The Eisenstein ideal with squarefree level

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UCSD number theory seminar, 7 May 2020

Plan

1. What is a congruence of modular forms?
2. Mazur's Eisenstein congruences at prime level
3. Eisenstein congruences at squarefree level (joint with Preston Wake)
    - Reference: `https://arxiv.org/abs/1804.06400`
4. Examples
5. A more subtle case (joint with P. Wake and Catherine Hsu)

Modular forms

Let $N = \ell_1 \cdots \ell_r$ be a squarefree integer, with ℓ_i prime.

"Modular forms" in this talk are modular forms of weight 2 and level $\Gamma_0(N)$, $M_2(N)$, though of as holomorphic functions in the complex variable z in the upper half plane of $\mathbb{C}$. We have a splitting

$$M_2(N) = \mathrm{Eis}_2(N) \oplus S_2(N)$$

into cusp forms and the span of Eisenstein series.

We represent them as q-series, $f(z) = \sum_{n \geq 0} a_n(f) q^n$, for $q = e^{2\pi i z}$.

Example (When N is prime)

$\mathrm{Eis}_2(N)$ is spanned by $E_{2,N}(z) = \frac{N-1}{24} + \sum_{n \geq 1} \left(\sum_{d|n,(N,d)=1} d \right) q^n$.
The least prime N with non-zero $S_2(N)$ is $N = 11$, spanned by
$f(z) = q \prod_{n=1}^{\infty} (1-q^n)^2 (1-q^{11 \cdot n})^2$.

Congruences of Hecke eigenvalues

Consider the Hecke action by Hecke operators

- T_n for $(n, N) = 1$, $n \geq 1$ ("unramified" operators)
- w_ℓ for primes $\ell \mid N$, the Atkin–Lehner operator at ℓ (an involution)

There is a common eigenbasis of $M_2(N)$ for the Hecke action, consisting of eigenforms, and the eigenvalues are algebraic integers.

$\rightsquigarrow$ We can make sense of "congruences between eigenforms."

Definition

Let $f, g \in M_2(N)$ be eigenforms. Let $\mathbb{Q}(f)$ denote the number field generated by the Hecke eigenvalues of f. A *congruence modulo p between f and g* is a prime v of $\mathbb{Q}(f, g)$ dividing p such that the Hecke eigenvalues of f and g under $\{T_n, w_\ell\}$ are equivalent modulo v.

Write "$f \equiv g \pmod{p}$" to express the existence of a cong. modulo p.

Eisenstein congruences

We focus on congruences between Eisenstein series and cusp forms.

Definition (Eisenstein congruences)

An *Eisenstein congruence modulo p* is a cusp form $f \in S_2(N)$ that is congruent to some Eisenstein series modulo p.

Convention: $f = f(z) = \sum_{n \geq 1} a_n(f) q^n$ denotes a normalized eigenform, meaning $a_1(f) = 1$. Then for $(n, N) = 1$, $T_n \cdot f = a_n(f) \cdot f$. So we can read the T_n-part of congruences off of q-series. E.g. $a_n(E_{2,N}) = \sigma(n)$.

Example

Let $N = 11$. Then $E_{2,11} = \frac{10}{24} + q + 3q^2 + 4q^3 + \cdots$ and $f = q - 2q^2 - q^3 + \cdots$ are congruent modulo p if and only if $p = 5$.

(More on w_ℓ later, but $w_N = -1$ on $M_2(N)$ when N is prime.)

Mazur's results on the Eisenstein ideal

Mazur's torsion theorem (mid 1970s) says the torsion subgroup $E(\mathbb{Q})_{\mathrm{tors}}$ of the rational points of an elliptic curve $E/\mathbb{Q}$ is one of 15 options. He proved this by analyzing Eisenstein congruences at *prime* level N.

The analysis was carried out by applying commutative algebra to the Hecke algebra of weight 2 and level $\Gamma_0(N)$.

