2 \cos(2\pi/9) - 1} = 1.4396\dots$$

Then $\ell_2 > \ell_1$; ie, the shortest return path in N is unique.

Proof. Suppose $\ell_2 = \ell_1$. Applying the right-angled hexagon rule as in the proof of Lemma 2.6, we find that each of d_{11} , d_{12} , and d_{22} is at least $2R$, for R the function of ℓ_1 defined there. Then a disk of radius R is embedded around each of the feet of λ_1 and λ_2 , so that none of these disks overlap. Boröczky's bound on the radius of four disks of equal area embedded without overlapping on a surface of genus 2 is the quantity R'' defined in Lemma 2.9. Setting $R = R''$ and solving for ℓ_1 yields the quantity of the bound above. \square

Lemma 6.4. *Let N be a compact hyperbolic 3-manifold with totally geodesic boundary. Suppose C and C' are distinct $(1, 1, 1)$ hexagons in \widetilde{N} with exterior edges on the same component of $\partial\widetilde{N}$. Then $C \cap C'$ is either empty or a single interior edge.*

Proof. Let e and e' be exterior edges of C and C' on the same component of $\partial\widetilde{N}$. The endpoints of e and e' are feet of lifts of the shortest return path; any such pair has distance at least d_{11} . Since C and C' are distinct, so are e and e' ; if they share an endpoint then $C \cap C'$ consists of an interior edge.

Otherwise, e is a geodesic arc of length d_{11} connecting its endpoints a and b and e' an arc of the same length connecting its endpoints a' and b' , with $d(a, a')$, $d(a, b')$, $d(b, a')$, and $d(b, b')$ all at least d_{11} . Some hyperbolic trigonometry shows that any point at distance at least d_{11} from a and b satisfies $\cosh \ell \geq \cosh d_{11} / \cosh(d_{11}/2)$, where ℓ is its distance from e . Twice this distance is larger than d_{11} ; thus e' does not cross e , and so $C \cap C' = \emptyset$. \square

Remark 6.5. Suppose N is a compact hyperbolic 3-manifold with totally geodesic boundary and $g : \widetilde{N} \rightarrow \widetilde{N}$ is a covering transformation. If C is a $(1, 1, 1)$ hexagon, and an external edge of $g(C)$ intersects an external edge of C , then by Lemma 6.4, either $g(C) = C$ or $g(C) \cap C$ is a single internal edge. In the former case, g fixes a point in C , a contradiction. It follows that the union of the interiors of external edges of C projects homeomorphically to N .

Lemma 6.6. *Let N be a compact hyperbolic 3-manifold with totally geodesic boundary, and suppose $\cosh \ell_1 \leq 1.215$. If C is a $(1, 1, 1)$ hexagon in \widetilde{N} and λ is a short cut of length ℓ_1 , then λ is an internal edge of C or $\lambda \cap C = \emptyset$.*

Proof. Suppose λ and C are as in the lemma, and $\lambda \cap C$ is nonempty. Let Π_1 and Π_2 be the components of $\partial\widetilde{N}$ containing the endpoints of λ , and let Π be the geodesic plane containing C . Suppose first that $\lambda \subset \Pi$. If λ is not contained in C , let x be an endpoint of $\lambda \cap C$ contained in the interior of λ . Then x is a point of intersection between λ and an internal edge of C , since the external edges of C are contained in $\partial\widetilde{N}$ and λ is properly embedded. But each internal edge of C is a short cut with length ℓ_1 , so this contradicts Lemma 6.1. It follows that if $\lambda \subset \Pi$, $\lambda \subset C$. But C intersects only three components of $\partial\widetilde{N}$ — the components joined by its internal edges. Hence since C intersects Π_1 and Π_2 , λ is an internal edge of C .

Now suppose λ is not contained in Π . Then λ intersects C transversely in a single point x . There is a component Π' of $\partial\widetilde{N}$ such that $\Pi' \cap \Pi$ contains an external edge of C and

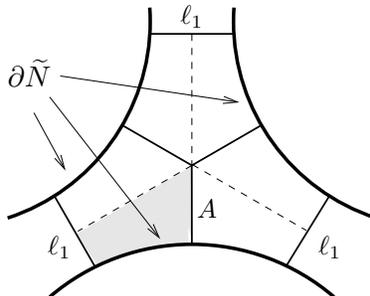


FIGURE 6.1. A $(1, 1, 1)$ hexagon in \tilde{N} , with a fundamental domain for its symmetry group shaded.

