PARTIAL DIFFERENTIAL EQUATIONS I & II

VERSION: August 1, 2023

ARMIN SCHIKORRA

Contents

In Analysis
there are no theorems
only proofs

A substantial part of these notes are strongly inspired on [Evans, 2010] and lectures by Heiko von der Mosel (RWTH Aachen). Parts of the parabolic part of these notes have been typed by Julian Scheuer (U Frankfurt). Other substantial inspirations are lectures and/or lecture notes by Lenzmann, Haslhofer, Struwe, Fernandez-Real & Ros-Oton [Fernández-Real and Ros-Oton, 2022].

if you find a typo, you can keep it

References

[Adams and Fournier, 2003] Adams, R. A. and Fournier, J. J. F. (2003). *Sobolev spaces*, volume 140 of *Pure and Applied Mathematics (Amsterdam)*. Elsevier/Academic Press, Amsterdam, second edition.

[Adimurthi and Banerjee, 2022] Adimurthi, K. and Banerjee, A. (2022). Unified approach to $C^{1,\alpha}$ regularity for quasilinear parabolic equations. *Atti Accad. Naz. Lincei Rend. Lincei Mat. Appl.*, 33(4):833–883.

[Bedrossian and Vicol, 2022] Bedrossian, J. and Vicol, V. ([2022] ©2022). *The mathematical analysis of the incompressible Euler and Navier-Stokes equations—an introduction*, volume 225 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI.

[Brasco et al., 2022] Brasco, L., Prinari, F., and Zagati, A. C. (2022). A comparison principle for the Lane-Emden equation and applications to geometric estimates. *Nonlinear Anal.*, 220:Paper No. 112847, 41.

[Brézis and Nirenberg, 1983] Brézis, H. and Nirenberg, L. (1983). Positive solutions of nonlinear elliptic equations involving critical Sobolev exponents. *Comm. Pure Appl. Math.*, 36(4):437–477.

[Caffarelli et al., 1982] Caffarelli, L., Kohn, R., and Nirenberg, L. (1982). Partial regularity of suitable weak solutions of the Navier-Stokes equations. *Comm. Pure Appl. Math.*, 35(6):771–831.

[Caffarelli and Silvestre, 2007] Caffarelli, L. and Silvestre, L. (2007). An extension problem related to the fractional Laplacian. *Comm. Partial Differential Equations*, 32(7-9):1245–1260.

[Chen, 1999] Chen, Z.-Q. (1999). Multidimensional symmetric stable processes. *Korean J. Comput. Appl. Math.*, 6(2):227–266.

[Colombo and Tione, 2022] Colombo, M. and Tione, R. (2022). Non-classical solutions of the p-Laplace equation. *arXiv e-prints*, page arXiv:2201.07484.

[Courant, 1950] Courant, R. (1950). *Dirichlet's Principle, Conformal Mapping, and Minimal Surfaces.* Interscience Publishers, Inc., New York, N.Y. Appendix by M. Schiffer.

[Derrick, 1964] Derrick, G. H. (1964). Comments on nonlinear wave equations as models for elementary particles. *J. Mathematical Phys.*, 5:1252–1254.

[DiBenedetto, 2010] DiBenedetto, E. (2010). *Partial differential equations.* Cornerstones. Birkhäuser Boston, Ltd., Boston, MA, second edition.

[Evans, 1982] Evans, L. C. (1982). A new proof of local $C^{1,\alpha}$ regularity for solutions of certain degenerate elliptic p.d.e. *J. Differential Equations*, 45(3):356–373.

[Evans, 2010] Evans, L. C. (2010). *Partial differential equations*, volume 19 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, second edition.

[Evans and Gariepy, 2015] Evans, L. C. and Gariepy, R. F. (2015). *Measure theory and fine properties of functions.* Textbooks in Mathematics. CRC Press, Boca Raton, FL, revised edition.

[Fernández-Real and Ros-Oton, 2022] Fernández-Real, X. and Ros-Oton, X. (2022). *Regularity Theory for Elliptic PDE.* Zurich Lecture Notes in Advanced Mathematics.

[Galdi, 1994a] Galdi, G. P. (1994a). *An introduction to the mathematical theory of the Navier-Stokes equations. Vol. I*, volume 38 of *Springer Tracts in Natural Philosophy*. Springer-Verlag, New York. Linearized steady problems.

[Galdi, 1994b] Galdi, G. P. (1994b). *An introduction to the mathematical theory of the Navier-Stokes equations. Vol. II*, volume 39 of *Springer Tracts in Natural Philosophy*. Springer-Verlag, New York. Nonlinear steady problems.

[Gazzola et al., 2010] Gazzola, F., Grunau, H.-C., and Sweers, G. (2010). *Polyharmonic boundary value problems*, volume 1991 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin. Positivity preserving and nonlinear higher order elliptic equations in bounded domains.

[Giaquinta and Martinazzi, 2012] Giaquinta, M. and Martinazzi, L. (2012). *An introduction to the regularity theory for elliptic systems, harmonic maps and minimal graphs*, volume 11 of *Appunti. Scuola Normale Superiore di Pisa (Nuova Serie) [Lecture Notes. Scuola Normale Superiore di Pisa (New Series)]*. Edizioni della Normale, Pisa, second edition.

[Gilbarg and Trudinger, 2001] Gilbarg, D. and Trudinger, N. S. (2001). *Elliptic partial differential equations of second order*. Classics in Mathematics. Springer-Verlag, Berlin. Reprint of the 1998 edition.

[Giusti, 2003] Giusti, E. (2003). *Direct methods in the calculus of variations*. World Scientific Publishing Co., Inc., River Edge, NJ.

[Han and Lin, 2011] Han, Q. and Lin, F. (2011). *Elliptic partial differential equations*, volume 1 of *Courant Lecture Notes in Mathematics*. Courant Institute of Mathematical Sciences, New York; American Mathematical Society, Providence, RI, second edition.

[Iwaniec and Manfredi, 1989] Iwaniec, T. and Manfredi, J. J. (1989). Regularity of p-harmonic functions on the plane. *Rev. Mat. Iberoamericana*, 5(1-2):1–19.

[Iwaniec and Sbordone, 1998] Iwaniec, T. and Sbordone, C. (1998). Riesz transforms and elliptic PDEs with VMO coefficients. *J. Anal. Math.*, 74:183–212.

[John, 1991] John, F. (1991). *Partial differential equations*, volume 1 of *Applied Mathematical Sciences*. Springer-Verlag, New York, fourth edition.

[Koike, 2004] Koike, S. (2004). *A beginner's guide to the theory of viscosity solutions*, volume 13 of *MSJ Memoirs*. Mathematical Society of Japan, Tokyo.

[Kuusi and Mingione, 2018] Kuusi, T. and Mingione, G. (2018). Vectorial nonlinear potential theory. *J. Eur. Math. Soc. (JEMS)*, 20(4):929–1004.

[Kuznetsov, 2019] Kuznetsov, N. (2019). Mean value properties of harmonic functions and related topics (a survey). *Preprint, arXiv:1904.08312*.

[Lewis, 1980] Lewis, J. L. (1980). Smoothness of certain degenerate elliptic equations. *Proc. Amer. Math. Soc.*, 80(2):259–265.

[Lewis, 1983] Lewis, J. L. (1983). Regularity of the derivatives of solutions to certain degenerate elliptic equations. *Indiana Univ. Math. J.*, 32(6):849–858.

[Li and Zhang, 2017] Li, D. and Zhang, K. (2017). A note on the Harnack inequality for elliptic equations in divergence form. *Proc. Amer. Math. Soc.*, 145(1):135–137.

[Lieberman, 1996] Lieberman, G. M. (1996). *Second order parabolic differential equations*. World Scientific Publishing Co., Inc., River Edge, NJ.

[Lin, 1998] Lin, F. (1998). A new proof of the Caffarelli-Kohn-Nirenberg theorem. *Comm. Pure Appl. Math.*, 51(3):241–257.

[Lindqvist, 2006] Lindqvist, P. (2006). *Notes on the p-Laplace equation*, volume 102 of *Report. University of Jyväskylä Department of Mathematics and Statistics*. University of Jyväskylä, Jyväskylä.

[Littman et al., 1963] Littman, W., Stampacchia, G., and Weinberger, H. F. (1963). Regular points for elliptic equations with discontinuous coefficients. *Ann. Scuola Norm. Sup. Pisa (3)*, 17:43–77.

[Llorente, 2015] Llorente, J. G. (2015). Mean value properties and unique continuation. *Commun. Pure Appl. Anal.*, 14(1):185–199.

[Malý and Ziemer, 1997] Malý, J. and Ziemer, W. P. (1997). *Fine regularity of solutions of elliptic partial differential equations*, volume 51 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI.

[Maz'ya, 2011] Maz'ya, V. (2011). *Sobolev spaces with applications to elliptic partial differential equations*, volume 342 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer, Heidelberg, augmented edition.

[Moser, 1961] Moser, J. (1961). On Harnack's theorem for elliptic differential equations. *Comm. Pure Appl. Math.*, 14:577–591.

[Pohozaev, 1965] Pohozaev, S. I. (1965). On the eigenfunctions of the equation $\Delta u + \lambda f(u) = 0$. *Dokl. Akad. Nauk SSSR*, 165:36–39.

[Seregin, 2015] Seregin, G. (2015). *Lecture notes on regularity theory for the Navier-Stokes equations.* World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ.

[Seregin, 2023] Seregin, G. (2023). Remarks on Type II blowups of solutions to the Navier-Stokes equations. *arXiv e-prints*, page arXiv:2304.04045.

[Simon, 1997] Simon, L. (1997). Schauder estimates by scaling. *Calc. Var. Partial Differential Equations*, 5(5):391–407.

[Struwe, 2008] Struwe, M. (2008). *Variational methods*, volume 34 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]*. Springer-Verlag, Berlin, fourth edition. Applications to nonlinear partial differential equations and Hamiltonian systems.

[Talenti, 1976] Talenti, G. (1976). Best constant in Sobolev inequality. *Ann. Mat. Pura Appl. (4)*, 110:353–372.

[Tsai, 2018] Tsai, T.-P. (2018). *Lectures on Navier-Stokes equations*, volume 192 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI.

[Uhlenbeck, 1977] Uhlenbeck, K. (1977). Regularity for a class of non-linear elliptic systems. *Acta Math.*, 138(3-4):219–240.

[Ural'ceva, 1968] Ural'ceva, N. N. (1968). Degenerate quasilinear elliptic systems. *Zap. Naucn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI)*, pages 184–222.

[user9464,] user9464. What does a "heat ball" look like? Mathematics Stack Exchange. URL:https://math.stackexchange.com/q/1169860 (version: 2019-12-12).

[Ziemer, 1989] Ziemer, W. P. (1989). *Weakly differentiable functions*, volume 120 of *Graduate Texts in Mathematics*. Springer-Verlag, New York. Sobolev spaces and functions of bounded variation.

Index

In PDE
notation is a mess

Basic formulas and concepts we'll use a lot

Integration by parts, Greens Theorem, Stokes theorem. If $\Omega \subset \mathbb{R}^n$ is a (nice) open bounded set with outwards facing unit normal $\nu = (\nu^1, \ldots, \nu^n) : \partial\Omega \to \mathbb{S}^{n-1}$ ($\mathbb{S}^{n-1}$ are simply the vectors $v \in \mathbb{R}^n$ with $|v| = 1$, i.e. the unit sphere) and f, g are (nice) functions then we have for any $\alpha \in \{1, \ldots, n\}$

$$\int_\Omega f\, \partial_\alpha g = \int_{\partial\Omega} f g\, \nu^\alpha - \int_\Omega \partial_\alpha f\, g \tag{0.1}$$

Observe that if $n = 1$ and $\Omega = (a, b)$ then $\nu(a) = -1$ and $\nu(b) = +1$, and then we have the usual one-dimension integration by parts formula

$$\int_{(a,b)} f\, \partial_\alpha g = f(b)g(b) - f(a)g(a) - \int_{(a,b)} \partial_\alpha f\, g \tag{0.2}$$

- works also for $\mathbb{R}^n$ or unbounded set Ω – as long as
$$\lim_{|x|\to\infty} f(x) = \lim_{|x|\to\infty} g(x) = 0$$
- Green's formula (divergence theorem) is normaly written for vector fields $G = (G^1, G^2, \ldots, G^n) : \Omega \to \mathbb{R}^n$,
$$\int_\Omega \operatorname{div}(G) = \sum_{\alpha=1}^n \int_\Omega 1\, \partial_\alpha G = \sum_{\alpha=1}^n \int_{\partial\Omega} \nu^\alpha G^\alpha - \sum_{\alpha=1}^n \int_\Omega \underbrace{(\partial_\alpha 1)}_{=0} G^\alpha = \int_{\partial\Omega} G \cdot \nu$$

Exercise 0.1. *Use Green's formula*
$$\int_\Omega \operatorname{div}(G) = \int_{\partial\Omega} G \cdot \nu$$
to prove the integration by parts formula (0.1) *.*

Exercise 0.2. *Use* (0.2) *to show* (0.1)

(You can use pictures and a simple set Ω – I care about the idea, not the most general case)

Polar coordinates. Let $f : B(0, R) \to \mathbb{R}^n$ (nice) then

$$\int_{B(0,R)} f(x)dx = \int_0^R \int_{\partial B(0,\rho)} f(\theta)\, d\theta\, d\rho \tag{0.3}$$

This is actually Fubini's theorem (or *Cavalieri's principle*), and really isn't that related to polar coordinates (well, there is a sphere...) We call it polar coordinates anyways. By a substitution we can write this as

$$\int_{B(0,R)} f(x)dx = \int_0^R \rho^{n-1} \int_{\partial B(0,1)} f(\rho\theta)\, d\theta\, d\rho \tag{0.4}$$

Exercise 0.3. *Prove* (0.4) *using* (0.3)

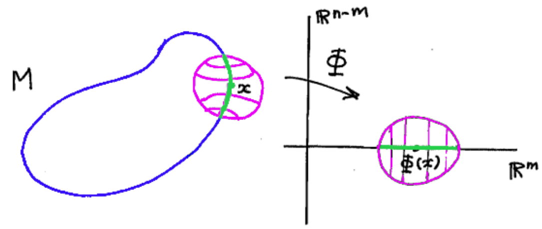


FIGURE 0.1. M is a (sub-)manifold iff around any point x there exist a chart Φ

A special case is the case when $f : B(0,R) \to \mathbb{R}$ is *radial*. f is called *radial* if $f(x) = f(Qx)$ for all rotation matrices $Q \in O(n)$ (i.e. $Q \in \mathbb{R}^{n\times n}$, $Q^t Q = I$).

Exercise 0.4. *Show that if f is radial then there exists $g : [0,\infty) \to \mathbb{R}$ such that*

$$f(x) = g(|x|).$$

Thus, one often writes "f radial" as "$f(x) = f(|x|)$" (this is an idiotic notation, but we'll still use it).

If f is radial then

$$\int_{B(0,R)} f(x)dx = \int_0^R \rho^{n-1} f(\rho) d\rho |\partial B(0,1)|, \tag{0.5}$$

where $|\partial B(0,1)|$ denotes the area of $\partial B(0,1)$, i.e. $|\partial B(0,1)| = \mathcal{H}^{n-1}(\partial B(0,1))$.

Exercise 0.5. *Show* (0.5) *using* (0.4) *or* (0.3)

REGULAR SETS

We are often going to talk about open sets Ω with *smooth boundary*, $\partial\Omega \in C^k$ or $\partial\Omega \in C^\infty$ or similar. When we say $\Omega \subset \mathbb{R}^n$ with $\partial\Omega \in C^k$ we mean that $M := \partial\Omega$ is a C^k-manifold. That is, for each $x \in \partial\Omega$ there exists a small ball $B(x,r) \subset \mathbb{R}^n$ and an associated *chart* $\Phi : B(x,r) \to \mathbb{R}^n$, which must be a C^k-diffeomorphism (Φ is invertible and Φ, Φ^{-1} are C^k-maps in their respective domain) and

$$\Phi(B(x,r) \cap \partial\Omega) \subset \mathbb{R}^n_+ = \{(x', x_n) \in \mathbb{R}^n : x_n > 0\}$$

and

$$\Phi(B(x,r) \setminus \partial\Omega) \subset \overline{\mathbb{R}^n_-} = \{(x', x_n) \in \mathbb{R}^n : x_n \leq 0\}$$

and $\Phi(x) = 0$. Cf. Figure 0.1.