- Let $\mathbb{T}_\mathbb{Z}$ denote the $\mathbb{Z}$-subalgebra of $\mathrm{End}_\mathbb{C}(M_2(N))$ generated by Hecke operators.
- Let $\mathbb{T}^0_\mathbb{Z}$ be its quotient arising from the action on $S_2(N)$.

Moduli-theoretic interpretation. We have a bijection

$$\begin{aligned} \{\text{normalized eigenforms } \in M_2(N)\} &\longleftrightarrow \{\mathbb{T}_\mathbb{Z} \to \overline{\mathbb{Q}}\} \\ f \quad &\mapsto \big(\text{homom. sending } T_n, w_\ell \mapsto \text{ their eigenval. on } f\big). \end{aligned}$$

Commutative algebra of Hecke algebras

We have the cuspidal version as well.

$$\{\text{normalized eigenforms } \in M_2(N)\} \longleftrightarrow \{\mathbb{T}_{\mathbb{Z}} \to \overline{\mathbb{Q}}\}$$
$$\{\text{normalized eigenforms } \in S_2(N)\} \longleftrightarrow \{\mathbb{T}^0_{\mathbb{Z}} \to \overline{\mathbb{Q}}\}$$

that is, "$\mathbb{T}_{\mathbb{Z}}$ is the moduli of eigenforms."

Let N be prime, so that there is a unique Eisenstein series $E_{2,N}$. Let

$$\mathbb{T}_{\mathbb{Z}} \supset I_{\mathbb{Z}} := \ker(\mathbb{T}_{\mathbb{Z}} \xrightarrow{\text{eigensys. of } E_{2,N}} \overline{\mathbb{Q}}).$$

We have $\mathbb{T}_{\mathbb{Z}}/I_{\mathbb{Z}} \cong \mathbb{Z}$, with $T_n \mapsto \sigma(n)$ there.
Its isomorphic image $I^0_{\mathbb{Z}} \subset \mathbb{T}^0_{\mathbb{Z}}$ is *Mazur's Eisenstein ideal.*

Idea (Mazur)**.** Relate commutative algebra of $I^0_{\mathbb{Z}} \subset \mathbb{T}^0_{\mathbb{Z}}$ to Eisenstein congruences.

Commutative algebra to measure Eisenstein congruences

Definition

Let $\mathbb{T} := (\mathbb{T}_\mathbb{Z})^\wedge_{(I_\mathbb{Z},p)}$, $\mathbb{T}^0 := (\mathbb{T}^0_\mathbb{Z})^\wedge_{(I^0_\mathbb{Z},p)}$ be the completions at the maximal ideals generated by $I_\mathbb{Z}$ and p.
Let $I := I_\mathbb{Z} \cdot \mathbb{T}$, $I^0 := I^0_\mathbb{Z} \cdot \mathbb{T}^0$.

Because $\mathbb{T}_\mathbb{Z} \twoheadrightarrow \mathbb{T}^0_\mathbb{Z}$ are reduced finite-rank free $\mathbb{Z}$-algebras, $\mathbb{T} \twoheadrightarrow \mathbb{T}^0$ have the same properties$/\mathbb{Z}_p$.

Moduli-theoretic interpretation. We have bijections

$$\{\text{Eisenstein congruences}\} \cup \{E_{2,N}\} \longleftrightarrow \{\mathbb{T} \to \overline{\mathbb{Q}}_p\}$$
$$\{\text{Eisenstein congruences}\} \longleftrightarrow \{\mathbb{T}^0 \to \overline{\mathbb{Q}}_p\}$$

$\mathbb{T}, \mathbb{T}^0$ are local rings. We say they *interpolate* the Eisenstein congruences.

Mazur's results on the Hecke algebras

Terminology. Today, we will call

- $\mathbb{T}$ the "Hecke algebra"
- $\mathbb{T}^0$ the "cuspidal Hecke algebra"
- $I \subset \mathbb{T}$ and its isomorphic image $I^0 \subset \mathbb{T}^0$ the "Eisenstein ideal."