x is distance at most A from Π' . (Recall that A , defined in (3.1.1), is a side length of a fundamental domain for the symmetry group of C ; see Figure 6.1.) Then the distance from Π' to each of Π_1 and Π_2 is less than $A + \ell_1$. By Lemma 3.8, ℓ_2 is at least twice A and ℓ_1 ; hence the distance from Π' to each of Π_1 and Π_2 is ℓ_1 , since it is less than ℓ_2 . By Lemma 2.3, there is a $(1, 1, 1)$ hexagon C' with λ as an internal edge and external edges in Π' , Π_1 , and Π_2 . But then $C' \cap C$ contains $\lambda \cap C$, contradicting Lemma 6.4. \square

Lemma 6.7. *Let N be a compact hyperbolic 3-manifold with totally geodesic boundary, and suppose $\cosh \ell_1 \leq 1.215$. If C and C' are distinct $(1, 1, 1)$ hexagons in \tilde{N} , then $C \cap C'$ is empty or a single internal edge of each.*

Proof. Suppose C and C' are distinct $(1, 1, 1)$ hexagons in \tilde{N} , and $C \cap C' \neq \emptyset$. By Lemma 6.4, the lemma holds if there is a component of $\partial\tilde{N}$ containing external edges of both C and C' , thus we may assume that this is not the case. It follows that no external edge of C contains a point of $C \cap C'$ and vice-versa, since by Lemma 2.4, $C \cap \partial\tilde{N}$ is precisely the union of its external edges. Let Π be the geodesic hyperplane containing C . If $C' \subset \Pi$, then $C \cap C'$ is a two-dimensional subpolyhedron of C , and each vertex of $C \cap C'$ is an intersection point between internal edges, by the above. But these are all short cuts of length ℓ_1 , contradicting Lemma 6.1.

If C' is not contained in Π , it intersects C transversely in a geodesic arc, whose endpoints are points of intersection of internal edges of one with the other. But such intersections violate Lemma 6.6, since the internal edges of each are short cuts of length ℓ_1 , a contradiction. \square

Proposition 6.8. *Let N be a compact, orientable hyperbolic 3-manifold with ∂N connected, totally geodesic, and of genus 2. Suppose that $\cosh \ell_1 \leq 1.215$, and there is a $(1, 1, 1)$ hexagon in \tilde{N} . There is a submanifold $X \subset N$ with $\partial N \subset X$, such that X is a trimonic manifold relative to ∂N .*

Proof. Let N be as in the hypotheses, and fix a $(1, 1, 1)$ hexagon $C \subset \tilde{N}$. Let $f : C \rightarrow N$ be the restriction of the universal covering, and let X be a regular neighborhood of $\partial N \cup f(C)$. Since $\cosh \ell_1 \leq 1.215$, Lemma 6.3 implies that N has a unique shortest return path α , hence each internal edge of C projects to α by f . The preimage of α is a union of short cuts with length ℓ_1 , hence Lemma 6.6 implies Property (1) of Definition 5.7. Remark 6.2 now

immediately implies Property (2) of the definition. Property (3) follows from Remark 6.5 and Lemma 6.7. Properties (4) and (5) follow from Lemma 2.4, and (6) holds by construction. \square

Proposition 6.9. *Let N be a compact, orientable hyperbolic 3-manifold with ∂N connected, totally geodesic, and of genus 2, such that $\cosh \ell_1 \leq 1.215$ and there is a $(1, 1, 1)$ hexagon in \widetilde{N} . The trimonic submanifold $X \subset N$ supplied by Proposition 6.8 is non-degenerate.*

Proof. Let N satisfy the hypotheses, and as in the proof of Proposition 6.8 let $C \subset \widetilde{N}$ be a $(1, 1, 1)$ hexagon, $f : C \rightarrow N$ the restriction of the universal cover, and $\alpha \subset N$ the shortest return path. Below we will borrow wholesale the constructions and notation from the proof of Lemma 5.9, with C here in the role of D there and ∂N in the role of S .

Recall that the internal edges a_i and external edges b_i of C , $i \in \{0, 1, 2\}$, are enumerated so that a_i shares a vertex with b_i and b_{i-1} for each i , and ∂C is oriented so that $\phi_0 \doteq f|_{a_0}$ induces the same orientation on α as $\phi_1 \doteq f|_{a_1}$. Then α is given the orientation induced by ϕ_0 and ϕ_1 , with initial and terminal vertices V and W , respectively. For each $i \in \{0, 1, 2\}$, the edge $\beta_i = f(b_i)$ of $\Gamma = \Gamma_f$ is given the induced orientation from b_i .

Cases (I) and (II) in the proof of Lemma 5.9 are distinguished according to whether $\phi_2 \doteq f|_{a_2}$ is orientation preserving or reversing, respectively. In Case (I), $\Gamma = \overline{\beta_0 \cup \beta_1 \cup \beta_2}$ is a theta graph, and in Case (II) it is an eyeglass. (Recall that $\beta_i = f(\text{int } b_i)$, $i \in \{0, 1, 2\}$.)