Part 1. PDE 1

1. Introduction and some basic notation

When studying Partial Differential Equations (PDEs) the first question that arises is: what are partial differential equations.

Let $\Omega \subset \mathbb{R}^n$ be an open set and $u : \Omega \to \mathbb{R}$ be differentiable. The partial derivatives ∂_1 is the directional derivative

$$\partial_1 u(x) \equiv \partial_{x_1} u(x) = \frac{d}{dx_1} u(x) = \frac{d}{dt}\Big|_{t=0} u(x + te_1),$$

where $e_1 = (1, 0, \ldots, 0)$ is the first unit vector. The partial derivatives ∂_2, $\ldots$ ∂_n are defined likewise.

Sometimes it is convenient to use multiinidces: an n-multiindex γ is a vector $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_n)$ where $\gamma_1, \ldots, \gamma_n \in \{0, 1, 2, \ldots, \}$. The order of a multiindex is $|\gamma|$ defined as

$$|\gamma| = \sum_{i=1}^{n} \gamma_i.$$

For a suitable often differentiable function $u : \Omega \to \mathbb{R}$ and a multiindex γ we denote with $\partial^\gamma u$ the partial derivatives

$$\partial^\gamma u(x) = \partial_{x_1}^{\gamma_1} \partial_{x_2}^{\gamma_2} \ldots \partial_{x_n}^{\gamma_n} u(x).$$

For example, for $\gamma = (1, 0, 0, \ldots, 0)$ we have

$$\partial^\gamma u(x) = \partial_{x_1} u,$$

i.e. a partial derivative of first order; and for $\gamma = (1, 2, 0, \ldots, 0)$ we have

$$\partial^\gamma u = \partial_{122} u \equiv \partial_1 \partial_2 \partial_2 u,$$

i.e. a partial derivative of 3rd order.

The collection of all partial derivatives of k-th order of u is usually denoted by $D^k u(x) \in \mathbb{R}^{n^k}$ or (the "gradient") $\nabla^k u$. Usually these are written in matrix form, namely

$$Du(x) = (\partial_1 u(x), \partial_2 u(x), \partial_3 u(x), \ldots, \partial_n u(x))$$

and

$$D^2 u(x) = (\partial_{ij} u)_{i,j=1,\ldots n} \equiv \begin{pmatrix} \partial_{11} u(x) & \partial_{12} u(x) & \partial_{13} u(x) & \ldots & \partial_{1n} u(x) \\ \partial_{21} u(x) & \partial_{22} u(x) & \partial_{23} u(x) & \ldots & \partial_{2n} u(x) \\ \vdots & & & \vdots & \\ \partial_{n1} u(x) & \partial_{n2} u(x) & \partial_{n3} u(x) & \ldots & \partial_{nn} u(x) \end{pmatrix}$$

Definition 1.1. Let $\Omega \subset \mathbb{R}^n$ an open set and $k \in \mathbb{N} \cup \{0\}$. A partial differential equation (PDE) of k-th order is an expression of the form

$$F(D^k u(x), D^{k-1} u(x), D^{k-2} u(x), \ldots, Du(x), u(x), x) = 0 \quad x \in \Omega, \tag{1.1}$$

where $u : \Omega \to \mathbb{R}$ is the unknown (also the "solution" to the PDE) and F is a given structure (i.e. map)

$$F : \mathbb{R}^{n^k} \times \mathbb{R}^{n^{k-1}} \times \ldots \times \mathbb{R}^n \times \mathbb{R} \times \Omega \to \mathbb{R}$$

- (1.1) is called *linear* if F is linear in u: meaning if we can find for every n-multiindex γ with $|\gamma| \leq k$ a function $a_\gamma : \Omega \to \mathbb{R}$ (independent of u) such that

$$F(D^k u(x), D^{k-1}u(x), D^{k-2}u(x), \ldots, Du(x), u(x), x) = \sum_{|\gamma| \leq k} a_\gamma(x) \partial^\gamma u(x)$$

- (1.1) is called *semilinear* if F is linear with respect to the highest order k, namely if

$$F(D^k u(x), D^{k-1}u(x), D^{k-2}u(x), \ldots, Du(x), u(x), x) = \sum_{|\gamma| = k} a_\gamma(x) \partial^\gamma u(x) + G(D^{k-1}u(x), D^{k-2}u(x), \ldots, Du(x$$

- (1.1) is called *quasilinear* if F is linear with respect to the highest order k but the coefficient for the highest order may depend on the lower order derivatives of u. Namely if we have a representation of the form

$$F(D^k u(x), D^{k-1}u(x), D^{k-2}u(x), \ldots, Du(x), u(x), x) = \sum_{|\gamma| = k} a_\gamma(D^{k-1}u(x), D^{k-2}u(x), \ldots, Du(x), u(x), x) \partial^\gamma u$$

- If all the above do not apply then we call F *fully nonlinear*.

We have a system of partial differential equations of order k, if $u : \Omega \to \mathbb{R}^m$ is a vector and/or the structure function F is also a vector

$$F : \mathbb{R}^{m\,n^k} \times \mathbb{R}^{m\,n^{k-1}} \times \ldots \times \mathbb{R}^{m\,n} \times \mathbb{R}^m \times \Omega \to \mathbb{R}^\ell$$

for $m, \ell \geq 1$.

The goal in PDE is usually (besides modeling what PDE describes what situation) to solve PDEs, possibly subject to side-condition (such as prescribed boundary data on $\partial\Omega$).

This is rarely possible explicitely (even in the linear case) – which is a huge contrast to ODE. E.g.

$$u''(x) = 2u(x) \quad x \in \mathbb{R},$$

then we know that $u(x) = e^{\sqrt{2}x}A$, and we can compute A by prescribing some initial value at $x = 0$ or similar.

Now if we try that in two dimensions, and consider

$$\Delta u(x) \equiv \partial_{11}u(x) + \partial_{22}u(x) = 2u(x) \quad x \in B(0,1) \subset \mathbb{R}^2,$$

it is really difficult to see what u is (observe that also the amound of initial data – e.g. values at $\partial B(0,1)$ is not one, but infinitely many!

So in general the best one can hope for is address the following main questions for PDEs are

- Is there a solution to a problem (and if so: in what sense? – we will learn the distributional/weak sense and strong sense)
- Are solutions unique (under reasonable assumptions like initial data, boundary data?)?
- What are properties of the solutions (e.g. does the solution depend continuously on the data of the problem)?

It is important to accept that there are PDEs without (classical) solutions and there is no general theory of PDEs. There is theory for several types of PDES.

Example 1.2 (Some basic linear equations)**.**

- Laplace equation
$$\Delta u := \sum_{i=1}^{n} u_{x_i x_i} = 0.$$
- Eigenvalue equation (aka Helmholtz equation)
$$\Delta u = \lambda u.$$
- Transport equation
$$\partial_t u - \sum_{i=1}^{n} b^i \partial_{x_i} u = 0$$
- Heat equation
$$\partial_t u - \Delta u = 0$$
- Schrödinger equation
$$i\partial_t u + \Delta u = 0$$
- Wave equation
$$u_{tt} - \Delta u = 0$$

Second order linear equations are classified into *elliptic*, *parabolic*, *hyperbolic* PDE. Roughly this is understood as follows. Assume that u depends on x and t, then

- *elliptic* means the equation is of the form
$$u_{xx} + u_{tt} = G(x, y, u, u_t, u_x)$$
- *parabolic* means
$$u_{xx} = G(x, y, u, u_t, u_x)$$
- *Hyperbolic*
$$u_{xx} - u_{tt} = G(x, y, u, u_t, u_x)$$
or
$$u_{x,t} = G(x, y, u, u_t, u_x)$$

Let us have a generic second order linear equation

$$Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = g$$

(for now let us assume that $A, B, \ldots$ are constant.) We can write the second-order part as

$$Au_{xx} + Bu_{xy} + Cu_{yy} = \begin{pmatrix} A & \frac{1}{2}B \\ \frac{1}{2}B & C \end{pmatrix} : \begin{pmatrix} \partial_{xx}u & \partial_{yx}u \\ \partial_{xy}u & \partial_{yy}u \end{pmatrix},$$

where : denotes the matrix scalar product (sometimes: Hilbert-Schmidt product). If $AC - \frac{1}{4}B^2 > 0$ the determinant of the coefficient matrix is positive, i.e. either the matrix has two positive eigenvalues $\lambda_1 > 0$ and $\lambda_2 > 0$ or two negative eigenvalues $\lambda_1 < 0$ and $\lambda_2 < 0$, and there exists orthogonal matrices $P \in SO(2)$ such that

$$P^T \begin{pmatrix} A & \frac{1}{2}B \\ \frac{1}{2}B & C \end{pmatrix} P = \operatorname{diag}(\lambda_1, \lambda_2).$$

Then we have

$$\begin{pmatrix} A & \frac{1}{2}B \\ \frac{1}{2}B & C \end{pmatrix} : \begin{pmatrix} \partial_{xx}u & \partial_{yx}u \\ \partial_{xy}u & \partial_{yy}u \end{pmatrix} = \operatorname{diag}(\lambda_1, \lambda_2) : P^T D^2 u P.$$

Now consider $\tilde{u}(x, y) := u(P(x, y)^t)$, then by the chain rule,

$$D^2\tilde{u}(x, y) = P^t(D^2u)(P(x, y))P,$$

so that if we set $(\tilde{x}, \tilde{y})^t := P(x, y)^t$ we have

$$\lambda_1 u_{\tilde{x}\tilde{x}} + \lambda_2 u_{\tilde{y}\tilde{y}} = G(u, u_x, u_y),$$

that is if $AC - \frac{1}{4}B^2 > 0$ we can transform our equation into an elliptic equation.

Similarly, if $AC - \frac{1}{4}B^2 = 0$, at least one eigenvalue of the matrix in question is negative, one is positive, so we can transform the equation into

$$\lambda_1 u_{\tilde{x}\tilde{x}} - \lambda_2 u_{\tilde{y}\tilde{y}} = G(u, u_x, u_y),$$

i.e. a hyperbolic equation.

And if $AC - \frac{1}{4}B^2 < 0$, one of the eigenvalues is zero, so that we have the structure

$$\lambda_1 u_{\tilde{x}\tilde{x}} = G(u, u_x, u_y),$$

i.e. we are parablic.

Whether one is elliptic, parabolic, hyperbolic is not purely an algebraic question – it often determines the ways we can understand properties of the equation in question. Often we think of elliptic equation as equilibrium or stationary equations, parabolic equations as a flow of an energy, and hyperbolic of a wave-like equation – but this is not really always the case, since the Schrödinger equation is parabolic in the previous sense, but it is wave-like. It generally holds: every type of equation warrants its own methods.

One can extend this theory, of course, to higher dimensions. If

$$\sum_{i,j=1}^{n} A_{ij}\partial_{x_i,x_j}u + \sum_{i=1}^{n} B_i\partial_{x_i}u + Cu = D,$$

then we may assume that A is symmetric (any antisymmetric part vanishes because $(\partial_{x_i,x_j}u)_{ij}$ is symmetric) – and thus we can discuss its eigenvalues.

- The equation is *elliptic* if all eigenvalues are nonzero and have the same sign.
- The equation is *parabolic* if exactly one eigenvalue is zero, all others are nonzero and have the same sign.
- The equation is *hyperbolic* if no eigenvalue is zero, and $n-1$ eigenvalues have the same sign, and the other one has the opposite sign.

Of course there are more cases (and they may be very challenging to treat). In principle: elliptic means the second order derivatives "move in the same direction", parabolic means "all but one direction move in the same direction and the remaining direction is of first order only", and hyperbolic "the second derivatives compete with each other".

Of course, since in general A and B are nonconstant, the type of equation may change and depend on the point x (e.g. $tu_{xx} + u_{tt} = 0$).

Example 1.3 (Some basic nonlinear equations)**.**

- Eikonal equation
$$|Du| = 1$$
- p-Laplace equation
$$\operatorname{div}(|Du|^{p-2}Du) \equiv \sum_{i=1}^{n} \partial_i(|Du|^{p-2}\partial_i u) = 0$$
- Minimal surface equation
$$\operatorname{div}\left(\frac{Du}{\sqrt{1+|Du|^2}}\right) = 0.$$
- Monge-Ampere
$$\det(D^2u) = 0.$$
- Hamilton-Jacobi
$$\partial_t u + H(Du, x) = 0$$

The notion of what constitutes a solution is important, as a too weak notion allows for too many solutions, and a too strong notion of solution may allow for no solutions at all. We illustrate this for the Eikonal equation:

Exercise 1.4. *We consider different notions of solutions for the Eikonal equation:*

(1) Consider

$$\begin{cases} |u'(x)| = 1 \quad x \in (-1,1) \\ u(-1) = u(1) = 0. \end{cases} \tag{1.2}$$

Show that there is no $u \in C^0([-1,1]) \cap C^1((-1,1))$ *such that* (1.2) *is satified.*

(2) Consider instead

$$\begin{cases} |u'(x)| = 1 \quad \textit{all but finitely many } x \in (-1,1) \\ u(-1) = u(1) = 0. \end{cases} \tag{1.3}$$

Show that there are infinitely many solutions $u \in C^0([-1,1])$ *that are differentiable in all but finitely many points in* $(-1,1)$ *such that* (1.3) *is satified.*

(3) Show that there is a sequence $u_k \in C^0([-1,1])$ *that are differentiable in all but finitely many points in* $(-1,1)$*, such that*

$$\sup_{x \in [-1,1]} |u_k(x) - 0| \xrightarrow{k \to \infty} 0.$$

(4) Consider instead

$$\begin{cases} |u'(x)| = 1 \quad \textit{in all but one } x \in (-1,1) \\ u(-1) = u(1) = 0. \end{cases} \tag{1.4}$$

Show that there are still two solutions $u \in C^0([-1,1])$ *that are differentiable in all but at most one points in* $(-1,1)$ *such that* (1.4) *is satified.*

In this course we will focus on the linear theory (the nonlinear theory is *almost* always based on ideas on the linear theory). Almost each of the linear and nonlinear equations warrants its own course, so we will focus on the basics (namely: mainly elliptic equations).

2. Laplace equation

2.1. **Sort of a physical motivation.** The following is often used to motivate Laplace's equation

Assume Ω is an open set in $\mathbb{R}^n$ (usually $\mathbb{R}^3$), and u describes the density of a fluid or heat that is at an equilibrium state, i.e. no fluid is moving in or our, or not heat is exchanged any more. This means that if we look at any subset $\Omega' \subset \Omega$ nothing flows out or in that would change the density, that is

$$\int_{\partial \Omega'} \nabla u \cdot \nu = 0.$$

By Green's divergence theorem this is equivalent to saying

$$\int_{\Omega'} \operatorname{div}(\nabla u) = 0.$$

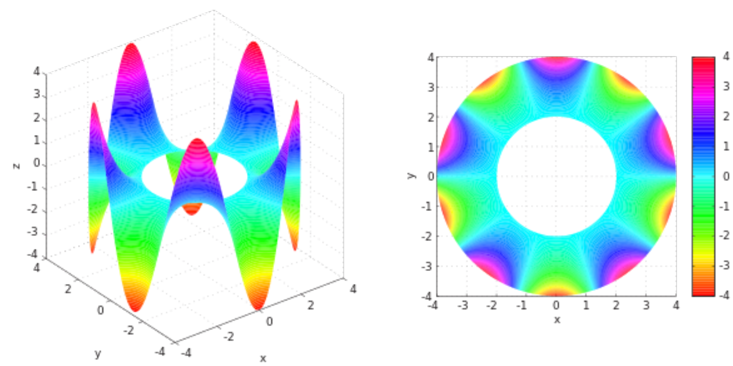


FIGURE 2.1. Solve $\Delta u = 0$ on the annulus (inner radius $r = 2$ and outer radius $R = 4$) with boundary condition $g(\theta) = 0$ if $|\theta| = 2$ and $g(\theta) = 4\sin(5\sigma)$ for $|\theta| = 4$ – where $\sigma \in [0, 2\pi)$ is the angle such that $(\sin(\sigma), \cos(\sigma)) = \theta/|\theta|$. Source: Fourthirtytwo/Wikipedia CC-SA 3

Since this happens for all Ω' we obtain that

$$\operatorname{div}(\nabla u) = 0$$

So we call $\operatorname{div}(\nabla u) =: \Delta u$ and observe that $\Delta u = \sum_{i=1}^n \partial_{x_i x_i} u = \operatorname{tr}(D^2 u)$.