Theorem (Mazur, case of prime N)

There exists an Eisenstein congruence modulo p (i.e. $\mathbb{T}^0 \neq 0$) if and only if $p \mid \frac{N-1}{12}$. Moreover, when $\mathbb{T}^0 \neq 0$,

1. $\mathbb{T}^0/I^0 \cong \mathbb{Z}_p/(\frac{N-1}{12})$ $\quad \rightsquigarrow \frac{N-1}{12}$ *is the "congruence number"*
2. I^0 *is principal (hence $\mathbb{T}^0$ is monogeneric/$\mathbb{Z}_p$)*
3. *(let $p \geq 3$) for a prime $q \neq N$, $T_q - (q+1)$ generates I^0 if and only if*
   - $q \not\equiv 1 \pmod{p}$, *and*
   - *q is not a pth power modulo N.*

Mazur's results on the Hecke algebras

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Mazur's application. For applications to rational points on the modular curve $X_0(N)$, what was especially useful is the *Gorenstein* property of $\mathbb{T}^0$. Gorensteinness follows from the complete intersection ("CI") property, which follows from $\mathbb{T}^0/\mathbb{Z}_p$ being flat and monogeneric.

Generalizing to squarefree level

Recall: Write $N = \ell_1 \cdots \ell_r$, and

$$M_2(N) = \mathrm{Eis}_2(N) \oplus S_2(N), \quad \text{with } \dim_{\mathbb{C}} \mathrm{Eis}_2(N) = 2^r - 1.$$

Definition (A-L signature ε)

In $M_2(N)$, there are exactly $2^r - 1$ realized possibilities for the eigenvalues of the r-tuple of Atkin–Lehner involution $(w_{\ell_1}, \dots, w_{\ell_r})$, denoted

$$\varepsilon = (\varepsilon_1, \cdots, \varepsilon_r) \in \{\pm 1\}^{\times r} \smallsetminus \{(+1, \dots, +1)\}$$

In particular, the Eisenstein part $\mathrm{Eis}_2(N)$ has a basis enumerated by the A-L signatures,

$$\{E^{\varepsilon}_{2,N}\}_{\varepsilon}, \quad \text{where } w_{\ell_i} \cdot E^{\varepsilon}_{2,N} = \varepsilon_i \cdot E^{\varepsilon}_{2,N}.$$

Why w_ℓ-operators instead of U_ℓ-operators?

One more frequently sees operators U_ℓ instead of w_ℓ. Their eigenvalues on $\mathrm{Eis}_2(N)$ are

$$w_\ell : \pm 1, \qquad U_\ell : 1,\, \ell.$$

We study w_ℓ because we view them as "desingularizing" the U_ℓ-action on Eisenstein series:

- the w_ℓ-eigenvalues on $\mathrm{Eis}_2(N)$ are distinct modulo any odd prime p
- the U_ℓ-eigenvalues on $\mathrm{Eis}_2(N)$ degenerate modulo p that divide $\ell - 1$

Moreover, Eisenstein congruences proliferate when $p \mid (\ell - 1)$, so using w_ℓ-operators is a useful refinement!

Commutative algebra - squarefree setup

Let ε be a choice of A-L signature, which singles out an Eisenstein series $E^{\varepsilon}_{2,N}$ (even modulo p for p odd).