For each $i \in \{0, 1, 2\}$, let Π_i be the component of $\partial \widetilde{N}$ containing b_i . Suppose that U is a component of $\partial N - \Gamma$ which is homeomorphic to an open disk, and let $U_0 \subset \Pi_0$ be a component of the preimage of U under the universal covering map. Then U_0 is projected homeomorphically to U , and \overline{U}_0 is a compact polygon in Π_0 with edges projecting to the β_i , hence covering translates of the b_i . Since the b_i are geodesic arcs, \overline{U}_0 has at least three edges. Since each vertex v of \overline{U}_0 projects to V or W , and U_0 projects homeomorphically, \overline{U}_0 has at most 6 vertices. Hence \overline{U}_0 is an n -gon for n between 3 and 6.

Suppose edges b and b' incident to a single vertex v of \overline{U}_0 were identified in Γ . The covering transformation f taking b to b' is orientation preserving on \widetilde{N} , so it preserves the boundary orientation on Π_0 . Give U_0 this orientation, and orient b and b' as arcs in the boundary of U_0 . Since $f(U_0)$ does not intersect U_0 , $f(\overline{U}_0) \cap \overline{U}_0 = b' = f(b)$. Since $f(b)$ has the boundary orientation from $f(U_0)$, its orientation is opposite that of b' . But then since v is the initial vertex of (say) b and the terminal vertex of b' , $f(v) = v$, a contradiction. Hence:

6.9.1. *If v is a vertex of \overline{U}_0 , the edges incident to v project to distinct edges of Γ .*

Now suppose \overline{U}_0 is a triangle. Then two of its vertices are identified in N , since Γ has only two vertices. The edge joining these vertices projects to an edge joining V to V or W to W , so in this case Γ is an eyeglass graph. On the other hand, the final vertex of \overline{U}_0 is not identified with the other two by 6.9.1, since an eyeglass graph has only one edge joining each vertex to itself. But then the two edges emanating from the final vertex yield distinct edges of Γ , joining V to W , which does not occur in an eyeglass graph. This is a contradiction.

When \overline{U}_0 is a pentagon, three of its vertices are identified to V (say) in Γ . Thus two of these are adjacent in \overline{U}_0 . The edge joining the adjacent vertices of \overline{U}_0 identified to V joins it to

itself in Γ , hence Γ is an eyeglass graph. The third vertex identified to V is not adjacent to either of the others, by 6.9.1, since Γ has only one edge joining V to itself. Then the edges adjacent to this vertex project to distinct β_i joining V to W , a contradiction.

To rule out the possibility that \bar{U}_0 is a quadrilateral or hexagon requires counting angles. Recall that $S = \partial N$ is oriented so that \mathcal{T}_V is a positive oriented triod (see 5.1). Here $\mathcal{T}_V = (\sigma_0, \sigma_1, \sigma_2)$ in Case (I) and $\mathcal{T}_V = (\sigma_0, \sigma_1, \rho_2)$ in Case (II), where for each $i \in \{0, 1, 2\}$, ρ_i is the closure of an initial segment of β_i and σ_i is the closure of a terminal segment of β_{i-1} . Define θ_1 to be the angle measure from σ_0 to σ_1 at V , in the direction prescribed by the orientation on ∂N . In Case (I), we take θ_2 to be the angle from σ_0 to σ_2 in the orientation direction, and in Case (II) we let θ_2 be the angle from σ_0 to ρ_2 . Then $0 < \theta_1 < \theta_2 < 2\pi$.

Recall that C is a totally geodesic hexagon in \tilde{N} by Lemma 2.3, and the covering projection f immerses C in N isometrically. Then appealing to Figure 5.2, we note that θ_1 is the angle from ρ_1 to ρ_0 at W , measured in the orientation direction. This is because the homeomorphism $\psi : \delta_V \rightarrow \delta_W$ defined in the proof of Lemma 5.9 is orientation reversing. Similarly, in Case (I) θ_2 is the angle from ρ_2 to ρ_0 at W in the orientation direction, and in Case (II) it is the angle from σ_2 to ρ_0 .

If \bar{U}_0 is a quadrilateral, then two of its edges are sent by f to the same edge of Γ . These are opposite, by 6.9.1; hence the image in Γ of each of the remaining edges joins a vertex to itself. Then Γ is an eyeglass graph, the pair of opposite edges identified by f project to β_0 , and the other two edges project to β_1 and β_2 . Abusing notation slightly, we label the edges projecting to β_0 by b_0 and b'_0 , and similarly label the edge projecting to β_i by b_i , $i = 1, 2$. (These are covering translates of the corresponding edges of C .) We give each edge of \bar{U}_0 an orientation matching that of its correspondent in Γ .