Often one thinks of Laplace equation $\Delta u = 0$ in Ω as a heat distribution. Take Ω a solid, apply some heat at its boundary: at $\theta \in \partial\Omega$ we apply $g(\theta)$ heat. Wait until the heat had time to fully propagate. Then the solution $u : \Omega \to \mathbb{R}$ to the *Dirichlet boundary problem*

$$\begin{cases} \Delta u = 0 & \text{in } \Omega \\ u = g & \text{on } \partial\Omega \end{cases}$$

describes the temperature $u(x)$ at the point $x \in \Omega$. Cf. Figure 2.1.

2.2. **Definitions.** Let $\Omega \subset \mathbb{R}^n$ be an open set (this will always be the case from now on).

- We consider the *homogeneous* Laplace equation

$$\Delta u = 0 \quad \text{in } \Omega \tag{2.1}$$

where we recall that $\Delta u = \operatorname{tr}(D^2 u) = \sum_{i=1}^n \partial_{ii} u$.
- The *inhomogeneous* equation (sometimes: *Poisson equation*) is, for a given function $f : \Omega \to \mathbb{R}$,

$$\Delta u = f \quad \text{in } \Omega \tag{2.2}$$

Two types of *boundary problems* are common:

- *Dirichlet-problem* or *Dirichlet-data* $g : \partial\Omega \to \mathbb{R}$

$$\begin{cases} \Delta u = f & \text{in } \Omega \\ u = g & \text{on } \partial\Omega \end{cases}$$

- *Neumann-problem* or *Neumann-data* $g : \partial\Omega \to \mathbb{R}$

$$\begin{cases} \Delta u = f & \text{in } \Omega \\ \partial_\nu u = g & \text{on } \partial\Omega \end{cases}$$

  Here $\nu : \partial\Omega \to R^n$ is the *outwards facing unit normal* of $\partial\Omega$. (Often this is combined with a normalizing assumption like $\int_\Omega u = 0$, because $u + c$ is otherwise a solution if u is a solution – i.e. non-uniqueness occurs).

Definition 2.1. A function $u \in C^2(\Omega)$ is called *harmonic* if u pointwise solves

$$\Delta u(x) = 0 \quad \text{in } \Omega$$

We also say, u is a solution to the *homogeneous Laplace equation*.

We say that u is a *subsolution* or *subharmonic* if

$$\Delta u(x) \geq 0 \quad \text{in } \Omega.$$

If

$$\Delta u(x) \leq 0 \quad \text{in } \Omega$$

we say that u is a *supersolution* or *superharmonic*.

This notion is very confusing, but it comes from the fact that $-\Delta u$ is a "positive operator" (i.e. has only positive eigenvalues).

2.3. **Fundamental Solution, Newton- and Riesz Potential.** There are many trivial solutions (polynomials of order 1) of Laplace equation. But these are not very interesting. There is a special type of solution which is called *fundamental solution* (which, funny enough, is actually not a solution).

It appears when we want to compute the solution to an equation on the whole space

$$\Delta u(x) = f(x). \tag{2.3}$$

For this we make a brief (formal) introduction to Fourier transform:

The Fourier transform takes a map $f : \mathbb{R}^n \to \mathbb{R}$ and transforms it into $\mathcal{F}u \equiv \hat{f} : \mathbb{R}^n \to \mathbb{R}$ as follows

$$\hat{f}(\xi) := \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{-i\langle \xi, x\rangle} \ f(x) \ dx.$$

The *inverse Fouriertransform* $f^\vee$ is defined as

$$f^\vee(\xi) := \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{+i\langle \xi, x\rangle} \ f(x) \ dx.$$

It has the nice property that $(f^\wedge)^\vee = f$.

One of the important properties (which we will check in exercises) is that derivatives become polynomial factors after Fourier transform:

$$(\partial_{x_i} g)^\wedge(\xi) = -i\xi_i \hat{g}(\xi).$$

For the Laplace operator Δ this implies

$$(\Delta u)^\wedge(\xi) = -|\xi|^2 \hat{u}(\xi).$$

(Side-remark: In this sense $-\Delta$ is positive operator).

This means that if we look at the equation (2.3) and apply Fourier transform on both sides we have

$$-|\xi|^2 \hat{u}(\xi) = \hat{f}(\xi),$$

that is

$$\hat{u}(\xi) = -|\xi|^{-2} \hat{f}(\xi),$$

Inverting the Fourier transform we get an explicit formula for u in terms of the data f.

$$u(x) = -\left(|\xi|^{-2}\hat{f}(\xi)\right)^\vee(x).$$

This is not a very nice formula, so let us simplify it. Another nice property of Fourier transform (and its inverse) is that products become convolutions. Namely

$$(g(\xi)f(\xi))^\vee(x) = \int_{\mathbb{R}^n} g^\vee(x-z) f^\vee(z)\, dz.$$

In our case, for $g(\xi) = -|\xi|^{-2}$ we get that

$$u(x) = \int_{\mathbb{R}^n} g^\vee(x-z)\, f(z)\, dz.$$

Now we need to compute $g^\vee(x-z)$, and for this we restrict our attention to the situation where the dimension is $n \geq 3$. In that case, just by the definition of the (inverse) Fourier transform we can compute that since g has homogeneity of order 2 (i.e. $g(t\xi) = t^{-2} g(\xi)$, then $g^\vee$ is homogeneous of order $2-n$. In particular

$$g^\vee(x) = |x|^{2-n} g^\vee(x/|x|).$$

Now an argument that radial functions stay radial under Fourier transforms leads us to conclude that

$$g^\vee(x) = c_1 |x|^{2-n}.$$

That is, we have arrived that (by formal computations) a solution of (2.3) should satisfy

$$u(x) = c_1 \int_{\mathbb{R}^n} |x-z|^{2-n}\, f(z)\, dz. \tag{2.4}$$

The constant c_1 can be computed explicitely, and we will check below that this *potential* representation of u really is true. This potential is called the *Newton potential* (which is a special case of so-called *Riesz potential*s). The kernel of the Newton potential is called the fundamental solution of the Laplace equation (which, let us stress this again, is **not** a solution)

Definition 2.2. The *fundamental solution* $\Phi(x)$ of the Laplace equation for $x \neq 0$ is given as

$$\Phi(x) = \begin{cases} -\frac{1}{2\pi} \log |x| & \text{for } n = 2 \\ -\frac{1}{n(n-2)\omega_n} |x|^{2-n} & \text{for } n \geq 2 \end{cases}$$

Here ω_n is the Lebesgue measure of the unit ball $\omega_n = |B(0,1)|$.

One can explicitely check that $\Delta\Phi(x) = 0$ for $x \neq 0$ (indeed, $\Delta\Phi(x) = \delta_0$ where δ_0 is the Dirac measure at the point zero, cf. remark 2.7).

Exercise 2.3. *Show that $\Phi \in C^\infty(\mathbb{R}^n \setminus \{0\})$ and compute that $\Delta\Phi(x) = 0$ for $x \neq 0$.*

The following statement justifies (somewhat) the notion of fundamental solution: the fundamental solution $\Phi(x)$ can be used to construct all solutions to the imhomogeneous Laplace equation:

Theorem 2.4. *Let u be the Newton-potential of $f \in C_c^2(\mathbb{R}^n)$, that is*

$$u(x) := \int_{\mathbb{R}^n} \Phi(x-y)\, f(y)\, dy.$$

Here $C_c^2(\mathbb{R}^n)$ are all those functions in $C^2(\mathbb{R}^n)$ such that f is constantly zero outside of some compact set.

We have

- $u \in C^2(\mathbb{R}^n)$
- $-\Delta u = f \quad$ *in* $\mathbb{R}^n$.

Proof. First we show differentiability of u. By a substitution we may write

$$u(x) := \int_{\mathbb{R}^n} \Phi(x-y)\, f(y)\, dy = \int_{\mathbb{R}^n} \Phi(z)\, f(x-z)\, dz.$$

Now if we denote the difference quotient

$$\Delta_h^{e_i} u(x) := \frac{u(x+he_i) - u(x)}{h}$$

where e_i is the i-th unit vector, then we obtain readily

$$\Delta_h^{e_i} u(x) := \int_{\mathbb{R}^n} \Phi(x-y)\, f(y)\, dy = \int_{\mathbb{R}^n} \Phi(z)\, (\Delta_h^{e_i} f)(x-z)\, dz.$$

One checks that Φ is locally integrable (it is not globally integrable!), that is for every bounded set $\Omega \subset \mathbb{R}^n$,

$$\int_\Omega |\Phi| < \infty. \tag{2.5}$$

Indeed, (we show this for $n \geq 3$, the case $n = 2$ is an exercise), if $\Omega \subset \mathbb{R}^n$ is a bounded set, then it is contained in some large ball $B(0, R)$.

$$\int_\Omega |\Phi| \leq C \int_{B(0,R)} |x|^{2-n}\, dx \tag{2.6}$$

Using Fubini's theorem,

$$\begin{aligned}
&\int_{B(0,R)} |x|^{2-n}\, dx \\
=&\int_0^R \int_{\partial B(0,r)} |\theta|^{2-n}\, d\mathcal{H}^{n-1}(\theta)\, dr \\
=&\int_0^R r^{2-n} \int_{\partial B(0,r)} d\mathcal{H}^{n-1}(\theta)\, dr \\
=&c_n \int_0^R r^{2-n} r^{n-1} dr \\
=&c_n \int_0^R r^1 dr \\
=&c_n \frac{1}{2} R^2 < \infty.
\end{aligned}$$

This establishes (2.5)

On the other hand $(\Delta_h^{e_i} f)$ has still compact support for every h. In particular, by dominated convergence we can conclude that

$$\lim_{h\to 0} \Delta_h^{e_i} u(x) = \int_{\mathbb{R}^n} \Phi(z)\, \lim_{h\to 0} (\Delta_h^{e_i} f)(x-z)\, dz.$$

that is

$$\partial_i u(x) = \int_{\mathbb{R}^n} \Phi(z)\, (\partial_i f)(x-z)\, dz.$$

In the same way

$$\partial_{ij} u(x) = \int_{\mathbb{R}^n} \Phi(z)\, (\partial_{ij} f)(x-z)\, dz.$$

Now the right-hand side of this equation is continuous (again using the compact support of f). This means that $u \in C^2(\mathbb{R}^n)$.

To obtain that $\Delta u = f$ we first use the above argument to get

$$\Delta u(x) = \int_{\mathbb{R}^n} \Phi(z)\, (\Delta f)(x-z)\, dz.$$

Observe that

$$(\Delta f)(x-z) = \Delta_x(f(x-z)) = \Delta_z(f(x-z)).$$

Now we fix a small $\varepsilon > 0$ (that we later send to zero) and split the integral, we have

$$\Delta u(x) = \int_{\mathbb{R}^n} \Phi(x-y)\, f(y)\, dy = \int_{B(0,\varepsilon)} \Phi(z)\, (\Delta f)(x-z)\, dz + \int_{\mathbb{R}^n\setminus B(0,\varepsilon)} \Phi(z)\, (\Delta f)(x-z)\, dz =: I_\varepsilon + II_\varepsilon.$$

The term I_ε contains the singularity of Φ, but we observe that

$$I_\varepsilon \xrightarrow{\varepsilon\to 0} 0.$$

Indeed, this follows from the absolute continuity of the integral and since Φ is integrable on $B(0,1)$:

$$|I_\varepsilon| \leq \sup_{\mathbb{R}^n} |\Delta f| \int_{B(x,\varepsilon)} |\Phi(z)| \xrightarrow{\varepsilon\to 0} 0.$$

The term II_ε does not contain any singularity of Φ which is smooth on $\mathbb{R}^n \backslash B_\varepsilon(0)$, so we can perform an integration by parts[1]

$$II_\varepsilon = \int_{\mathbb{R}^n\backslash B(0,\varepsilon)} \Phi(z)\,(\Delta f)(x-z)\,dz = \int_{\partial B(0,\varepsilon)} \Phi(z)\,\partial_\nu f(x-z)\,d\mathcal{H}^{n-1}(z) - \int_{\mathbb{R}^n\backslash B(0,\varepsilon)} \nabla\Phi(z)\cdot\nabla f(x-z)\,dz.$$

Here ν is the unit normal to the ball $\partial B(0,\varepsilon)$, i.e. $\nu = \frac{-z}{\varepsilon}$.

By the definition of Φ one computes that (using (2.5))

$$\left|\int_{\partial B(0,\varepsilon)} \Phi(z)\,\partial_\nu f(x-z)\,d\mathcal{H}^{n-1}(z)\right| \leq \sup_{\mathbb{R}^n} |\nabla f| \int_{\partial B(0,\varepsilon)} |\Phi(z)| \xrightarrow{\varepsilon\to 0} 0.$$

So we perform another integration by parts and have

$$\begin{aligned} II_\varepsilon =& o(1) - \int_{\partial B(0,\varepsilon)} \partial_\nu \Phi(z)\,f(x-z)\,dz + \int_{\mathbb{R}^n\backslash B(0,\varepsilon)} \underbrace{\Delta\Phi(z)}_{=0}\,f(x-z)\,dz \\ =& o(1) - \int_{\partial B(0,\varepsilon)} \partial_\nu \Phi(z)\,f(x-z)\,dz \end{aligned}$$

Here in the last step we used that $\Delta\Phi = 0$ away from the origin, Exercise 2.3.

Now we observe that the unit normal on $\partial B(0,\varepsilon)$ is $\nu(z) = -\frac{z}{\varepsilon}$ and

$$D\Phi(z) = \begin{cases} -\frac{1}{2\pi}\frac{1}{|z|}\frac{z}{|z|} & n=2, \\ -\frac{1}{n(n-2)\omega_n}(2-n)|z|^{1-n}\frac{z}{|z|} & n\geq 3. \end{cases}$$

Thus, for $|z| = \varepsilon$,

$$\partial_\nu \Phi(z) = \nu\cdot D\Phi(z) = \frac{1}{n\omega_n}\varepsilon^{1-n}$$

[1]

$$\int_\Omega f\,\partial_i g = \int_{\partial\Omega} f\,g\,\nu^i - \int_\Omega \partial_i f\,g,$$

where ν is the normal of $\partial\Omega$ pointing outwards (from the point of view of Ω). ν^i is the i-th component of ν. Fun exercise: Check this rule in 1D, to see the relation what we all learned in Calc 1.