- Let $I^{\varepsilon}_{\mathbb{Z}} \subset \mathbb{T}_{\mathbb{Z}}$ be the annihilator of $E^{\varepsilon}_{2,N}$, so $\mathbb{T}_{\mathbb{Z}}/I^{\varepsilon}_{\mathbb{Z}} \cong \mathbb{Z}$. We have

$$I^{\varepsilon}_{\mathbb{Z}} = (T_n - \sigma(n), w_\ell - \varepsilon_\ell).$$

- Let $\mathbb{T}^{\varepsilon} := (\mathbb{T}_{\mathbb{Z}})^{\wedge}_{(I^{\varepsilon}_{\mathbb{Z}},p)}$.
  - Bijection $\{\mathbb{T}^{\varepsilon} \to \overline{\mathbb{Q}}_p\} \longleftrightarrow \{\text{Eis. congruences with } E^{\varepsilon}_{2,N}\} \cup \{E^{\varepsilon}_{2,N}\}$.

- Let $\mathbb{T}^{\varepsilon,0}$ be the completion of $\mathbb{T}^0_{\mathbb{Z}}$ at the image of $(I^{\varepsilon}_{\mathbb{Z}}, p)$.
  - Bijection $\{\mathbb{T}^{\varepsilon,0} \to \overline{\mathbb{Q}}_p\} \longleftrightarrow \{\text{Eis. congruences with } E^{\varepsilon}_{2,N}\}$.

Goal. Generalize Mazur's results on $\mathbb{T}^0_{\text{prime}}$ to the squarefree level $\mathbb{T}^{\varepsilon,0}$ and $\mathbb{T}^{\varepsilon}$.

Results at squarefree level

Mazur's results on $\mathbb{T}^0_{\text{prime}}$ (level N):

1. the congruence number is $(N-1)/12$
2. I^0_{prime} is principal, so $\mathbb{T}^0/\mathbb{Z}_p$ is monogeneric and complete intersection
3. Determination of which choices of single Hecke operator generate I^0

Ohta has established the analogue of (1) for odd p.

Theorem (Ohta, stated here when $p \geq 5$)

The congruence number of $I^{\varepsilon,0} \subset \mathbb{T}^{\varepsilon,0}$ is $a_0(E^{\varepsilon}_{2,N})$, which equals

$$\frac{1}{24}\prod_{i=1}^{r}(\ell_i + \varepsilon_i).$$

In other words, $\mathbb{T}^{\varepsilon,0} \neq 0$ iff $p \mid a_0(E^{\varepsilon}_{2,N})$; moreover, $\mathbb{T}^{\varepsilon,0}/I^{\varepsilon,0} \cong \dfrac{\mathbb{Z}_p}{a_0(E^{\varepsilon}_{2,N})}$.

New results at squarefree level, Case I

We present analogues to (2) and (3) in two Cases ($p \geq 5$ throughout).

Case I. $\varepsilon = (-1,-1)$, $N = \ell_1\ell_2$, $\ell_1 \equiv \ell_2 \equiv 1 \pmod{p}$, and

$$\log_{\ell_1}(\ell_2) \neq 0 \quad \text{and} \quad \log_{\ell_2}(\ell_1) \neq 0,$$

where $\log_\ell : \mathbb{F}_\ell^\times \twoheadrightarrow \mathbb{Z}/p\mathbb{Z}$ is a choice of surjection.

Theorem (Wake–WE, Case I.)

1. $\mathbb{T}^\varepsilon$ *is complete intersection*
2. $\mathbb{T}^{\varepsilon,0}$ *is not Gorenstein*
3. *There is an isomorphism* $I^\varepsilon/I^{\varepsilon 2} \cong \mathbb{Z}_p/(\ell_1 - 1) \oplus \mathbb{Z}_p/(\ell_2 - 1)$
4. *For primes* $q_1, q_2 \nmid N$*, the Hecke operators* $\{T_{q_i} - (q_i + 1)\}$ *generate* I^ε *if and only if*

$$(q_1 - 1)(q_2 - 1)\det\begin{pmatrix}\log_{\ell_1}(q_1) & \log_{\ell_1}(q_2)\\ \log_{\ell_2}(q_1) & \log_{\ell_2}(q_2)\end{pmatrix} \neq 0 \in \mathbb{F}_p.$$

