Differential Geometry: homework # 1
Due day: September 16

All problems are graded in the scale 0–10. You need to show all your work. Answer is not enough. I will not return the homework, so you might want to keep a copy.

Problem 1. Prove that $\{v_1, v_2, v_3\}$ is a basis of $\mathbb{R}^3$ if and only if $\det(v_1, v_2, v_3) \neq 0$.
Hint: *Use the knowledge how determinants are related to solvability of linear equations.*

Problem 2. Prove that the sign of the determinant (3×3) changes if we change the order of two columns, i.e.

$$\det(u, v, w) = -\det(v, u, w) = -\det(u, w, v) = -\det(w, v, u).$$

For simplicity prove only the first equality.

The transpose of a matrix is

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}^T = \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{bmatrix}.$$

Problem 3. Prove that if A is a 3×3 matrix, then $\det A = \det A^T$.

Problem 4. Prove that $|u \times v|^2 = |u|^2|v|^2 - (u \cdot v)^2$.

Problem 5. Prove that $(u(t) \times v(t))' = u'(t) \times v(t) + u(t) \times v'(t)$.
Hint: *Learn the proof that $(fg)' = f'g + fg'$ and mimic the argument.*

Problem 6. Let $\alpha, \beta : I \to \mathbb{R}^3$ be C^1 curves. Prove that if

$$u'(t) = au(t) + bv(t), \qquad v'(t) = cu(t) - av(t)$$

for all t and some constants a, b, c, then $u(t) \times v(t)$ is a constant vector.

Problem 7. Let $\alpha(t) = (\sin t, 2t^2, t)$ and $t = \log \theta$. Find $d\alpha/d\theta$ (a) as a function of θ; (b) as a function of t.

Problem 8. The sphere $x^2 + y^2 + z^2 = 4$ and the cylinder $(x-1)^2 + y^2 = 1$ intersect along a curve whose shape is similar to that of the digit 8. The curve has a self-intersection. Find the angle at which the curve intersects with itself.

Problem 9. Compute the length of $\alpha(t) = (3\cosh(2t), 3\sinh(2t), 6t)$, $t \in [0, \pi]$.

Problem 10. The conchoid od Nicomedes in polar coordinates is

$$r = \frac{a}{\cos\theta} + c, \quad a \neq 0,\ c \neq 0,\ \theta \in [-\pi, \pi].$$

Find a representation of this curve in the Euclidean coordinates, i.e. of the form

$$\alpha(t) = (x(t), y(t)).$$

Problem 11. Find the arc-length parametrization of $\alpha(t) = (t, \sqrt{2}\ln t, t^{-1})$, $t > 0$.