Thus we arrive at

$$\begin{aligned}
II_\varepsilon =& o(1) - \int_{\partial B(0,\varepsilon)} \frac{1}{n\omega_n \varepsilon^{n-1}} f(x-z)\, d\mathcal{H}^{n-1}(z)\\
=& o(1) - \fint_{\partial B(0,\varepsilon)} f(x-z)\, d\mathcal{H}^{n-1}(z)\\
=& o(1) - f(x) + \fint_{\partial B(0,\varepsilon)} (f(x)-f(x-z))\ d\mathcal{H}^{n-1}(z)
\end{aligned}$$

Here we use the mean value notation

$$\fint_{\partial B(0,\varepsilon)} = \frac{1}{\mathcal{H}^{n-1}(\partial B(0,\varepsilon)} \int_{\partial B(0,\varepsilon)}.$$

Now one shows (exercise!) that for continuous f

$$\lim_{\varepsilon\to 0} \fint_{\partial B(0,\varepsilon)} (f(x)-f(x-z))\ d\mathcal{H}^{n-1}(z) = 0.$$

(Indeed this is essentially Lebesgue's theorem). That is

$$II_\varepsilon = o(1) - f(x) \quad \text{as } \varepsilon \to 0$$

and thus

$$\Delta u(x) = -f(x) + o(\varepsilon),$$

and letting $\varepsilon \to 0$ we have

$$\Delta u(x) = -f(x),$$

as claimed. □

Exercise 2.5. *Show that* $\log|x|$ *is locally integrable, i.e. that for any bounded set* $\Omega \subset \mathbb{R}^n$ *we have*

$$\int_\Omega \log|x| < \infty.$$

Exercise 2.6. *Assume* f *is continuous. Show that*

$$\lim_{\varepsilon\to 0^+} \fint_{\partial B(0,\varepsilon)} |f(x)-f(x-z)| d\mathcal{H}^{n-1}(z) = 0.$$

Remark 2.7. One can argue (in a distributional sense, which we learn towards the end of the semester)

$$-\Delta\Phi = \delta_0,$$

where δ_0 denotes the Dirac measure at 0, namely the measure such that

$$\int_{\mathbb{R}^n} f(x)\, d\delta_0 = f(0) \quad \text{for all } f \in C^0(\mathbb{R}^n).$$

Observe that δ_0 is not a function, only a measure. In this sense one can justify that

$$\begin{aligned}-\Delta u(x) =&\Delta \int_{\mathbb{R}^n} \Phi(x-z)f(z)\\ =&\int_{\mathbb{R}^n} \Delta\Phi(x-z)f(z)\,dz\\ =&\int_{\mathbb{R}^n} f(z)\,d\delta_x(z)\\ =&f(x)\end{aligned}$$

2.4. **Green Functions.** Our next goal are *Green's functions*. In some way Green functions are a restriction of the fundamental solution to domains $\Omega \subset \mathbb{R}^n$ factoring in also boundary data. Recall that for the fundamental solution Φ we showed in Theorem 2.4 that for the *Newton potential*

$$u(x) := \int_{\mathbb{R}^n} \Phi(x-y)f(y)\,dy \tag{2.7}$$

we have $\Delta u = f$. It is an interesting observation that (for reasonable f) we have

$$\lim_{|x|\to\infty} u(x) = 0.$$

That is the Newton potential approach solves an equation of

$$\begin{cases}\Delta u = f & \text{in } \mathbb{R}^n\\ u = 0 & \text{on the boundary, i.e. for } |x| \to \infty.\end{cases}$$

The *Greens function* is a way to restrict this construction to domains Ω. Instead of the Fundamental solution $\Phi(x-y)$ we get the Green kernel $G(x,y)$. Instead of the Newton potential we consider

$$u(x) = \int_\Omega G(x,y)\,f(y)\,dy$$

and hope that this object solves

$$\begin{cases}\Delta u = f & \text{in } \Omega\\ u = 0 & \text{on } \partial\Omega.\end{cases}$$

The Greens function G (which depends on Ω) can be computed explicitely only for very specific Ω (balls, half-spaces) – which is somewhat related to the fact that there is not necessarily a reasonable Fourier transform for generic sets Ω.

But one can abstractly show that the Green functions exists for reasonable sets Ω. The idea is as follows: We know that the Newton potential as in (2.7) solves the right equation $\Delta u = f$, but it does not satisfy $u = 0$ on $\partial\Omega$. So let us try to correct the Newton potential and choose the Ansatz

$$u(x) := \int_\Omega \Phi(x-y)\,f(y)\,dy - \int_\Omega H(x,y)\,f(y)\,dy$$

By our computatins for Theorem 2.4 we have that then for $x \in \Omega$

$$\Delta u(x) := f(x) - \int_\Omega \Delta_x H(x,y)\, f(y)\, dy,$$

so it would be nice if

$$\Delta_x H(x,y) = 0 \quad \forall\, x, y \in \Omega.$$

Moreover we would like that $u(x) = 0$ on $\partial\Omega$, which would be satisfied if

$$\Phi(x-y) = H(x,y) \quad \forall x \in \partial\Omega, y \in \Omega.$$

That is, for each fixed $y \in \Omega$ we should try to find a function $H(\cdot, y)$ that solves

$$\begin{cases} \Delta_x H(\cdot,y) = 0 & \text{in } \Omega, \\ H(\cdot,y) = \Phi(\cdot - y) & \text{on } \partial\Omega. \end{cases} \tag{2.8}$$

Observe that for fixed $y \in \Omega$ the boundary condition $\Phi(\cdot - y) \in C^\infty(\partial\Omega)$ is a smooth function, since for $y \in \Omega$ we clearly have

$$\inf_{x \in \partial\Omega} |x-y| > 0.$$

That is, there is a good chance to solve this equation (2.8) (and from Theorem 2.22 we know that there is *at most one* solution).

Definition 2.8 (Green function)**.** For given Ω, if there exists H as in (2.8) then we call

$$G(x,y) := \Phi(x-y) - H(x,y)$$

the Green function on Ω.

One can show that G is symmetric, i.e. that

$$G(x,y) = G(y,x) \quad \forall\, x \neq y \in \Omega \tag{2.9}$$

While the Green function are usually not explicit, some properties and estimates can be shown, and there is an extensive research literature on the subject, e.g. see [Littman et al., 1963]. The Green function is also specially important from the point of view of stochastic processes, see e.g. [Chen, 1999].

We will only investigate the most basic property:

Theorem 2.9. *Let $\Omega \subset\subset \mathbb{R}^n$, $\partial\Omega \in C^1$ $f \in C^0(\Omega)$ and $g \in C^0(\partial\Omega)$. Assume that $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ is a solution to*

$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ u = g & \text{on } \partial\Omega \end{cases} \tag{2.10}$$

Then if G is the Green function for Ω from Definition 2.8 we have for any $x \in \Omega$,

$$u(x) = \int_\Omega G(x,y)\, f(y)\, dy - \int_{\partial\Omega} g(\theta) \partial_{\nu(\theta)} G(x,\theta)\, d\mathcal{H}^{n-1}(\theta).$$

Proof. Recall the Gauss-Green formula[2] on (smooth enough) domains A,

$$\int_A u(y)\Delta v(y) - \Delta u(y)\, v(y)\, dy = \int_{\partial A} u(\theta)\partial_\nu v(\theta) - \partial_\nu u(\theta)\, v(\theta)\, d\mathcal{H}^{n-1}(\theta). \tag{2.11}$$

We apply this to formula to $A = \Omega \backslash B(x,\varepsilon)$ and $v(y) := G(x,y)$. Observe that by symmetry of G, (2.9),

$$\Delta_y G(x,y) = \Delta_x G(x,y) = 0 \quad x \neq y,$$

so, also in view of (2.10), (2.11) becomes

$$-\int_A G(x,y)\, f(y)\, dy = \int_{\partial A} u(\theta)\partial_\nu G(x,\theta) - \partial_\nu u(\theta)\, G(x,\theta)\, d\mathcal{H}^{n-1}(\theta). \tag{2.12}$$

Now we argue as in the proof of Theorem 2.4. Observe that H is a smooth function.

We have

$$\begin{aligned}
&\int_{\partial A} u(\theta)\partial_\nu G(x,\theta) d\mathcal{H}^{n-1}(\theta)\\
=&\int_{\partial\Omega} g(\theta)\partial_\nu G(x,\theta) - \int_{\partial B(x,\varepsilon)} u(\theta)\partial_\nu \Phi(x-\theta)\, d\mathcal{H}^{n-1}(\theta) + \int_{\partial B(x,\varepsilon)} u(\theta)\partial_\nu H(x-\theta)\, d\mathcal{H}^{n-1}(\theta)\\
\xrightarrow{\varepsilon\to 0}&\int_{\partial\Omega} g(\theta)\partial_\nu G(x,\theta) - u(x) + 0.
\end{aligned}$$

and

$$\begin{aligned}
&\int_{\partial A} \partial_\nu u(\theta) G(x,\theta) d\mathcal{H}^{n-1}(\theta)\\
=&\int_{\partial\Omega} \partial_\nu u(\theta) G(x,\theta) - \int_{\partial B(x,\varepsilon)} \partial_\nu u(\theta) G(x,\theta)\, d\mathcal{H}^{n-1}(\theta)\\
=&0 - \int_{\partial B(x,\varepsilon)} \partial_\nu u(\theta) G(x,\theta)\, d\mathcal{H}^{n-1}(\theta)\\
\xrightarrow{\varepsilon\to 0}&0.
\end{aligned}$$

This proves the claim. $\square$

In special situations one can actually construct explicit Green's function. Let us firstly consider the Half-space

$$\mathbb{R}^n_+ = \left\{x = (x_1,\ldots,x_n) \in \mathbb{R}^n : \ x_n > 0\right\}.$$

So we need to find a solution to the equation

$$\begin{cases}\Delta_x H(\cdot,y) = 0 & \text{in } \mathbb{R}^n_+,\\ H(\cdot,y) = \Phi(\cdot - y) & \text{on } \mathbb{R}^{n-1}\times\{0\} \equiv \partial\mathbb{R}^n_+.\end{cases}$$

Since H at the boundary has to coincide with Φ it is likely that H should be somewhat of the form of Φ – only the singularity has to be getten rid of – the idea is a reflection:

$$H(x,y) := \Phi(x - y^*)$$

[2]this is a special case of the integration by parts formula

where

$$y^* = (y_1, \ldots, y_n)^* = (y_1, \ldots, y_{n-1}, -y_n).$$

It is a good exercise to check that

(1) H is symmetric, $H(x,y) = H(y,x)$
(2) H is smooth in $\mathbb{R}^n_+ \times \mathbb{R}^n_+$ (since $x^* = y$ implies $x_n = -y_n$, so x and y cannot both lie in the upper half-space if this happens)
(3) The reflection does not change the PDE, namely $\Delta_x H = 0$ for $x, y \in \mathbb{R}^n_+$.
(4) Indeed $H(x,y) = \Phi(x-y)$ for $x \in \mathbb{R}^{n-1} \times \{0\}$ and $y \in \mathbb{R}^n_+$.

So we set

$$G(x,y) := \Phi(x-y) - \Phi(x-y^*) = \Phi(x-y) - \Phi(x^*-y)$$

When we now use the representation formula, as in Theorem 2.9, then we need to compute $\partial_{\nu(y)} G(x,y)$ for $y \in \mathbb{R}^{n-1} \times \{0\}$. Observe that the outwards unit normal $\nu(y) = (0, \ldots, 0, -1)$, so we compute

$$\partial_{\nu(y)} G(x,y) = -\partial_{y_n} \Phi(x-y) + \partial_{y_n} \Phi(x^*-y) = c_n \frac{x_n - y_n}{|x-y|^n} - c_n \frac{x_n + y_n}{|x-y|^n} = \tilde{c}_n \frac{x_n}{|x-y|^n}.$$

If we write the variables in $\mathbb{R}^n_+$ as $x = (x', x_n)$, $x' \in \mathbb{R}^{n-1}$ and $x_n > 0$, then as in Theorem 2.9 we indeed obtain, e.g., if

$$u(x) := c_n \int_{\mathbb{R}^{n-1}} \frac{x_n}{(|x'-y'|^2 + |x_n|^2)^{\frac{n}{2}}} g(y')\, dy' \tag{2.13}$$

then u satisfies indeed (for "reasonable" g)

$$\begin{cases} \Delta u = 0 & \text{in } \mathbb{R}^n_+ \\ \lim_{x_n \to 0} u(x) = g(x') \\ \lim_{x_n \to \infty} u(x) = 0. \end{cases}$$

The formula for u is called the *Poisson formula* on the Half-space $\mathbb{R}^n_+$, also the *harmonic extension of g from $\mathbb{R}^{n-1}$ to $\mathbb{R}^n_+$*.

Exercise 2.10. *(1) Show that the constant c_n in (2.13) is*

$$c_n = \left(\int_{\mathbb{R}^{n-1}} \frac{1}{(|x'-y'|^2 + 1^2)^{\frac{n}{2}}}\, dy' \right)^{-1}.$$

Hint: Use the maximum principle for u assuming that $g \equiv 1$.

(2) Show that for any $x_n > 0$

$$c_n = \left(\int_{\mathbb{R}^{n-1}} \frac{x_n}{(|x'-y'|^2 + |x_n|^2)^{\frac{n}{2}}}\, dy' \right)^{-1}$$

Example 2.11 (Dirichlet-to-Neumann formula)**.** Let $g \in C_c^\infty(\mathbb{R}^{n-1})$. Define u via (2.13).

We consider the Neumann-data of u:

$$\begin{aligned}
\partial_n u\Big|_{x_n=0} &= \lim_{x_n\to 0^+} \frac{u(x',x_n)-u(x',0)}{x_n}\\
&= \lim_{x_n\to 0^+} \frac{u(x',x_n)-g(x')}{x_n}\\
&\overset{E.\ 2.10}{=} c_n \lim_{x_n\to 0^+} \int_{\mathbb{R}^{n-1}} \frac{x_n}{(|x'-y'|^2+|x_n|^2)^{\frac{n}{2}}} \frac{g(y')-g(x')}{x_n}\, dy'\\
&= c_n \lim_{x_n\to 0^+} \int_{\mathbb{R}^{n-1}} \frac{1}{(|x'-y'|^2+|0|^2)^{\frac{n}{2}}} \left(g(y')-g(x')\right) dy'\\
&= c_n \lim_{x_n\to 0^+} \int_{\mathbb{R}^{n-1}} \frac{(g(y')-g(x'))}{(|x'-y'|^2)^{\frac{n}{2}}}\, dy'
\end{aligned}$$

This looks nice, but it has the problem that the integral does not converge absolutely (only in a *principal value* sense).

We try this again: Observe by substituting $h' := x'-y'$ we can write

$$u(x) := c_n \int_{\mathbb{R}^{n-1}} \frac{x_n}{(|h'|^2+|x_n|^2)^{\frac{n}{2}}}\, g(x'-h')\, dh'.$$

By substituting h' with $-h'$ we also have

$$u(x) := c_n \int_{\mathbb{R}^{n-1}} \frac{x_n}{(|h'|^2+|x_n|^2)^{\frac{n}{2}}}\, g(x'+h')\, dh'$$

So we can write

$$\begin{aligned}
\frac{u(x',x_n)-g(x')}{x_n} &= \frac{1}{2}\frac{2u(x',x_n)-2g(x')}{x_n}\\
&= \frac{1}{2} c_n \int_{\mathbb{R}^{n-1}} \frac{x_n}{(|h'|^2+|x_n|^2)^{\frac{n}{2}}} \frac{g(x'+h')+g(x'-h)-2g(x')}{x_n} dh'\\
&= \frac{1}{2} c_n \int_{\mathbb{R}^{n-1}} \frac{g(x'+h')+g(x'-h)-2g(x')}{(|h'|^2+|x_n|^2)^{\frac{n}{2}}}\, dh'\\
&\xrightarrow{x_n\to 0} \frac{1}{2} c_n \int_{\mathbb{R}^{n-1}} \frac{g(x'+h')+g(x'-h)-2g(x')}{|h'|^{n-1+1}}\, dh'.
\end{aligned}$$

In the last step we used that this integral really converges, Exercise 2.12.