New results at squarefree level, Case I

Generators of I^ε, $N = \ell_1\ell_2$ and $\varepsilon = (-1,-1)$

For primes $q_1, q_2 \nmid N$, the Hecke operators $\{T_{q_i} - (q_i+1)\}$ generate I^ε if and only if

$$(q_1-1)(q_2-1)\det\begin{pmatrix}\log_{\ell_1}(q_1) & \log_{\ell_1}(q_2)\\ \log_{\ell_2}(q_1) & \log_{\ell_2}(q_2)\end{pmatrix} \neq 0 \in \mathbb{F}_p.$$

This neatly generalizes Mazur's condition at prime level.

Generator of I at prime level ℓ

$T_q - (q+1)$ generates the Eisenstein ideal at level $\Gamma_0(\ell)$ if and only if

$$(q-1)\log_\ell(q) \neq 0 \in \mathbb{F}_p.$$

New results at squarefree level, Case II

Case II. $\varepsilon = (-1, +1, +1, \ldots, +1)$, $\ell_1 \equiv 1 \pmod{p}$, $\ell_i \equiv -1 \pmod{p}$ for $2 \leq i \leq r$.

For what levels and signatures do Eisenstein congruent newforms exist?

There can be new congruences at any prime ℓ when $\varepsilon_\ell = -1$. In contrast, when $\varepsilon = +1$, there are new congruences only when $\ell \equiv -1 \pmod{p}$.

We need to set up a certain number field for each prime ℓ_i for $2 \leq i \leq r$. Let $K_{\ell_i}/\mathbb{Q}(\zeta_p)$ be the unique $\mathbb{Z}/p$-extension such that

- ramified only at primes over ℓ_i
- the prime $(1 - \zeta_p)$ splits completely
- the $\mathrm{Gal}(\mathbb{Q}(\zeta_p)/\mathbb{Q})$-action on $\mathrm{Gal}(K_{\ell_i}/\mathbb{Q}(\zeta_p))$ is given by ω^{-1}, where ω is the modulo p cyclotomic character $\omega : \mathrm{Gal}(\mathbb{Q}(\zeta_p)/\mathbb{Q}) \xrightarrow{\sim} \mathbb{F}_p^\times$.

New results at squarefree level, Case II

Case II, meaning $\varepsilon = (-1, +1, \dots, +1)$, and $\ell_i \equiv -\varepsilon_i \pmod{p}$.

Theorem (Wake–WE, Case II.)

1. $\mathbb{T}^\varepsilon$ *is complete intersection*
2. $\mathbb{T}^{\varepsilon,0}$ *is Gorenstein iff* I^ε *is principal*
3. *There is a SES*

$$0 \to \bigoplus_{i=2}^{r} \frac{\mathbb{Z}_p}{(\ell_i + 1)} \to I^\varepsilon / I^{\varepsilon 2} \to \mathbb{Z}_p/(\ell_1 - 1) \to 0$$

4. *The minimal number of generators of* I^ε *is* $r - \delta$*, where* $\delta \in \{0, 1\}$ *equals* 0 *if and only if* ℓ_1 *splits completely in* K_{ℓ_i} *for all* $2 \le i \le r$*.*
5. *There is a sufficient condition for a* $(r - \delta)$*-tuple of primes* (q_j) *such that* $\{T_{q_j} - (q_j + 1)\}$ *generate* I^ε*. It is explicit, and we omit it here.*

Implications for "multiplicity one"

Let $J_0(N)$ be the Jacobian of the closed modular curve. "Multiplicity one" (modulo p) means that

$$\dim_{\mathbb{F}_p} J_0(N)[p]_{\mathfrak{m}^\varepsilon} = 2$$

where $(-)_{\mathfrak{m}^\varepsilon}$ refers to the $\mathfrak{m}^\varepsilon = (I^\varepsilon_{\mathbb{Z}}, p)$ part under the $\mathbb{T}_{\mathbb{Z}}$-action on $J_0(N)$.