Orient U_0 so that $f|U_0$ preserves orientation. We may assume, by switching the labels of b_0 and b'_0 and/or replacing C with a covering translate if necessary, that the prescribed orientation on $b_0 \subset C$ matches the boundary orientation which it inherits from \bar{U}_0 . The terminal endpoint of b_0 is sent to V , and a terminal segment is sent to the interior of σ_1 by definition. Since the orientation on ∂N is chosen so that $(\sigma_0, \sigma_1, \rho_2)$ is a positively oriented triod, the orientation σ_1 inherits from β_0 matches the boundary orientation from the component of $\delta_V - (\sigma_0 \cup \sigma_1 \cup \sigma_2)$ bounded by it and ρ_2 . It follows that the terminal endpoint of b_0 is the initial endpoint of b_2 , and the dihedral angle of \bar{U}_0 at this vertex is $\theta_2 - \theta_1$.

Since the orientation on b'_0 is opposite that induced by \bar{U}_0 , the dihedral angle of \bar{U}_0 at the common terminal endpoint of b'_0 and b_2 is θ_1 , the dihedral angle in δ_V between σ_0 and σ_1 . Arguing as above, we find that the dihedral angle of \bar{U}_0 at the initial endpoint of b_0 , which is the terminal endpoint of b_1 , is $2\pi - \theta_2$. (Recall that the homeomorphism $\psi : \delta_V \rightarrow \delta_W$ visible in Figure 5.2 as projection upward is orientation reversing.) The dihedral angle at the final vertex is then θ_1 . Thus the sum of the dihedral angles is $2\pi + \theta_1$, contradicting the well known fact that the dihedral angle sum of a hyperbolic quadrilateral is less than 2π .

Now suppose \bar{U}_0 is a hexagon. Then the projection of \bar{U}_0 contains each of δ_V and δ_W , since \bar{U}_0 has 6 vertices and each of V and W has valence 3. Thus the sum of the dihedral angles

around vertices of \bar{U}_0 is 4π . But a hyperbolic hexagon has dihedral angle sum less than 4π , a contradiction. It follows that no component of $\partial N - \Gamma$ is homeomorphic to an open disk. Since $f : C \rightarrow N$ is a defining hexagon (in the sense of Definition 5.10) for the submanifold X defined in Proposition 6.8, and since $\Gamma = \Gamma_f$, the trimonic manifold X is non-degenerate. \square

7. PUTTING IT ALL TOGETHER

In this section we prove the theorems stated in the introduction. Here we make much use of terminology and results from [6]. Of particular importance is the term “ (g, h) -small”, from Definition 1.2 there:

Definition 7.1. Let g be an integer ≥ 2 , let h be a positive real number, and let M be an orientable, irreducible 3-manifold. We shall say that M is (g, h) -small if every connected closed incompressible surface in M has genus at least h and every separating connected closed incompressible surface in M has genus at least g .

Lemma 7.2. Let \mathcal{W} be a normal book of I -bundles, set $W = |\mathcal{W}|$, and suppose that ∂W is connected and has genus 2. Then $\text{Hg}(W) = 3$.

Proof. Since $\bar{\chi}(W) = \frac{1}{2}\bar{\chi}(\partial W) = 1$, there is a unique page P of \mathcal{W} which has negative Euler characteristic; and furthermore, $\bar{\chi}(P) = 1$. Since $P \cap \partial W = \partial_h P$ is a π_1 -injective subsurface of ∂W with Euler characteristic -2 , its complement in ∂W is a disjoint union of annuli. Thus if B is a component of $\overline{W - P}$, then ∂B is a union of annuli in $\overline{\partial W - P}$ and vertical annuli in the frontier of P , and is therefore a torus. Since the frontier of B in W consists of essential annuli, B is π_1 -injective in W , and since W is simple, B is $(2, 2)$ -small. It now follows from [6, Proposition 2.3] that each component of $\overline{W - P}$ is a solid torus.