This defines an operator

$$(-\Delta)^{\frac{1}{2}} g(x') \equiv \sqrt{-\Delta} g(x') := \frac{1}{2} c_n \int_{\mathbb{R}^{n-1}} \frac{g(x'+h')+g(x'-h)-2g(x')}{|h'|^{n-1+1}}\, dh'$$

which is called the *half-Laplacian*. Indeed using the Fourier transform on $\mathbb{R}^{n-1}$ one can check that

$$\mathcal{F}\left((-\Delta)^{\frac{1}{2}} g\right)(\xi') = c|\xi'|\,\mathcal{F}g(\xi') = c\sqrt{|\xi'|^2}\,\mathcal{F}g(\xi')$$

So this is really the square-root of the Laplacian.

We have proven the *Dirichlet-to-Neumann principle*

If

$$\begin{cases} \Delta u = 0 & \text{in } \mathbb{R}^n_+ \\ u(x') = g(x') & \text{on } \mathbb{R}^{n-1} \times \{0\} \end{cases}$$

then

$$\partial_n u\Big|_{\mathbb{R}^{n-1}\times\{0\}} = c(-\Delta)^{\frac{1}{2}} g \quad \text{on } \mathbb{R}^{n-1} \times \{0\}$$

In 2007, [Caffarelli and Silvestre, 2007], this formula was generalized for $\sigma \in (0,2)$ to

$$\begin{cases} \operatorname{div}\left((x_n)^{1-\sigma} \nabla u\right) = 0 & \text{in } \mathbb{R}^n_+ \\ u(x') = g(x') & \text{on } \mathbb{R}^{n-1} \times \{0\} \end{cases}$$

then

$$\lim_{x_n \to 0^+} (x_n)^{1-\sigma} \partial_n u\Big|_{\mathbb{R}^{n-1}\times\{0\}} = c(-\Delta)^{\frac{\sigma}{2}} g \quad \text{on } \mathbb{R}^{n-1} \times \{0\}$$

This paper has more than 1500 citations and is often referred to as the Caffarelli-Silvestre extension formula.

Exercise 2.12. *Let $g \in C_c^\infty(\mathbb{R}^d)$.*

(1) *For $s \in (0,1)$ show that for each fixed $x \in \mathbb{R}^d$*

$$y \mapsto \frac{g(y) - g(x)}{|x-y|^{d+s}} \in L^1(\mathbb{R}^d),$$

i.e.

$$\int_{\mathbb{R}^d} \frac{|g(y) - g(x)|}{|x-y|^{d+s}}\, dy < \infty.$$

(2) *For $s \in (0,2)$ show that for each fixed $x \in \mathbb{R}^d$*

$$y \mapsto \frac{g(x+h) - g(x-h)}{|h|^{d+s}} \in L^1(\mathbb{R}^d),$$

i.e. that

$$\int_{\mathbb{R}^d} \frac{|g(x+h) + g(x-h) - 2g(x)|}{|x-y|^{d+s}}\, dy < \infty.$$

2.4.1. *On a ball.* The other situation where we can compute the Green's function is the ball. For simplicity let us consider $\Omega = B(0,1)$, the unit ball centered at zero. Again the first goal is to find $H(x,y)$ that corrects the fundamental solution. In the case of the half-space $\mathbb{R}^n_+$ we set $H(x,y) = \Phi(x-\tilde{y})$, i.e. we reflected the y-variable in a way that did not interfere with the PDE but removed the singularity (and coincided with $\Phi(x-y)$ on the boundary.

So lets do the same for the ball. The canonical operation that reflects points from the ball $B(0,1)$ outside and vice versa is called the inversion at a sphere,

$$y^* := \frac{y}{|y|^2} : B(0,1) \to B(0,1)^c.$$

(Although it is not explicitely used here, it is good to know: the inversion at the sphere is a conformal transform, i.e. it preserves angles). So a first attempt would be to set

$$H(x,y) := \Phi\left(\left|x - \frac{y}{|y|^2}\right|\right),$$

which takes care of the singularity of Φ (for $y,x \in B(0,1)$ we have $|x - \frac{y}{|y|^2}| \neq 0$, and does not disturb the PDE for $G(x,y)$. However such a $G(x,y)$ is not equal to $\Phi(x-y)$ for $|x| = 1$. So we need to adapt G to the boundary data. Observe that for $|x| = 1$,

$$\begin{aligned}
&|y|^2 \left|x - \frac{y}{|y|^2}\right|^2\\
=&|y|^2 \left(|x|^2 + \frac{1}{|y|^2} - 2\langle x, \frac{y}{|y|^2}\rangle\right)\\
=&\left(|y|^2|x|^2 + 1 - 2\langle x,y\rangle\right)\\
\overset{|x|=1}{=}&\left(|y|^2 + |x|^2 - 2\langle x,y\rangle\right)\\
=&|x-y|^2.
\end{aligned}$$

That is why we set

$$G_{B(0,1)}(x,y) := \Phi\left(|y|\left|x - \frac{y}{|y|^2}\right|\right), \tag{2.14}$$

which satisfies all the requested properties.

From this we obtain (without proof)

Theorem 2.13 (Poisson's formula for the ball)**.** *Assume $g \in C^0(\partial B(0,r))$. Define*

$$u(x) := c_n \int_{\partial B(0,r)} \frac{1}{r} \frac{r^2 - |x|^2}{|x-\theta|^n} g(\theta)\, d\mathcal{H}^{n-1}(\theta)$$

Then

(1) $u \in C^\infty(B(0,r))$

(2) $\Delta u = 0$ *in* $B(0,r)$

(3)

$$\lim_{B(0,r)\ni x\to x_0} u(x) = g(x_0) \quad \textit{for any } x_0 \in \partial B(0,r)$$

2.5. **Mean Value Property for harmonic functions.** An important property (but very special to the "base Operator Δ", i.e. not that easily generalizable to more general PDEs) is the *mean value property*

Theorem 2.14 (Harmonic functions satisfy Mean Value Property)**.** *Let* $u \in C^2(\Omega)$ *such that* $\Delta u = 0$*, then*

$$u(x) = \fint_{\partial B(x,r)} u(z)\, d\mathcal{H}^{n-1}(z) = \fint_{B(x,r)} u(y)\, dy \tag{2.15}$$

holds for all balls $B(x,r) \subset \Omega$*.*

If $\Delta u \leq 0$ *then we have "*$\geq$*"in (2.15). If* $\Delta u \geq 0$ *then we have "*$\leq$*" in (2.15).*

Proof. Set

$$\varphi(r) := \fint_{\partial B(x,r)} u(y)\, d\mathcal{H}^{n-1}(y).$$

Observe that by substitution $z := \frac{y-x}{r}$ we have

$$\varphi(r) := \fint_{\partial B(0,1)} u(x+rz)\, d\mathcal{H}^{n-1}(z).$$

Taking the derivative in r we have

$$\varphi'(r) = \fint_{\partial B(0,1)} Du(x+rz)\cdot z\, d\mathcal{H}^{n-1}(z).$$

Transforming back we get

$$\varphi'(r) = \fint_{\partial B(x,r)} Du(y)\cdot \frac{y-x}{r}\, d\mathcal{H}^{n-1}(y).$$

Observe that $\frac{y-x}{r}$ is the outer unit normal of $\partial B(x,r)$. That is

$$\varphi'(r) = |\partial B(x,r)|^{-1} \int_{\partial B(x,r)} \partial_\nu u(y) d\mathcal{H}^{n-1}(y).$$

By Stokes or Green's theorem (aka, integration by parts)

$$\varphi'(r) = |\partial B(x,r)|^{-1} \int_{B(x,r)} \Delta u(y) dy \overset{(2.15)}{=} 0.$$

That is,

$$\varphi'(r) = 0 \quad \forall r \text{ if } B(x,r) \subset \Omega.$$

which implies that φ is constant, and in particular

$$\varphi(r) = \lim_{\rho\to 0} \varphi(\rho).$$

But (Exercise 2.6) for continuous functions u,

$$\lim_{\rho\to 0}\varphi(\rho) = \lim_{\rho\to 0}\fint_{\partial B(x,\rho)} u(y)\, d\mathcal{H}^{n-1}(y) = u(x),$$

we have shown that

$$u(x) = \fint_{\partial B(x,r)} u(y)\, d\mathcal{H}^{n-1}(y) \tag{2.16}$$

holds whenever $B(x,r) \subset \Omega$.

Moreover, by Fubini's theorem

$$\begin{aligned}
\fint_{B(x,r)} u(y)\, dy = &\frac{1}{|B(x,r)|}\int_0^r \int_{\partial B(x,\rho)} u(\theta)\, d\mathcal{H}^{n-1}(\theta)\, d\rho\\
=&\frac{1}{|B(x,r)|}\int_0^r |\partial B(x,\rho)| \fint_{\partial B(x,\rho)} u(\theta)\, d\mathcal{H}^{n-1}(\theta)\, d\rho\\
\overset{(2.16)}{=}&\frac{1}{|B(x,r)|}\int_0^r |\partial B(x,\rho)|\, u(x) d\rho\\
=&u(x)\frac{1}{|B(x,r)|}\int_0^r \int_{\partial B(x,\rho)} 1\, d\mathcal{H}^{n-1}(\theta) d\rho\\
=&u(x)\frac{|B(x,r)|}{|B(x,r)|}\\
=&u(x).
\end{aligned}$$

Together with (2.16) we have shown the claim for $\Delta u = 0$. The inequality arguments are left as an exercise. □

The converse holds as well (and there is actually a whole literature on "how many balls" one has to assume the mean value property to get harmonicity, cf. [Llorente, 2015, Kuznetsov, 2019])

Theorem 2.15 (Mean Value property implies harmonicity)**.** *Let $u \in C^2(\Omega)$. If for all balls $B(x,r) \subset \Omega$,*

$$u(x) = \fint_{\partial B(x,r)} u(\theta)\, d\mathcal{H}^{n-1}(\theta) \tag{2.17}$$

then

$$\Delta u = 0 \quad \textit{in } \Omega$$

Proof. Assume the claim is false.

Then there exists some $x_0 \in \Omega$ such that $\Delta u(x_0) \neq 0$, so (by continuity of Δu) w.l.o.g. $\Delta u > 0$ in a small neighborhood $B(x_0, R)$ of x_0.

On the other hand, setting as above

$$\varphi(r) := \fint_{\partial B(x_0,r)} u(\theta) \overset{(2.17)}{\equiv} u(x_0)$$

we have $\varphi'(r) = 0$ for all $r > 0$ such that $B(x_0, r) \subset \Omega$. But as computed before, for $r < R$,

$$\varphi'(r) = C(r) \int_{B(x_0,r)} \Delta u \, dy > 0.$$

This $(0 = \varphi'(r) > 0)$ is a contradiction, so the claim is established. □

2.6. **Maximum and Comparison Principles.** The mean value property as above is very rigid in the sense that it holds only for very special operators such as the Laplacian. A much more general property (which for the Laplacian Δ is a direct consequence of the mean value property) are *maximum principles*, which should be seen as a "forced convexity/concavity property" for sub-/supersolutions of a large class of PDEs (2nd order elliptic).

In one-dimension a subsolution of Laplace's equation satisfies

$$u'' \geq 0$$

that is, subsolutions are exactly the convex C^2-functions.

On the hand, if $u : \Omega \to \mathbb{R}$ is a smooth convex function, then $D^2u(x) \geq 0$ (in the sence of matrices), so $\Delta u = \mathrm{tr} D^2 u = \sum(eigenvalues of D^2 u) \geq 0$.

On the other hand, the converse does not hold: if we take $u(x, y) = 2x^2 - y^2$ then u is not convex, but $\Delta u \geq 0$.

Still, subsolutions have some properties of convex functions (and supersolutions have some properties of concave functions): comparison principles:

Convexity means that on any interval (a, b) the maximum of u is obtained at a or at b – and if the maximum is obtained in a point $c \in (a, b)$ then u is constant. The curious fact is that these properties still hold in arbitrary dimension for solutions of the Laplace equation (and later a large class of elliptic 2nd order equations), they are the so-called weak maximum principle and strong maximum principle.

There is also a "physical" way to explain maximum principles: For example, assume that a solid Ω is heated from the sides with a heat source $g : \partial\Omega \to \mathbb{R}$ and assume there is some heat source from the middle, but it only subtracts heat, $-\Delta u \leq 0$, then what is the maximal heat at any point in the interior (letting the system become stationary)? well the maximum heat in the inside is the heat at the boundary (*weak maximum principle*). And if the heat at any point in the interior is exactly the maximum value of the heat, since the system is stationary, if it is colder at any other point then the heat would have distributed to that point – meaning any other point must have the same heat (*strong maximum principle*).

Corollary 2.16 (Strong Maximum-principle)**.** *Let $u \in C^2(\Omega)$ be subharmonic, i.e. $\Delta u \geq 0$ in Ω. If there exists $x_0 \in \Omega$ at which u attains a global maximum then u is constant in the connected component of Ω containing x_0.*

Proof. By taking a possibly smaller Ω we can assume w.l.o.g. Ω is connected and u still attains its global maximum in $x_0 \in \Omega$.

Let

$$A := \{y \in \Omega : u(y) = u(x_0)\}.$$

We will show that $A = \Omega$ (and thus u is constant) by showing that the following three properties hold

- A is nonempty
- A is relatively closed (in Ω).
- A is open

Then A is an open and closed set in Ω, and since A is not the empty set it is all of Ω.

Clearly A is nonempty since $x_0 \in A$.

Also A is relatively closed by continuity of u: If $\Omega \ni y_k \xrightarrow{k\to\infty} y_0 \in \Omega$ then

$$u(y_0) = \lim_{k\to\infty} u(y_k) = u(x_0)$$

and thus $y_0 \in A$.

To show that A is open let $y_0 \in A$. Since Ω is open we can find a small ball $B(y_0, \rho) \subset \Omega$.

Observe that x_0 is a global maximum of u in $B(y_0, \rho)$.

The mean value property, Theorem 2.14, and then the fact that $u(x_0) \geq u(y)$ for all y in $B(y_0, \rho)$, imply

$$u(x_0) = u(y_0) \leq \fint_{B(y_0,\rho)} u(y)\, dy \leq \fint_{B(y_0,\rho)} u(x_0)\, dy = u(x_0).$$

Since left-hand side and right-hand side coincide the inequality is actually an equality.

That is, we have

$$u(x_0) = \fint_{B(y_0,\rho)} u(y)\, dy,$$

in other words

$$\fint_{B(y_0,\rho)} u(y) - u(x_0)\, dy = 0.$$

Since $u(y) - u(x_0)$ by assumption ≤ 0 the above integral becomes

$$-\fint_{B(y_0,\rho)} |u(y) - u(x_0)|\, dy = 0.$$

that is

$$u(y) \equiv u(x_0) \quad \text{in } B(y_0, \rho),$$

that is $B(y_0, \rho) \subset A$. That is, A is open. $\square$

Remark 2.17. The statement of Corollary 2.16 is false if one replaces global with local maximum (even though local maxima are locally global maxima). A counterexample is for example

$$u(x) := \begin{cases} 0 & x \leq 0 \\ x^3 & x > 0 \end{cases}$$

Then $u \in C^2(\mathbb{R})$ and

$$\Delta u = u'' \geq 0 \quad \text{in } (-1, 1)$$

Clearly u attains several local maxima, namely in $(-1, 0)$ we have $u \equiv 0$, but also clearly u is not constant. The argument above in the proof of Corollary 2.16 fails, since the point 0 is not a local maximum, and thus the set

$$A := \{x \in (-1, 1) : u(x) = 0\}$$

is not open.

For the next statement, and henceforth, we use the notation $A \subset\subset B$ ("A is compactly contained in B) which means that A is bonded and its closure $\overline{A} \subset B$. I.e. for two open sets A, B the condition $A \subset\subset B$ means in particular that ∂A has positive distance from ∂B.

Corollary 2.18 (Weak maximum principle)**.** *Let $\Omega \subset\subset \mathbb{R}^n$ and $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ be subharmonic, i.e. $-\Delta u \leq 0$ in Ω. Then*

$$\sup_{\Omega} u = \sup_{\partial\Omega} u,$$

i.e. "the maximal value is attained at the boundary"[3].