Corollary (Wake–WE)

The $\mathbb{F}_p$-dimension above is $(\#\text{gens. of } I^\varepsilon) + 1$. Consequently,

1. *Multiplicity one fails in Case I (the $\#$gens. is 2).*
2. *Multiplicity one holds in Case II iff $r - \delta = 1$.*
3. *Generally, multiplicity one $\iff$ $\mathbb{T}^{\varepsilon,0}$ is Gorenstein.*

Statement (2) confirms a conjecture of Ribet, when $r = 2$. (But this requires a comparison between w-based and U-based theory!)

Examples

Example of Case II. Let $\varepsilon = (-1, +1)$, $p = 5$, $\ell_1 = 41$, $\ell_2 = 19$.

We find the number field K_{19} and check that 41 splits completely in it. Hence $\mathbb{T}^\varepsilon$ (and I^ε) has two generators. Using Sage, we find

$$\frac{\mathbb{T}^\varepsilon}{5\mathbb{T}^\varepsilon} \cong \frac{\mathbb{F}_5[x, y]}{(y^2 - 2x^2, xy)},$$

where $x = T_2 - 3$ and $y = T_{11} - 12$.

Also, $\mathbb{T}^\varepsilon$ is complete intersection, $\mathbb{T}^{\varepsilon,0}$ is not Gorenstein, multiplicity one fails, and the congruence number is $\frac{1}{24}(41 - 1) \cdot (19 + 1)$.

Examples

Example of Case I. Let $\varepsilon = (-1, -1)$, $p = 5$, $\ell_1 = 11$, $\ell_2 = 41$.
The theorem applies, since

- 11 is not a 5th power modulo 41, and
- 41 is not a 5th power modulo 11.

We get that $\mathbb{T}^\varepsilon$ and I^ε have two generators, $\mathbb{T}^{\varepsilon,0}$ is not Gorenstein, multiplicity one fails, and the congruence number is $\frac{1}{24}(11-1)(41-1)$.

What primes (q_1, q_2) give generators $\{ T_{q_i} - (q_i + 1)\}$ of I^ε?
Among $q \in \{2, 3, 7, 13\}$, we get

$$\log_{\ell_i}(q) = 0 \iff \ell_i = 41 \text{ and } q = 3.$$

Thus for $q \in \{2, 7, 13\}$, we get that

$$\begin{pmatrix} \log_{11}(q) & \log_{11}(3) \\ \log_{41}(q) & \log_{41}(3) \end{pmatrix}$$

has non-zero determinant $\in \mathbb{F}_5$, and $(q, 3)$ give generators of I^ε.

Proofs of the theorems: comparison with Galois representations

We know the shape of the 2-dim. p-adic $G_{\mathbb{Q}} = \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-representations ρ_f associated to new eigenforms $f \in M_2(N)$: when restricted to a decomposition group G_ℓ at a prime $\ell \mid Np$,

$$\rho|_{G_\ell} \simeq \nu(-\varepsilon_\ell) \otimes \begin{pmatrix} \kappa & * \\ 0 & 1 \end{pmatrix}, \quad \text{where}$$

- κ is the p-adic cyclotomic character
- $\nu(\alpha)$ is the unramified character of G_ℓ sending $\mathrm{Frob}_\ell \mapsto \alpha$.
- $*$ denotes a non-trivial extension

Method.

1. Produce a ring R^ε that parameterizes $G_{\mathbb{Q}}$-**pseudo**reps. as above
2. Verify Wiles's numerical criterion to prove $R^\varepsilon \cong \mathbb{T}^\varepsilon$.
   - Byproduct: $\mathbb{T}^\varepsilon$ is complete intersection.
3. Relate arithmetic properties to structure of R^ε, deducing theorems about $\mathbb{T}^\varepsilon, \mathbb{T}^{\varepsilon,0}$.

