Differential Geometry: homework # 3
Due day: October 11

All problems are graded in the scale 0–10. You need to show all your work. Answer is not enough. I am not sure if I will not return the homework, so you might want to keep a copy.

Problem 18. Prove that if w_1, w_2, w_3 are quadratic polynomials, then the curve

$$\alpha(t) = (w_1(t), w_2(t), w_3(t))$$

is contained in a plane.

Problem 19. Let $\alpha : I \to \mathbb{R}^3$ be a Frenet curve. Prove that a plane P has contact of order 2 with α at s_0 if and only if P is the osculating plane of α at $\alpha(s_0)$.

The next result was stated in class without proof, so you need to prove it now.

Problem 20. Prove that if $\alpha : I \to \mathbb{R}^2$ is parametrized by arc-length, then there is a smooth function $\underline{\kappa} : I \to \mathbb{R}$ such that

$$\begin{cases} \mathbf{T}'(s) = \underline{\kappa}(s)\mathbf{N}(s) \\ \mathbf{N}'(s) = -\underline{\kappa}(s)\mathbf{T}(s)\,. \end{cases}$$

Problem 21. Find the radius of the osculating sphere of the helix:

$$\alpha(t) = (a\cos t, a\sin t, bt), \quad a, b > 0.$$

Problem 22. Let $\alpha : [a, b] \to \mathbb{R}^3$ be a regular curve. For any $\lambda \in \mathbb{R}$ define $\alpha_\lambda(t) = \alpha(t) + \lambda\mathbf{T}(t)$. Prove that α_λ is regular for every $\lambda \in \mathbb{R}$.

Problem 23. Let $\alpha : [a, b] \to \mathbb{R}^3$ be a Frenet curve. For $\lambda \in \mathbb{R}$ we define $\alpha_\lambda(s) = \alpha(s) + \lambda\mathbf{N}(s) : [a, b] \to \mathbb{R}^3$.

1. Prove that there is a positive number λ_0 such that the curve α_λ is regular for all $\lambda < \lambda_0$.
2. Prove that if $\lambda < 0$, then the length of α_λ is greater than the length of α.
3. Show that if in addition the torsion of α is different than zero at every point, then α_λ is regular for all $\lambda \in \mathbb{R}$.

Problem 24. Prove that the curve

$$\alpha(t) = \left(\frac{4}{5}\cos t, 1 - \sin t, -\frac{3}{5}\cos t\right)$$

is a circle, and find its center, radius and the plane in which it lies. **Hint:** *compute curvature and torsion.*