Remark 2.19. This statement also holds on unbounded sets Ω, one just has to define the meaning of $\sup_{\partial\Omega}$ in a suitable sense (i.e. "$\sup_{\partial\mathbb{R}^n}$" should be interpreted as $\limsup_{|x|\to\infty}$).

Proof of Corollary 2.18. Clearly by continuity

$$\sup_{\Omega} u \geq \sup_{\partial\Omega} u.$$

To prove the converse let us argue by contradiction and assume that

$$\sup_{\Omega} u > \sup_{\partial\Omega} u. \tag{2.18}$$

Since u is continuous and Ω bounded this must mean that there exists a local maximum point $x_0 \in \Omega$ such that

$$u(x_0) = \sup_{\Omega} u > \sup_{\partial\Omega} u. \tag{2.19}$$

[3] again: think of convex functions which do have this property

But in view of Corollary 2.16 (strong maximum principle) u is then constant on the connected component of Ω containing x_0. But this implies that on the boundary of this connected component the value of u is still $u(x_0)$, which implies

$$\sup_{\partial\Omega} u \geq u(x_0).$$

But this contradicts the assumption (2.19). □

Remark 2.20. A particular consequence of the strong maximum principle is the following. If for $\Omega \subset\subset \mathbb{R}^n$ we have $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ satisfying

$$\begin{cases} \Delta u \geq 0 & \text{in } \Omega \\ u = g & \text{on } \partial\Omega \end{cases}$$

for some $g \in C^0(\partial\Omega)$. Then the following (equivalent) statements are true:

- If $g \leq 0$ but $g \not\equiv 0$ on $\partial\Omega$ then we have that $u < 0$ in all of Ω.
- If $g \leq 0$ then either $u \equiv 0$ or $u < 0$ everywhere in Ω.

Such a behaviour is special to the PDEs of order two. Even for

$$\Delta^2 u = \Delta(\Delta u) = 0 \quad \text{in } \Omega$$

the above statement may not hold (see e.g. [Gazzola et al., 2010]).

Corollary 2.21 (Strong Comparison Principle)**.** *Let $\Omega \subset\subset \mathbb{R}^n$ open and connected. Assume that $u_1, u_2 \in C^2(\Omega) \cap C^0(\overline{\Omega})$ satisfy*

$$\Delta u_1 \geq \Delta u_2 \quad \text{in } \Omega.$$

If $u_1 \leq u_2$ on $\partial\Omega$, then exactly one of the following statements is true

1. *either $u_1 \equiv u_2$*
2. *or $u_1(x) < u_2(x)$ for all $x \in \Omega$.*

Proof. Let $w := u_1 - u_2$, then we have

$$\begin{cases} \Delta w \geq 0 & \text{in } \Omega \\ w \leq 0 & \text{in } \partial\Omega \end{cases}$$

The claim now follows from Remark 2.20. □

The maximum principle is a great tool to get uniqueness for linear equations!

Theorem 2.22 (Uniqueness for the Dirichlet problem)**.** *Let $\Omega \subset\subset \mathbb{R}^n$, $f \in C^0(\Omega)$ and $g \in C^0(\partial\Omega)$ be given. Then there is at most(!) one solution $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ of*

$$\begin{cases} \Delta u = f & \text{in } \Omega \\ u = g & \text{on } \partial\Omega \end{cases}$$

Proof. Assume there are two solutions, u, v solving this equation. If we set $w := u - v$ then w is a solution to the equation

$$\begin{cases} \Delta w = 0 & \text{in } \Omega \\ w = 0 & \text{on } \partial\Omega \end{cases}$$

In view of Corollary 2.18 we then have

$$\sup_{\Omega} w \leq \sup_{\partial\Omega} w = 0.$$

That is, $w \leq 0$ in Ω. But observe that $-w$ solves the same equation, which implies that

$$\sup_{\Omega}(-w) \leq \sup_{\partial\Omega}(-w) = 0,$$

that is $-w \leq 0$ in Ω. But this readily implies that $w \equiv 0$ in Ω, that is $v \equiv w$. $\square$

So comparison principles are a fantastic tool for obtaining uniqueness for PDEs. Let us also note that via the so-called *Perron's method* (which relies heavily on maximum principles) we also can obtain existence, Section 2.8. But first we need another comparison principle: Harnack inequality

2.7. **Harnack Principle.** Above we learned, e.g. in Corollary 2.16 of the strong maximum principle. Another type of maximum principle is the Harnack inequality.

Theorem 2.23. *Let $\Omega \subset \mathbb{R}^n$ open. For any open, connected, and bounded $U \subset\subset \Omega$ there exists a constant $C = C(U, \Omega)$ such that for any solution $u \in C^2(\Omega)$ with $u \geq 0$ and such that*

$$\Delta u = 0 \quad \textit{in } \Omega$$

we have

$$\sup_{U} u \leq C \inf_{U} u$$

Proof. The proof is based on the mean value formula, Theorem 2.14, namely for any $x \in U$ and any $r < \operatorname{dist}(U, \partial\Omega)$ we have

$$u(x) = \fint_{B(x,r)} u(z)\, dz$$

Let now $R := \frac{1}{4}\operatorname{dist}(U, \partial\Omega)$. For any $x_0 \in U$ and any $x \in B(x_0, R)$ we have (here we use $u \geq 0$ and that $B(x, R) \subset B(y, 2R)$ for $x, y \in B(x_0, R)$)

$$u(x) = \fint_{B(x,R)} u(z)\, dz \leq 2^n \fint_{B(y,2R)} u(z)\, dz = 2^n\, u(y).$$

Again, this holds for any $x, y \in B(x_0, R)$. Taking the supremum for $x \in B(x_0, R)$ and the infimum on $y \in B(x_0, R)$ we get

$$\sup_{B(x_0,R)} u \leq 2^n \inf_{B(x_0,R)} u. \tag{2.20}$$

That is we have the Harnack principle on any Ball $B(x_0,R)$. Since U is bounded, open and compactly contained in Ω we can now cover all of U by finitely many balls $(B_\ell)_{\ell=1}^N$ which lie inside Ω centered at points in U and of radius R.

Take any $i_1, i_2 \in \{1,\dots,N\}$ and assume that $B_{i_1} \cap B_{i_2} \neq \emptyset$. Since then $\inf_{B_{i_1}} u \le \sup_{B_{i_2}} u$ Harnack's principle on the ball B_{i_1} and the ball B_{i_2} implies

$$\sup_{B_{i_1}} u \le 2^{2n} \inf_{B_{i_2}} u \quad \text{whenever } B_{i_1} \cap B_{i_2} \neq \emptyset.$$

Repeating the same argument, assume now that $i_1, i_2, i_3 \in \{1,\dots,N\}$ such that $B_{i_1} \cap B_{i_2} \neq \emptyset$ and $B_{i_2} \cap B_{i_3} \neq \emptyset$. Then

$$\sup_{B_{i_1}} u \le 2^{2n} \inf_{B_{i_2}} u \le 2^{2n} \sup_{B_{i_3}} u \le 2^{4n} \inf_{B_{i_3}} u \quad \text{whenever } B_{i_1} \cap B_{i_2} \neq \emptyset \text{ and } B_{i_2} \cap B_{i_3} \neq 0.$$

By induction we readily conclude the following fact: Whenever we have $i, j \in \{1,\dots,N\}$ such that there are $i_1,\dots,i_K \in \{1,\dots,N\}$ with $i_1 = i$ and $i_K = j$ and $B_{i_\ell} \cap B_{i_{\ell+1}} \neq \emptyset$ for all ℓ then we have

$$\sup_{B_i} u \le 2^{2nK} \inf_{B_j} u.$$

Cf. Figure 2.2. Since U is connected and it is covered by N balls we conclude that

$$\sup_U u \le \sup_{i\in\{1,\dots,N\}} \sup_{B_i} u \le 2^{2nN} \inf_{j\in\mathbb{N}} \inf_{B_j} u \le 2^{2nN} \inf_U u.$$

Observe that N heavily depends on $U \subset\subset \Omega$ – and the closer the boundary of U is to Ω, the larger N tends to be. Thus we have shown that

$$\sup_U u \le C(U,\Omega) \inf_U u.$$

□

We observe from the proof above that we can proof Harnack inequality on a ball with a uniform constant.

Corollary 2.24. *For any dimension $n \in \mathbb{N}$ there exists a constant $C = C(n)$ such that the following holds:*

Let $B(x_0,R)$ be a ball. If $u \in C^2(B(x_0,R))$ with $u \ge 0$ in $B(x_0,R)$ satisfies

$$\Delta u = 0 \quad \text{in } B(x_0,R)$$

then

$$\sup_{B(x_0,R/2)} u \le C \inf_{B(x_0,R/2)} u$$

Exercise 2.25. *Let $\Omega \subset \mathbb{R}^n$ be any open set. Assume there is $u \in C^0(\Omega)$ such that*

$$u \ge 0 \quad \text{in } \Omega$$

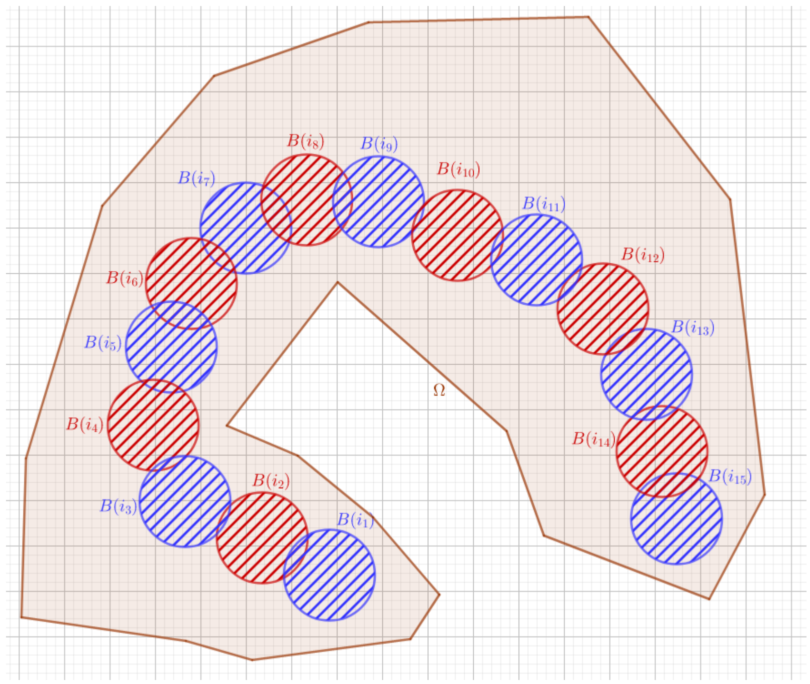


FIGURE 2.2. From Harnack's inequality on balls we can conclude Harnack's inequality on any set $U \subset\subset \Omega$: Harnack's principle repeatedly applied on balls implies $\sup_{B_{i_1}} u \leq 2^{2\cdot 15} \inf_{B_{i_{15}}} u$ (as long as each ball is small enough, so that e.g. twice the ball is in Ω). Any set $U \subset\subset \Omega$ can be covered by finitely many such small balls. So we have $\sup_U u \leq C(U,\Omega) \inf_U u$.

.

and for some $\lambda \in (0,1)$ *and* $\Lambda > 1$ *we know that*

$$u(x) \leq \Lambda \fint_{B(x,r)} u$$

and

$$u(x) \geq \lambda \fint_{B(x,r)} u$$

holds for all $x \in \Omega$ *with* $B(x,r) \subset\subset \Omega$.

Show that there exists a constant $C > 0$ *only depending on* n, λ, Λ *such that*

$$\sup_{B(y,\rho)} u \leq C \inf_{B(y,\rho)} u$$

holds for all balls $B(y, 2\rho) \subset \Omega$.

An important consequence of Harnack inequality is that it implies Hölder continuity. This is of course more relevant if we do not a priori that $u \in C^2$ – but we still illustrate this, because this principles applies to many equations.

Example 2.26 (Harnack implies Hölder estimates)**.** Assume

$$\Delta u = 0 \quad \text{in } \Omega$$

For $r > 0$ and any $x\Omega$ such that $B(x, 2r) \subset \Omega$.

$$M(x_0, r) := \sup_{B(x_0,r)} u, \quad m(x_0, r) := \inf_{B(x_0,r)} u.$$

(*We assume both values are finite*)

Then

$$\Delta(M(x_0, r) - u) = 0$$

and $M(x_0, r) - u \geq 0$ in $B(x_0, r)$ so we have from Harnack's inequality Corollary 2.24 for a *uniform* constant C,

$$\sup_{B(x_0,r/2)} (M(x_0, r) - u) \leq C \inf_{B(x_0,r/2)} (M(x_0, r) - u),$$

and thus

$$M(x_0, r) - m(x_0, r/2) \leq C (M(x_0, r) - M(x_0, r/2)).$$

Similarly,

$$\sup_{B(x_0,r/2)} (u - m(x_0, r)) \leq C \inf_{B(x_0,r/2)} (u - m(x_0, r)),$$

and thus

$$M(x_0, r/2) - m(x_0, r) \leq C (m(x_0, r/2) - m(x_0, r)).$$

We add those two equations

$$M(x_0, r/2) - m(x_0, r) + M(x_0, r) - m(x_0, r/2) \leq C (m(x_0, r/2) - m(x_0, r) + M(x_0, r) - M(x_0, r/2)).$$

and thus

$$\begin{aligned} M(x_0, r/2) - m(x_0, r/2) \leq& M(x_0, r/2) - m(x_0, r/2) \underbrace{-m(x_0, r) + M(x_0, r)}_{\geq 0} \\ \leq& C (M(x_0, r) - m(x_0, r) - (M(x_0, r/2) - m(x_0, r/2))). \end{aligned}$$

And thus we have

$$M(x_0, r/2) - m(x_0, r/2) \leq C (M(x_0, r) - m(x_0, r) - (M(x_0, r/2) - m(x_0, r/2))).$$

whic by absorbing becomes

$$(C + 1) (M(x_0, r/2) - m(x_0, r/2)) \leq C (M(x_0, r) - m(x_0, r)).$$

That is

$$(M(x_0, r/2) - m(x_0, r/2)) \leq \frac{C}{C+1} (M(x_0, r) - m(x_0, r)).$$

Set

$$\gamma := \frac{C}{C+1} < 1.$$

If we then set the *oscillation*

$$\operatorname*{osc}_{B(x_0,r)} u := M(x_0, r) - m(x_0, r),$$

we have shown

$$\operatorname*{osc}_{B(x_0,r/2)} u \leq \gamma \operatorname*{osc}_{B(x_0,r)} u.$$

We can *iterate* this: for any $k \in \mathbb{N}$ we have

$$\operatorname*{osc}_{B(x_0,r/2^k)} u \leq \gamma^k \operatorname*{osc}_{B(x_0,r)} u.$$

Now let $\rho < r$, then there exists exactly one $k \in \mathbb{N}$ such that $\rho \in [r/2^{k-1}, r/2^k)$. And we have (the oscillation is monotone, Exercise 2.28)

$$\operatorname*{osc}_{B(x_0,\rho)} u \leq \operatorname*{osc}_{B(x_0,2^k r)} u \leq \gamma^k \operatorname*{osc}_{B(x_0,r)} u.$$

Now observe that $\gamma = 2^{-\sigma}$ for some $\sigma > 0$. So,

$$\gamma^k = (2^k)^{-\sigma} \lesssim_\sigma \left(\frac{r}{\rho}\right)^{-\sigma} = \frac{\rho^\sigma}{r^\sigma}.$$

Thus we have shown, for any $\rho < r$

$$\operatorname*{osc}_{B(x_0,\rho)} u \leq \frac{\rho^\sigma}{r^\sigma} \operatorname*{osc}_{B(x_0,r)} u.$$

If $B(x_0, 2r) \subset \Omega$ we in particular have

$$\sup_{x_1 \in B(x_0,r)} \operatorname*{osc}_{B(x_1,\rho)} u \leq \frac{\rho^\sigma}{r^\sigma} \operatorname*{osc}_{B(x_0,2r)} u.$$

This implies Hölder continuity, Exercise 2.27.