By definition P is an I -bundle over a surface T . We let $p : P \rightarrow T$ denote the bundle map. If C is a closed disk contained in the interior of T , then $\mathcal{H} = p^{-1}(C)$ is a 1-handle in P joining $\partial_h P \subset \partial W$ to itself. Let δ_0 be an arc embedded in $\overline{T - C}$, so that $\partial\delta_0 = \delta_0 \cap \partial(\overline{T - C}) \subset \partial C$ and no arc of ∂C bounds a disk embedded in $\overline{T - C}$ together with δ_0 . The existence of such an arc can be established using the fact that $\chi(\overline{T - C}) = -2$ and standard Morse theory arguments. If D_0 is a regular neighborhood of δ_0 in $\overline{T - C}$, then $\overline{T - C - D_0}$ is a possibly disconnected surface with Euler characteristic -1 and no component which is a disk. Let T_0 be the component with $\chi(T_0) = -1$, and let β be a component of ∂T_0 containing an arc of the frontier in $\overline{T - C}$ of D_0 . There is an arc δ_1 embedded in T_0 , so that $\partial\delta_1 = \delta_1 \cap \partial T_0 \subset \beta$ and no arc of β bounds a disk in T_0 together with δ_1 . This follows as above, and we may further assume, after sliding $\partial\delta_1$ along β if necessary, that $\partial\delta_1$ does not intersect the frontier of D_0 . Let D_1 be a regular neighborhood in T_0 of δ_1 which does not intersect D_0 , and let $D = D_0 \sqcup D_1 \subset \overline{T - C}$.

By construction, $\overline{T - C - D}$ is a disjoint union of surfaces with Euler characteristic 0; that is, annuli and Möbius bands. Since T is connected, each component of $\overline{T - C - D}$ has at least one component of its boundary containing arcs of the frontier of D . Thus if α is a component of ∂T , the component T' of $\overline{T - C - D}$ containing α is an annulus, and α is

the unique component of $\partial T'$ contained in ∂T . For $i = 0$ or 1 , let $\mathcal{D}_i = p^{-1}(D_i) \subset P$, and let $\mathcal{D} = \mathcal{D}_0 \sqcup \mathcal{D}_1$. Each of \mathcal{D}_0 and \mathcal{D}_1 is an I -bundle over a disk, hence a ball, and each component of $\overline{P - \mathcal{H} - \mathcal{D}}$ is an I -bundle over an annulus or Möbius band, hence is a solid torus.

Let B be a component of $\overline{W - P}$. Then the component of $\overline{W - \mathcal{H} - \mathcal{D}}$ containing B is the union of B with a collection of solid torus components of $\overline{P - \mathcal{H} - \mathcal{D}}$. If B_1 is such a component, by the above B_1 is an I -bundle over an annulus component of $\overline{T - C - \overline{D}}$ with a unique boundary component $\alpha \subset \partial T$. Let $A = p^{-1}(\alpha) = B \cap B_1$, a vertical annulus in the frontier of P . Since A is a degree one annulus in ∂B_1 it follows that $B \cup B_1$ is a solid torus. It follows that each component of $\overline{W - \mathcal{H} - \mathcal{D}}$ is a solid torus, so $\overline{W - \mathcal{H}}$ is obtained from a collection of solid tori by adding \mathcal{D}_0 and \mathcal{D}_1 .

Let \mathcal{N}_0 be a regular neighborhood of ∂W such that $\mathcal{N}_0 \cap P$ is a regular neighborhood of $\partial_h P$ in T with horizontal frontier. Then $\mathcal{N}_0 \cap \mathcal{H}$ is a disjoint union of two solid cylinders. Let $V_0 = \mathcal{N}_0 \cup \mathcal{H}$. V_0 is a compression body in W with frontier a surface S of genus 3. Our description above shows that $V_1 = \overline{W - V_0}$ is the union of a collection of solid tori with $\mathcal{D}_0 \cap V_1$ and $\mathcal{D}_1 \cap V_1$. Each of these has the structure of a 1-handle, since it is a ball and its intersection with its complement in V_1 consists of two disks. Hence V_1 is a handlebody and S is a Heegaard surface for W . \square

Lemma 7.3. *Let N be a compact, irreducible, orientable 3-manifold such that ∂N is a connected non-empty surface of genus > 1 . Suppose that N contains no incompressible torus. Let Y be a connected, 3-dimensional submanifold of N such that $\partial N \subset Y$, and such that every component of the frontier of Y in N is a torus. Then $\text{Hg}(N) \leq \text{Hg}(Y)$.*

Proof. We argue by induction on the number $m \geq 0$ of components of the frontier of Y in N . If $m = 0$ we have $N = Y$ and the assertion is trivial. Now suppose that $m > 0$, and that for every connected, 3-dimensional submanifold $Z \supset \partial N$ of N whose frontier consists of strictly fewer than m tori, we have $\text{Hg}(N) \leq \text{Hg}(Z)$. Let T denote the frontier of Y . Since $T \neq \emptyset$ and each component of T is a torus, the hypothesis implies that T is compressible in N . Hence there is a disk $D \subset N$ with $D \cap T = \partial D$, such that ∂D is homotopically non-trivial in the component T_0 of T containing it. Since T is the frontier of Y in N , we must have either $D \cap Y = \partial D$ or $D \subset Y$.