Beyond complete intersection cases

We suspect we have addressed all complete intersection cases...
$\rightsquigarrow$ what lies beyond?
Simplest case. $p \geq 5$, $N = \ell_1 \ell_2$, $\varepsilon = (-1, -1)$,

$$\ell_1 \equiv 1 \pmod{p}, \qquad \text{but} \quad \ell_2 \not\equiv \pm 1 \pmod{p}.$$

Theorem (A particular case of a more general result of Ribet)

There is a new Eisenstein congruence at level N with A-L sig. $(-1, -1)$
$\iff \log_{\ell_1}(\ell_2) = 0 \in \mathbb{F}_p$.

When this new Eisenstein congruence exists, $\mathbb{T}_N^\varepsilon$ is not Gorenstein. (So we can't hope to apply the numerical criterion to prove $R_N^\varepsilon = \mathbb{T}_N^\varepsilon$.)

With C. Hsu and P. Wake, we study the simplest non-Gorenstein case.

Non-Gorenstein "$R = \mathbb{T}$"

Assume $\log_{\ell_1}(\ell_2) = 0$ and consider the *strict* surjection $\mathbb{T}_N^\varepsilon \twoheadrightarrow \mathbb{T}_{\ell_1}$. We understand Mazur's prime level setting: $R_{\ell_1} \cong \mathbb{T}_{\ell_1}$ there.

For simplicity assume $\mathrm{rank}_{\mathbb{Z}_p} \mathbb{T}^0_{\ell_1} = 1$, so $\mathbb{T}^0_{\ell_1} \cong \mathbb{Z}_p$. Then $\mathrm{rank}_{\mathbb{Z}_p} \mathbb{T}_{\ell_1} = 2$.

An argument for $R_N^\varepsilon \cong \mathbb{T}_N^\varepsilon$ (Hsu–Wake–WE)

- Since $\mathbb{T}_N^\varepsilon \twoheadrightarrow \mathbb{T}_{\ell_1}$ is strict, $\mathrm{rank}_{\mathbb{Z}_p} \mathbb{T}_N^\varepsilon \geq 3$.
- It is easy to produce $R_N^\varepsilon \twoheadrightarrow \mathbb{T}_N^\varepsilon$
- Prove that an arithmetic condition implies that $\dim_{\mathbb{F}_p} R_N^\varepsilon / pR_N^\varepsilon \leq 3$.

Put together, the arithmetic condition implies $R_N^\varepsilon \cong \mathbb{T}_N^\varepsilon$, a non-Gorenstein case of $R_N^\varepsilon \cong \mathbb{T}_N^\varepsilon$.

Sufficient condition for non-Gorenstein $R_N^\varepsilon \cong \mathbb{T}_N^\varepsilon$

What is this arithmetic condition that we prove to imply that $R_N^\varepsilon \cong \mathbb{T}_N^\varepsilon$, under the assumptions above?

- We start with the $\mathbb{Z}/p$-extension $K_{\ell_1}/\mathbb{Q}(\zeta_p)$ defined above.
- Let $M := K_{\ell_1}(\ell_2^{1/p})$
- Let $C := \mathrm{Cl}_M/p\mathrm{Cl}_M$ be the p-cotorsion of its ideal class group.
- Take the Galois co-invariants $C_{\mathrm{Gal}(M/\mathbb{Q})}$, which we can show to be 1-dimensional/$\mathbb{F}_p$.

Arithmetic condition. ℓ_2 is inert in the $\mathbb{Z}/p$-extension of M cut out by $C_{\mathrm{Gal}(M/\mathbb{Q})}$. (Ongoing work: simplify the condition.)

Reason for interest in the non-Gorenstein rings "$R = \mathbb{T}$"

The techniques above allow us to understand a non-Gorenstein case of "$R = \mathbb{T}$" very explicitly. These cases are difficult in general, so we hope to obtain some general/theoretical insight.