Exercise 2.27. *Show that if for any $\rho \in (0, r)$ we have*

$$\sup_{x_1 \in B(x_0,r)} \operatorname*{osc}_{B(x_1,\rho)} u \leq C\rho^\sigma,$$

then u is Hölder continuous, namely

$$\sup_{x,y \in B(x_0,r)} \frac{|u(x) - u(y)|}{|x-y|^\sigma} < \infty.$$

Exercise 2.28. *Show that if u is a bounded function then if we set*

$$\operatorname*{osc}_A u := \sup_A u - \inf_A u.$$

Show that if $A \subset B$ then

$$\operatorname*{osc}_A u \leq \operatorname*{osc}_B u.$$

2.8. **Perrons method (illustration).** Comparison principles (weak, strong maximum principle, and Harnack) are not only great for estimates – they can also be used to show existence (for equations that have these comparison principles – which many have not.

To illustrate this we jump a little bit ahead, and recall that we can already solve the Laplace equation in a ball $B(x,R)$ (via the Green's function method, Theorem 2.13). Namely, we shall accept that if $f \in C^0(\partial B_R(y))$ then for a certain constant $c_n > 0$ if we set

$$w(x) := \frac{R^2 - |x-y|^2}{c_n R} \int_{\partial B_R(y)} \frac{f(z)}{|z-x|^n} dz, \quad x \in B_R(y)$$

then $w \in C^0(\overline{B_R(y)}) \cap C^2(B_R(y))$ and

$$\begin{cases} \Delta w = 0 & \text{in } B(y,r) \\ w = f & \text{on } \partial B(y,R). \end{cases}$$

For general open and bounded sets set with smooth boundary $\partial\Omega$, it is not so easy to get an explicity formula. But one can use Perrons method and *local* replacements to *show existence of solutions* of

$$\begin{cases} \Delta u = 0 & \text{in } \Omega \\ u = g & \text{on } \partial\Omega \end{cases}$$

where $g \in C^0(\partial\Omega)$.

First we extend the notion of solution and subsolution to upper- and lowercontinuous functions.

Definition 2.29. Let $\Omega \subset \mathbb{R}^n$ open.

(1) A function $f : \Omega \to (-\infty, \infty)$ is called *subharmonic* in Ω if it is continuous and for any $x \in \Omega$, $r > 0$ $B_r(x) \subset \Omega$ we have

$$f(x) \leq \frac{1}{|\partial B_r|} \int_{\partial B_r(x)} f(y) dy.$$

(2) A function $f : \Omega \to [-\infty, \infty)$ is called harmonic in Ω if it is continuous and for any $x \in \Omega$, $r > 0$ $B_r(x) \subset \Omega$

$$f(x) = \frac{1}{|\partial B_r|} \int_{\partial B_r(x)} f(y) dy.$$

Similar to Theorem 2.15 one can show that if $u \in C^2$ then subharmonicity as defined above coincides with subharmonicity in the sense of $-\Delta u \leq 0$. One can show that any harmonic function as defined above must be C^2 and thus Theorem 2.15 says that indeed ourt notion of harmonicity coincides with the earlier one.

We now need a first important ingredient: Perron's method works locally, so somehow one has to pass from the notion of local subsolutions to global subsolutions.

Lemma 2.30. *Let $f : \Omega \to \mathbb{R}$ be continuous and assume that for any $x \in \Omega$ ther exists $\overline{r} = \overline{r}(x)$ such that for any $r \in (0, \overline{r}(x))$ we have*

$$f(x) \leq \frac{1}{|\partial B_r|} \int_{\partial B_r(x)} f(y)dy.$$

Then f is subharmonic.

Proof. Denote $\rho(x)$ the maximal value such that

$$\rho(x) := \sup\{\rho > 0 : \quad f(x) \leq \frac{1}{|\partial B_r|} \int_{\partial B_r(x)} f(y)dy \quad \text{for all } r \in (0, \rho)\}.$$

We observe that

$$f(x) \leq \frac{1}{|\partial B_{\rho(x)}|} \int_{\partial B_{\rho(x)}(x)} f(y)dy,$$

which follows from the continuity (for each fixed x and $r > 0$)

$$r \mapsto \frac{1}{|\partial B_r|} \int_{\partial B(x,r)} f(y)dy.$$

Observe also that

$$\lim_{r \to 0^+} \frac{1}{|\partial B_r|} \int_{\partial B(x,r)} f(y)dy = f(x).$$

Now we show that ρ is lower semicontinus, i.e.

$$\liminf_{\Omega \ni y \to x} \rho(y) \geq \rho(x)$$

Indeed, assume that there exists a sequence $y_k \to x_k$ and some $\varepsilon > 0$ such that $\rho(y_k) \leq \rho(x) - \varepsilon$ then there must be some $r_k \leq \rho(x) - \varepsilon$ such that

$$f(y_k) \geq \frac{1}{|\partial B_{r_k}|} \int_{\partial B_{r_k}(x)} f(z)dz + \varepsilon.$$

Clearly $r_k > 0$. Up to taking a subsequence we can assume that $r_k \xrightarrow{k \to \infty} \bar{r} \in [0, \rho(x) - \varepsilon]$. Then we have (by continuity of f)

$$f(x) \geq \frac{1}{|\partial B_{\bar{r}}|} \int_{\partial B_{\bar{r}}(x)} f(z)dz + \varepsilon.$$

This is a contradiction, since $\bar{r} < \rho(x)$. The contradiction also holds if $\bar{r} = 0$, then the integral on the right-hand side would be replaced with $f(x)$.

So we indeed have

$$\liminf_{\Omega \ni y \to x} \rho(y) \geq \rho(x)$$

In particular on any compact subset $K \subset \Omega$, ρ attains its global minimum in some $x_0 \in K$, and $\rho(x_0) > 0$. Call this minimum ρ_{min}.

We need to show that $\rho(x) = \operatorname{dist}(x, \partial\Omega)$ for all $x \in K$ (since $K \subset\subset \Omega$ is arbitrary this implies the claim.

Assume that $x \in K$ and $\rho(x) < \operatorname{dist}(x, \partial\Omega)$. Take $\delta \in (0, \rho_{min})$ such that $R < \rho(x) + \delta < \operatorname{dist}(x, \partial\Omega)$.

Let h be the solution to

$$\begin{cases} \Delta h = 0 & \text{in } B(x,R) \\ h = f & \text{on } \partial B(x,R). \end{cases}$$

We know that h exists, since we are in a ball and have the explicit Poisson formula. We then have that h satisfies the mean value equality, and thus

$$h(x) = |\partial B(R)| \int_{\partial B(x,R)} h = |\partial B(R)| \int_{\partial B(x,R)} f.$$

If only we could show that $h(x) \geq f(x)$ we'd have that

$$f(x) \leq |\partial B(R)| \int_{\partial B(x,R)} f \quad \forall R < \rho(x) + \delta,$$

which contradicts the definition of $\rho(x)$.

How do we show $h(x) \geq f(x)$? This is the maximum principle.

Consider $f - h$. We then have for any $y \in B(x,R)$ and any $r \leq \min\{\rho(y), B(x,R)\}$

$$(f-h)(y) \leq |\partial B(r)|^{-1} \int_{\partial B(r)} (f-h)(z)dz.$$

This rules out that there is any local maximum of $f - h$ anywhere in $B(r)$, and thus there is no local maximum of $f - h$ in $B(x,R)$. In particular we have that

$$f(x) - h(x) \leq \sup_{y \in B(x,R)} (f-h)(y) \leq \sup_{\partial B(x,R)} f - h = 0.$$

Thus $f(x) \leq h(x)$, thus we have shown

$$f(x) \leq |\partial B(R)| \int_{\partial B(x,R)} f \quad \forall R < \rho(x) + \delta,$$

a contradiction to $\rho(x)$. Thus $\rho(x) = \operatorname{dist}(x, \partial\Omega)$ and we can conclude.

□

(Very roughly) the idea of Perrons method is as follows.

Perron: Step 1

Consider the collection of *all* subsolutions (which is a nonempty set)

$$S_g := \left\{ v \in C^0(\overline{\Omega}) : \quad v \leq g \quad \text{on } \partial\Omega,\ v \text{ is subharmonic in } \Omega \right\}$$

We need to show S_g is nonempty. This is easy. Take $v := \min_{x \in \partial\Omega} g(x)$. Then v is constant, so $-\Delta u = 0$ (in particular v is subharmonic). And clearly $v \leq g$ on $\partial\Omega$.

Perron: Step 2

Here comes the trick: let u be simply the *largest* subsolution, for $x \in \overline{\Omega}$

$$u(x) := \sup_{v \in S_g} v(x).$$

The idea is that since u is the largest subsolution, then even locally there cannot be a larger one. Howver if locally u was not harmonic, then we can use a harmonic replacement technique on a ball to get a contradiction.

First we need to ensure that u is well-defined. Here we use the maximum principle, Corollary 2.18 and Corollary 2.16. Observe that these arguments were based on the continuity of a subsolution v and the mean value formula so they still apply to our situation, and we have

$$v(x) \leq \sup_{\partial\Omega} g \quad \forall v \in S_g, \quad \forall x \in \overline{\Omega}.$$

This implies that for each $x \in \overline{\Omega}$ the family $\{v(x) : \quad v \in S_g\}$ has an upper bound, so the supremum is well-defined. That is u is well-defined.

Next we observe that (formally) u is still subharmonic. Let $x \in \Omega$ and consider any ball $B_r(x) \subset \Omega$.

$$u(x) = \sup_{v \in S_g} v(x) \leq \sup_{v \in S_g} \frac{1}{|B_r|} \int_{B_r(x)} v(y)dy \leq \frac{1}{|B_r|} \int_{B_r(x)} \sup_{v \in S_g} v(y)dy \leq \frac{1}{|B_r|} \int_{B_r(x)} u(y)dy.$$

Alas the integral of u may not exist (for all we know u could be non-measurable!). That won't happen, indeed we have

Lemma 2.31. *u is lower semicontinuous, that is*

$$u(x) \leq \liminf_{y \to x} u(y)$$

Think of $u(t) := \sup_{r>0} t^r$ for $t \in [0,1]$ to see that u may not be continuous!

Proof. Fix any $x \in \overline{\Omega}$ and let $\varepsilon > 0$. Then there must be some $\bar{v} \in S_g$ such that

$$u(x) \leq \bar{v}(x) + \varepsilon.$$

Since $\bar{v}$ is continuous, there exists $\delta > 0$ such that for any

$$|\bar{v}(x) - \bar{v}(y)| \leq \varepsilon \quad \forall y \in B(x, \delta) \cap \overline{\Omega}.$$

Consequently, for any $y \in \overline{\Omega}$,

$$u(x) - u(y) \leq \bar{v}(x) - \bar{v}(y) + \varepsilon \leq 2\varepsilon \quad \forall y \in B(x, \delta) \cap \overline{\Omega}.$$

Observe that we cannot do the same argument in the other direction, since x is fixed and y is variable. In any case, now we have

$$u(x) \leq u(y) + 2\varepsilon \forall y \in B(x, \delta) \cap \overline{\Omega}$$

which implies

$$u(x) \leq \liminf_{y \to x} u(y) + \varepsilon.$$

□

The above lemma makes u measurable, and since it is bounded

$$\min_{\partial\Omega} g \leq u(x) \leq \sup_{\partial\Omega} g.$$

u is integrable. But still it does not say that u is a subsolution (because we haven't shown that u is continuous).

Fix now $\bar{x} \in \Omega$. Then there must be a sequence of subharmonic $\tilde{v}_n \in S_g$ such that $\lim_{n\to\infty} \tilde{v}_n(\bar{x}) = u(\bar{x})$. Set

$$v_n(z) := \max\{\tilde{v}_1(x), \tilde{v}_2(x), \ldots, \tilde{v}_n(x), \min_{\partial\Omega} g\}.$$

As a (finite) maximum of continuous functions $v_n \in C^0(\overline{\Omega})$. As we did for u above, we can also easily check that v_n is still a subharmonic function. Moreover we have monotonicity

$$v_n(x) \leq v_{n+1}(x) \quad \forall x \in \overline{\Omega},$$

all while still ensuring $\lim_{n\to\infty} \tilde{v}_n(\bar{x}) = u(\bar{x})$.

Take now a ball $B(\overline{x}, R) \subset \Omega$ ($\overline{x}$ is in the interior of Ω!). We now replace now v_n inside of $B(\overline{x}, R)$ with its harmonic replacment, i.e. we set

$$w_n(x) := \begin{cases} \frac{R^2-|x-\overline{x}|^2}{c_n R} \int_{\partial B_R(\overline{x})} \frac{v_n(z)}{|z-x|^n} dz & x \in B_R(\bar{x}) \\ v_n(x) & x \in \Omega \setminus B_R(\bar{x}). \end{cases}$$

Then we have $w_n \in C^0(\overline{\Omega})$. Since v_n was monotonically increasing, so is w_n,

$$w_n(x) \leq w_{n+1}(x) \quad \forall x \in \overline{\Omega}.$$

Lemma 2.32. *We have the following properties*

(1) $w_n(x) \geq v_n(x)$ *and*
(2) $w_n \in S_g$.

Proof. (1) We have $w_n \equiv v_n$ in $\overline{\Omega} \setminus B(\overline{x}, R)$. Since w_n is harmonic in $B(\overline{x}, R)$ we have $(v - w)$ is subharmonic in $B(\overline{x}, R)$, and since $v - w = 0$ on $\partial B(\overline{x}, R)$ the maximum principle implies $v - w \leq 0$ in $B(\overline{x}, R)$, i.e.

$$v(x) \leq w(x) \quad \forall x \in B(\overline{x}, R).$$

(2) Since $w_n \equiv v_n$ in $\overline{\Omega} \setminus B(\overline{x}, R)$ we have that $w_n(x) \leq g(x)$ for all $x \in \partial\Omega$. We have

$$v_n(x) \leq \frac{1}{|B(r)|} \int_{B(x,r)} v_n(y) dy \overset{(1)}{\leq} \frac{1}{|B(r)|} \int_{B(x,r)} w_n(y) dy$$

So for all $x \in \mathbb{R}^n \setminus B(\bar{x}, R)$, $v_n = w_n$ is subharmonic.

Let now $x \in B(\bar{x}, R)$ (which is open). Since w_n is harmonic for all $r < \operatorname{dist}(x, \partial B(\bar{x}, R))$ we have

$$w_n(x) \le \frac{1}{|B(r)|} \int_{B(x,r)} w_n(y).$$

We conclude that w_n is subharmonic in Ω by Lemma 2.30.

□

Since $w_n \in S_g$ we conclude that

$$v_n(x) \le w_n(x) \le u(x) \quad \forall x \in \overline{\Omega}$$

and thus in particular

$$\lim_{n\to\infty} w_n(\bar{x}) = u(\bar{x}).$$

Lemma 2.33. *For $x \in \overline{B(\bar{x}, R/2)}$ set*

$$w(x) := \lim_{n\to\infty} w_n(\bar{x}).$$

(This exists since w_n is bounded by u and monotonicity). Then w is harmonic in $B(\bar{x}, R/2)$ and $w \le u$ in $B(\bar{x}, R/2)$.

Proof. For each $n \in \mathbb{N}$ we know that w_n is harmonic in $B(\bar{x}, R)$ (by definition).