In the case where $D \cap Y = \partial D$, we let E denote a regular neighborhood of D in $\overline{N - Y}$. The component of $\partial(Y \cup E)$ which meets T_0 is a sphere, which by irreducibility bounds a ball $B \subset N$. Since $\partial N \subset Y$ we cannot have $B \supset Y \cup E$; hence we must have $B \cap (Y \cup E) = \partial B$. It follows that $J \doteq B \cup E$ is a solid torus with $J \cap Y = \partial J = T_0$. A Heegaard surface for Y is also a Heegaard surface for $Y \cup J$ (cf. [6, Lemma 2.1]), so that $\text{Hg}(Y \cup J) \leq \text{Hg}(Y)$; but the induction hypothesis implies that $\text{Hg}(N) \leq \text{Hg}(Y \cup J)$. Hence $\text{Hg}(N) \leq \text{Hg}(Y)$.

In the case where $D \subset Y$, we let E denote a regular neighborhood of D in Y . The component of the boundary of $\overline{Y - E}$ which meets T_0 is a sphere, and is isotopic by a small ambient isotopy of N to a sphere $S \subset \text{int } Y$ which separates T_0 from ∂N in Y . Given a Heegaard surface F for Y , Haken's lemma (cf. [5, Lemma 1.1]) asserts that there is a 2-manifold

S' obtained from S by 1-surgeries and isotopies such that each component of S' meets F transversally in a single simple closed curve. We will take F to be a minimal-genus Heegaard surface for Y . Since S separates T_0 from ∂N , some component S_0 of S' must separate T_0 from ∂N . Let C denote the simple closed curve $S_0 \cap F$.

Since N is irreducible, S_0 bounds a ball $B_0 \subset \text{int } N$. We have $T_0 \subset \text{int } B_0$. Hence the number k of components of T contained in B_0 is strictly positive. If we set $Z = Y \cup B_0$ then $Y \supset \partial N$, and the frontier of Z consists of m' tori, where $m' = m - k < m$. Hence the induction hypothesis implies that $\text{Hg}(N) \leq \text{Hg}(Z)$. We will complete the proof in this case by showing that $\text{Hg}(Z) \leq \text{Hg}(Y)$.

Let H_1 and H_2 denote the closures of the components of $Y - F$. Then $D_i = S_0 \cap H_i$ is a disk with $\partial D_i = C$ for $i = 1, 2$. Set $Z' = \overline{Z - B_0} = \overline{Y - (Y \cap B_0)}$, for $i = 1, 2$ set $H'_i = H_i \cap Z'$, and let $F' = F \cap Z' = H'_1 \cap H'_2$. Then H'_i is a submanifold of H_i whose frontier is D_i . But H_i is a compression body with $\partial_+ H_i = F$, and D_i is a properly embedded disk in H_i whose boundary lies in F . Hence H'_i is a compression body with $\partial_+ H'_i = F' \cup D_i$ (cf. [3, Corollary B.3]), and $\text{genus}(F') \leq \text{genus}(F) = \text{Hg}(Y)$.

If we choose a properly embedded disk Δ in B_0 with $\partial \Delta = C$, we may write $B_0 = B_1 \cup B_2$, where B_i is a ball containing D_i for $i = 1, 2$, and $B_1 \cap B_2 = \Delta$. It now follows that $H_i^* \cdot H'_i \cup B_i$ is a compression body with $\partial_+ H_i^* = F^* \doteq F' \cup \Delta$, and $Z = H_1^* \cup H_2^*$. Hence F^* is a Heegaard surface for Z , and therefore $\text{Hg}(Z) \leq \text{genus}(F^*) = \text{genus}(F') \leq \text{Hg}(Y)$, as required. \square

Theorem 7.4. *Let N be a compact, orientable hyperbolic 3-manifold with ∂N a connected totally geodesic surface of genus 2. If $\cosh \ell_1 \leq 1.215$ and there is a $(1, 1, 1)$ hexagon in \widetilde{N} , then $\text{Hg}(N) \leq 4$ or $\text{vol}(N) > 7.32$.*

Proof. Let N satisfy the hypotheses of the theorem, and let $X \subset N$ be the codimension-0 submanifold supplied by Proposition 6.8, which is a trimonic manifold with respect to ∂N , nondegenerate by Proposition 6.9. Let T be the frontier of X in N . By Lemma 5.13, T is a surface of genus 2 which is π_1 -injective in X .

Let $V = \overline{N - X}$. Then V is a compact, connected, irreducible 3-dimensional submanifold of N with $\partial V = T$. Therefore $\bar{\chi}(V) = 1$. By Lemma 5.11, X has Heegaard genus equal to 3.

We first consider the case in which $V = |\mathcal{W}|$ for some normal book of I -bundles \mathcal{W} . By Lemma 7.2, we have $\text{Hg}(V) = 3$. Amalgamating the Heegaard splittings of V and X , each of genus 3, across T yields a Heegaard splitting of N with genus 4 (cf. [20], Remark 2.7 and the definition above it).

Next we consider the case in which V is boundary-irreducible and V does not have the form $|\mathcal{W}|$ for any normal book of I -bundles \mathcal{W} . Since V is boundary-irreducible and T is π_1 -injective in X , the surface T is incompressible in N . Hence V and X are simple. By Lemma 5.14, X is not homeomorphic to $|\mathcal{W}|$ for any book of I -bundles \mathcal{W} . Since V is also not homeomorphic to $|\mathcal{W}|$ for any book of I -bundles \mathcal{W} , it follows from Proposition 4.7 that $\chi(\overline{X - \Sigma_X}) < 0$ and $\chi(\overline{V - \Sigma_V}) < 0$, where Σ_V and Σ_X denote the characteristic submanifolds of V and X relative to their boundaries. According to [6, Definition 1.1],

kish(V) (or kish(X)) is the union of all components of $\overline{V - \Sigma_V}$ (or respectively $\overline{X - \Sigma_X}$) having negative Euler characteristic. We therefore have $\text{kish } V \neq \emptyset$ and $\text{kish } X \neq \emptyset$, so that $\bar{\chi}(\overline{X - \Sigma_X}) \geq 1$ and $\bar{\chi}(\overline{V - \Sigma_V}) \geq 1$. Hence $\bar{\chi}(\text{kish } N \setminus T) = \bar{\chi}(\text{kish}(X)) + \bar{\chi}(\text{kish}(V)) \geq 2$, and by [2, Theorem 9.1], the volume of N is greater than 7.32.

There remains the case in which V is boundary-reducible. Let D be an essential properly embedded disk in V , and let E denote a regular neighborhood of D relative to V . Set $W = \overline{V - E}$ and $Y = X \cup E$. Then $N = Y \cup W$ and $Y \cap W = \partial W$. Furthermore, ∂W has at most two components, and each component of ∂W is a torus. A genus-3 Heegaard surface F for X is also a Heegaard surface for Y (cf. [6, Lemma 2.1]), and so $\text{Hg}(Y) \leq 3$. On the other hand, since N is a compact hyperbolic 3-manifold with totally geodesic boundary, it is irreducible and contains no incompressible torus. It therefore follows from Lemma 7.3 that $\text{Hg}(N) \leq \text{Hg}(Y) \leq 3$. □

Theorem 1.1. *Let N be a compact, orientable hyperbolic 3-manifold with ∂N a connected totally geodesic surface of genus 2. If $\text{Hg}(N) \geq 5$, then N has volume greater than 6.89.*

Proof. Let N satisfy the hypotheses of the theorem, and let ℓ_1 be the length of the shortest return path of N . If $\cosh \ell_1 \geq 1.215$, then by Proposition 3.7, N has volume greater than 6.89. If \tilde{N} contains no $(1, 1, 1)$ hexagon, then by Proposition 3.9, $\ell_1 \geq 1.215$, and Proposition 3.7 again gives the desired volume bound. We thus suppose that N has a $(1, 1, 1)$ hexagon and $\cosh \ell_1 < 1.215$. But in this case Theorem 7.4 gives a better volume bound of 7.32, since by hypothesis $\text{Hg}(N) \geq 5$. □

Theorem 1.4. *Suppose that M is a closed, simple 3-manifold which contains a connected closed incompressible surface of genus 2 or 3, and that $\text{Hg}(M) \geq 8$. Then M contains a connected closed incompressible surface S of genus at most 4, such that either $\bar{\chi}(\text{kish}(M \setminus S)) \geq 2$, or S is separating and $M \setminus S$ has an acylindrical component N with $\text{Hg}(N) \geq 7$.*

Proof. Let M satisfy the hypotheses of Theorem 1.4, and note that since M is simple, it is $(2, 2)$ -small by definition. Suppose M contains a connected closed incompressible surface of genus 2. If M is $(3, 2)$ -small, the hypothesis on $\text{Hg}(M)$ and [6, Theorem 3.2] imply that for any such surface S , $\bar{\chi}(\text{kish}(M \setminus S)) \geq 2$, satisfying the first conclusion of Theorem 1.4. Otherwise, [6, Theorem 5.8] provides a separating, connected, closed incompressible surface S of genus 2 satisfying one of the conditions below.

- (1) At least one component of $M \setminus S$ is acylindrical; or
- (2) For each component B of $M \setminus S$ we have $\text{kish}(B) \neq \emptyset$.

If S satisfies condition (2), then since each component B of $M \setminus S$ has $\text{kish } B \neq \emptyset$ we have $\bar{\chi}(\text{kish}(M \setminus S)) \geq 2$, which implies the first conclusion of Theorem 1.4. We address the other case below.

Now suppose that M contains no connected closed incompressible surface of genus 2 but contains a connected closed incompressible surface of genus 3. If M is $(5, 3)$ -small, the hypothesis on $\text{Hg}(M)$ and [6, Theorem 3.2] imply that for any such surface S , $\bar{\chi}(\text{kish}(M \setminus S)) \geq 4$, satisfying the first conclusion of Theorem 1.4. The remaining possibilities are that M contains

a separating incompressible surface of genus g and is $(g, 3)$ -small, for $g = 3$ or 4 . In either case, the hypothesis on Heegaard genus ensures that [6, Theorem 5.8] provides a separating connected closed incompressible surface S of genus g satisfying condition (1) or (2) above. As above, if S satisfies conclusion (2), then the first conclusion of Theorem 1.4 follows.

In the remaining cases, we have a separating, connected, closed incompressible surface $S \subset M$ of genus 2, 3, or 4, satisfying condition (1) above, and we may assume that S does not satisfy condition (2) there. Let N be an acylindrical component of $M \setminus S$ and B the remaining component. Since N is acylindrical, $N = \text{kish}(N)$. Therefore $\text{kish}(B) = \emptyset$, since otherwise S would satisfy condition (2) above. Then $B = |\mathcal{B}|$ for some book of I -bundles \mathcal{B} (cf. [6, §5.1]), and so by [6, Lemma 5.3], B is “shallow relative to S ” ([6, Definition 4.3]). It now follows from [6, Lemma 4.4] that $\text{Hg}(M) \leq 1 + \text{Hg}(N)$, hence by hypothesis that $\text{Hg}(N) \geq 7$. Thus in this case the second conclusion of Theorem 1.4 holds. \square

Theorem 1.3. *Let M be a closed, orientable hyperbolic 3-manifold containing a closed, connected incompressible surface of genus 2 or 3, and suppose that $\text{Hg}(M) \geq 8$. Then M has volume greater than 6.89.*

Proof. We apply Theorem 1.4 to M , yielding a connected closed surface S of genus at most 4 satisfying its conclusion. If $\bar{\chi}(\text{kish}(M \setminus S)) \geq 2$, then Theorem 9.1 of [2] implies that the volume of M is greater than 7.32. Thus we assume S is separating and $M \setminus S$ has an acylindrical component X with $\text{Hg}(X) \geq 7$. It is a standard result (cf. [6, Proposition 6.3]) that X is homeomorphic to a hyperbolic 3-manifold N with totally geodesic boundary, and $\text{vol}(N) = \text{geodvol}(X)$ (see Definition 6.2 of [6]).

If S has genus at least 3, then by Miyamoto’s Theorem [18, Theorem 5.4], $\text{vol}(N) > 10.4$. If S has genus 2, then Theorem 1.1 implies that N has volume greater than 6.89. Theorem 1.3 now follows from [6, Proposition 6.4] (which is in turn derived from results in [2]). \square

Theorem 1.2. *Let M be a closed, orientable hyperbolic 3-manifold with*

$$\dim_{\mathbb{Z}_2} H_1(M; \mathbb{Z}_2) \geq 5,$$

and suppose that the cup product map $H^1(M; \mathbb{Z}_2) \otimes H^1(M; \mathbb{Z}_2) \rightarrow H^2(M; \mathbb{Z}_2)$ has image of dimension at most one. Then M has volume greater than 3.44.

Proof. If $\pi_1 M$ is 4-free, then Theorem 1.2 of [7] implies that M has volume greater than 3.44. Otherwise there is a subgroup G of $\pi_1 M$ which has rank at most 4 and is not free. The homological hypotheses and Proposition 3.5 of [9] ensure that there is a twofold cover $\tilde{M} \rightarrow M$, with $\dim_{\mathbb{Z}_2} H_1(\tilde{M}; \mathbb{Z}_2) \geq 8$, such that $G < \pi_1 \tilde{M}$. Theorem 1.1 of [7] implies that \tilde{M} contains an incompressible surface of genus 2 or 3. Since $\text{Hg}(\tilde{M})$ bounds above the dimension of its \mathbb{Z}_2 -homology, we have $\text{Hg}(\tilde{M}) \geq 8$. Theorem 1.3 now implies that \tilde{M} has volume greater than 6.89; hence that M has volume greater than 3.445. \square

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