So $w_n - w_m$ for $n, m \in \mathbb{N}$ is harmonic in $B(\bar{x}, R)$. We want to apply Harnack's inequality, Theorem 2.23, so let us assume $n \ge m$, then we have $w_n - w_m \ge 0$, and thus

$$\sup_{x\in B(\bar{x},R/2)} (w_n(x) - w_m(x)) \le C \inf_{y\in B(\bar{x},R/2)} (w_n(y) - w_n(y)) \quad \forall n \ge m,$$

i.e.

$$\sup_{x\in B(\bar{x},R/2)} |w_n(x) - w_m(x)| \le C \inf_{y\in B(\bar{x},R/2)} |w_n(y) - w_n(y)| \quad \forall n \ge m.$$

In particular,

$$\sup_{x\in B(\bar{x},R/2)} |w_n(x) - w_m(x)| \le C\, |w_n(\bar{x}) - w_m(\bar{x})| \xrightarrow{n,m\to\infty} 0.$$

That is, w_n is a Cauchy sequence with respect to uniform convergence in $\overline{B(\bar{x}, R/2)}$, and since w_n is continuous we conclude that there must be some $w \in C^0(\overline{B(\bar{x}, R/2)})$ such that w is the uniform limit of w_n in $B(\bar{x}, R/2)$.

Since w_n is harmonic in $B(\bar{x}, R/2)$ (in the sense of Definition 2.29(2)), so is w (by the uniform convergence).

Since $w_n(x) \le u(x)$ for all $x \in \overline{\Omega}$ (because $w_n \in S_g$), we conclude that $w(x) = \lim_{n\to\infty} w_n(x) \le u$ for all $x \in \overline{B(\bar{x}, R/2)}$. □

Lemma 2.34. *Take w from Lemma 2.32. Then $w = u$ in $B(\bar{x}, R/2)$.*

Proof. We already know $w \leq u$ from Lemma 2.32.

So assume that there is $\tilde{y} \in B(\bar{x}, R/2)$ such that $w(\tilde{y}) > u(\tilde{y})$.

Since $w(\tilde{y}) = \lim_{n\to\infty} w_n(\tilde{y})$ there must be some n such that

$$w_n(\tilde{y}) > u(\tilde{y}).$$

But this is a contradiction since $w_n \in S_g$, and thus

$$u(\tilde{y}) = \sup_{v\in S_g} v(\tilde{y}) \geq w_n(\tilde{y}) > u(\tilde{y}).$$

We can conclude. □

Corollary 2.35. *Let $u(x) := \sup_{v\in S_g} v(x)$. Then $u \in C^0(\Omega)$ and $\Delta u = 0$ in Ω*

Proof. For every $\bar{x} \in \Omega$ there exists a small neighborhood $B(\bar{x}, R/2)$ where u equals a harmonic function, Lemma 2.34. So u must be harmonic and continuous around any point $x \in \Omega$. □

It remains to show that $u = g$ on $\partial\Omega$.

Lemma 2.36. *Assume that $\partial\Omega \in C^\infty$ and g is continuous in $\partial\Omega$. Let $u(x) := \sup_{v\in S_g} v(x)$, $u \in C^0(\Omega)$ (not yet up to the boundary!) be the harmonic function from before.*

Then $u \in C^0(\overline{\Omega})$ and for any $\theta \in \partial\Omega$ we have

$$\lim_{x\to\theta} u(x) = g(\theta).$$

Proof. Since $u(x) := \sup_{v\in S_g} v(x)$ and $v \in S_g$ must satisfy $v \leq g$ on $\partial\Omega$ we conclude that To see the other direction, we build what is called a *barrier*. A barrier at θ is a continuous function $b \in C^0(\overline{\Omega})$ which is superharmonic (i.e. $-b$ is subharmonic) in Ω and $b(x) \geq 0$ for all $x \in \overline{\Omega}$ and $b(x) = 0$ if and only if $x = \theta$.

Fix $\theta \in \partial\Omega$. Since $\partial\Omega$ is smooth, there exists (nontrivial exercise!) a ball $B(\bar{z}, R) \subset \mathbb{R}^n \setminus \Omega$ such that $\overline{B(\bar{z}, R)} \cap \overline{\Omega} = \{\theta\}$ (this is called the *exterior sphere condition* of $\partial\Omega$).

Here is our barrier function

$$b(x) := \begin{cases} R^{2-n} - |x - \bar{z}|^{2-n} & \text{if dimension } n \geq 3 \\ -\log(R) + \log(|x - \bar{z}|) & \text{if } n = 2. \end{cases}$$

Then $b \in C^\infty(\mathbb{R}^n \setminus \{\bar{z}\})$ and since it involves the fundamental solution we know that $\Delta b(x) = 0$ for all $x \in \mathbb{R}^n \setminus \{\bar{z}\}$. Since $\bar{z} \notin \overline{\Omega}$ we conclude that $\Delta b = 0$ in Ω.

For $x \in \mathbb{R}^n \setminus \overline{B(\bar{z}, R)}$ we have $b(x) > 0$ (observe that $2 - n$ is a negative power!) and we have $b(\theta) = 0$. Since $\overline{B(\bar{z}, R)} \cap \overline{\Omega} = \{\theta\}$ this satisfies the barrier definition in Ω.

Fix $\varepsilon > 0$. Since g is continuous on $\partial\Omega$ there exists $\delta > 0$ such that

$$|g(x) - g(\theta)| < \varepsilon \quad \forall x \in \partial\Omega, |x - \theta| < \delta.$$

Set

$$\lambda := \inf_{z \in \partial\Omega \setminus B(\theta,\delta)} \beta(z) > 0.$$

and

$$\Lambda := 2 \sup_{\partial\Omega} |g| < \infty.$$

$$\bar{v}(x) := g(\theta) - \varepsilon - b(x)\frac{\Lambda}{\lambda}.$$

Then $\bar{v}$ is still harmonic in Ω (in particular it is subharmonic). Moreover for $x \in \partial\Omega$, if $|x - \theta| < \delta$ then

$$\bar{v}(x) - g(x) = g(\theta) - g(x) - \varepsilon - b(x)\frac{\Lambda}{\lambda} \leq |g(\theta) - g(x)| - \varepsilon \leq 0.$$

If on the other hand $|x - \theta| \geq \delta$ then

$$\begin{aligned}\bar{v}(x) - g(x) =& g(\theta) - \varepsilon - g(x) - b(x)\frac{\Lambda}{\lambda} \overset{\frac{b}{\lambda} \geq 1}{\leq} g(\theta) - g(x) - \Lambda \\ \leq& 2\sup_{\partial\Omega} |g| - \Lambda \leq 0.\end{aligned}$$

So we have $\bar{v} \leq g$ on $\partial\Omega$, and thus $\bar{v} \in S_g$. Since $u = \sup_{v \in S_g} v$ we find that

$$u(\theta) \geq \bar{v}(\theta) = g(\theta) - \varepsilon.$$

Since this holds for all $\theta \in \partial\Omega$ we have shown

$$u(x) \geq g(x) - \varepsilon \quad \text{for all } x \in \partial\Omega.$$

This again holds for any $\varepsilon > 0$ so that we have

$$u(x) \geq g(x) \quad \text{for all } x \in \partial\Omega.$$

We conclude that $u(x) = g(x)$ for all $x \in \partial\Omega$ and we can conclude. $\square$

We finally can conclude

Corollary 2.37. *Let $u(x) := \sup_{v \in S_g} v(x)$. Then $u \in C^0(\overline{\Omega})$ and $\Delta u = 0$ in Ω and $u = g$ on $\partial\Omega$.*

Let us summarize some features of Perron's method.

- Perron's method shows existence of solutions via obtaining "a largest subsolution" (a "smallest supersolution" would work similarly) .
- it relies on the ability to locally improve a subsolution to obtain a global solution (but observe that we worked hard to show that a local subsolution everywhere is a global subsolution)

- Perron's method relies *extremely* on comparison principles. Take an equation without comparison principle (e.g. 4th order, or systems of equations), and there is essentially no hope of running this idea.
- Perron's method likes to work with some form of continuity, not differentiability; in particular we need to define a notion of "weak" subsolution that makes sense for continuous functions. This works for second order equations with comparison principles often via theories like *Viscosity solutions*

2.9. **Weak Solutions, Regularity Theory.** Now we look at our first encounter with *distributional solutions*. Let $u \in L^1_{loc}(\Omega)$, that is u is a measurable function on Ω which is integrable on every compactly contained set $K \subset \Omega$, i.e.

$$\int_K |u| < \infty.$$

u certainly has no reason to be differentiable, it might not even be continuous. How on earth are we going to define

$$\Delta u = 0 \quad \text{in } \Omega?$$

The idea is that if $u \in C^2(\Omega)$ then

$$\Delta u = 0 \quad \text{in } \Omega \tag{2.21}$$

is equivalent to saying that

$$\int_\Omega u\, \Delta\varphi = 0 \quad \text{for all } \varphi \in C_c^\infty(\Omega). \tag{2.22}$$

(Recall that $C_c^\infty(\Omega)$ are those smooth functions that have compact support $\operatorname{supp}\varphi \subset\subset \Omega$).

Indeed, for $\varphi \in C_c^\infty(\Omega)$ and $u \in C^2(\Omega)$ we have by integration by parts

$$\int_\Omega u\, \Delta\varphi = \int_\Omega \Delta u\, \varphi.$$

So for $u \in C^2(\Omega)$ we clearly have that (2.22) is equivalent to

$$\int_\Omega \Delta u\, \varphi = 0 \quad \text{for all } \varphi \in C_c^\infty(\Omega). \tag{2.23}$$

Now if (2.21) holds then clearly (2.23) holds.

On the other hand assume that (2.23) holds, but (2.21) is false. That is assume there is $x_0 \in \Omega$ such that (w.l.o.g.)

$$\Delta u(x_0) > 0.$$

Since $u \in C^2(\Omega)$ we have $\Delta u \in C^0(\Omega)$ and thus there exists a ball $B(x_0, r) \subset\subset \Omega$ such that

$$\Delta u > 0 \quad \text{on } B(x_0, r) \tag{2.24}$$

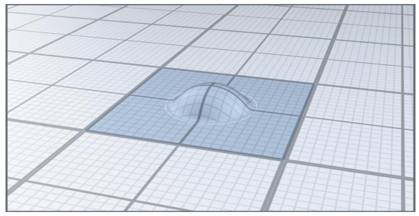


FIGURE 2.3. A bump function

Now let $\varphi \in C_c^\infty(\Omega)$ a *bump function* (or *cutoff function*), namely a function φ such that $\varphi \geq 1$ in $B(x_0, r/2)$ and $\varphi \equiv 0$ in $\Omega \backslash B(x_0, r)$, and $\varphi \geq 0$ everywhere. These bump functions really exist: they can be build by essentially scaled and glued versions of

$$\eta(x) := \begin{cases} e^{-\frac{1}{1-|x|^2}} & \text{for } |x| < 1 \\ 0 & \text{for } |x| > 1 \end{cases}$$

See Figure 2.9.

For this bump function φ we have from (2.24)

$$\int_\Omega \varphi \Delta u > 0$$

which contradicts (2.23). This proves the equivalence of (2.22) and (2.21) for C^2-functions u.

However, we notice that while (2.21) only makes sense for functions u that are twice differentiable, the statement (2.22) makes sense for all functions $u \in L^1_{loc}(\Omega)$. This warrants the following definition:

Definition 2.38 (Weak solutions of the Laplace equation)**.** For a function $u \in L^1_{loc}(\Omega)$ we say that (2.21) is satisfied *in the weak sense* (or in the *distributional* sense) if

$$\int_\Omega u \, \Delta \varphi = 0 \quad \text{for all } \varphi \in C_c^\infty(\Omega). \tag{2.22}$$

holds. The functions φ used to "test" the equation are for this very reason called *test-functions*.

To distinguish the notion of solution we used before, we say that if $\Delta u = 0$ in a differentiable function sense tjem u is a *strong solution* or *classical solution*.

Above, we already have shown the following statement

Proposition 2.39. *Let $u \in C^2(\Omega)$. Then the following two statements are equivalent:*

(1) u is a weak solution to the Laplace equation $\Delta u = 0$ in Ω
(2) u is a classical solution of $\Delta u = 0$ in Ω.

Weyl proved that this equivalence holds for $u \in L^1_{loc}$ (i.e. with no a priori differentiablity at all) – this is our first result of *regularity theory*: showing that weak solutions which are a priori only integrable are actually differentiable. Observe: the reason this works here is that we have a homogeneous equation $\Delta u = 0$, and that Δ is a constant-coefficient linear elliptic operator (and one can spend much more time for proving similar results for more general linear elliptic operators). Having said that, in some sense, the regularity theory for elliptic equations is always somewhat based on the following Theorem, Theorem 2.40 (albeit in a hidden way).

Theorem 2.40 (Weyl's Lemma)**.** *Let $u \in L^1_{loc}(\Omega)$ for $\Omega \subset \mathbb{R}^n$ open. If u is a weak solution of Laplace equation, i.e.*

$$\int_\Omega u\, \Delta\varphi = 0 \quad \text{for all } \varphi \in C^\infty_c(\Omega). \tag{2.22}$$

then $u \in C^\infty(\Omega)$ and Δu in the classical sense.

Observe that this theorem (rightfully) does not say anything about u on $\partial\Omega$, this is a purely interior result!

The proof of Theorem 2.40 exhibits the structure that many proofs in PDE have. First on obtains some *a priori estimates* (namely under the assumption that everything is smooth we find good estimates). Then we show that these estimates hold also for rough solutions by an approximation argument.

The a priori estimates for the Laplace equations are called the Cauchy estimates. These are truly amazing: They say that if we solve the Laplace equation we can estimate *all derivatives, in pretty much any norm* simply by the L^1-norm of the function.

Lemma 2.41 (Cauchy estimates)**.** *Let $u \in C^\infty(\Omega)$ be harmonic, $\Delta u = 0$ in Ω. Then we have for any ball $B(x_0, r) \subset \Omega$ and for any multiindex γ of order $|\gamma| = k$,*

$$|\partial^\gamma u(x_0)| \leq \frac{C_k}{r^{n+k}} \|u\|_{L^1(B(x_0,r))}.$$

In particular we have for any $\Omega_2 \subset\subset \Omega$ that

$$\sup_{\Omega_2} |D^k u| \leq C(\operatorname{dist}(\Omega_2, \Omega), k) \|u\|_{L^1(\Omega)}$$

Proof of the Cauchy estimates, Lemma 2.41. For $k = 0$ we argue with the mean value property for harmonic functions, Theorem 2.15. We have for any ρ such that $B(x_0, \rho) \subset \Omega$ and any $x \in B(x_0, \rho/2)$,

$$|u(x)| = \left| \fint_{B(x,\rho/2)} u(z)\, dz \right| \leq \frac{C}{\rho^n} \int_{B(x,\rho/2)} |u(z)|\, dz \leq \frac{C}{\rho^n} \int_{B(x_0,\rho)} |u(z)|\, dz.$$

That is, we have obtained that for if $\Delta u = 0$ on $B(x_0, \rho)$ then

$$\sup_{B(x_0,\rho/2)} |u| \leq \frac{C}{\rho^n} \|u\|_{L^1(B(x_0,\rho))}. \tag{2.25}$$

This proves in particular the case $k = 0$ (taking $\rho =: r$).

For the case $k = 1$ we use a technique called "*differentiating the equation*" (and in more general situations where this is used in a discretized version we will study later is due to Nirenberg, cf. Section 5.2). Observe that $\Delta u = 0$ in Ω implies

$$\Delta \partial_i u = \partial_i \Delta u = 0 \quad \text{in } \Omega$$

So if we set $v := \partial_i u$ we have that $\Delta v = 0$ in Ω. For $x \in B(x_0, \rho/4)$, again from the mean value property for harmonic functions, Theorem 2.15, we get with an additional integration by parts

$$\begin{aligned}
|\partial_i u(x)| =& \left| \fint_{B(x,\rho/4)} \partial_i u(z)\, dz \right| = \frac{C}{\rho^n} \left| \int_{\partial B(x,\rho/4)} u(\theta)\, \nu^i d\mathcal{H}^{n-1}(\theta) \right| \\
\leq& \frac{C}{\rho^n} \rho^{n-1} \sup_{B(x,\rho/4)} |u| \\
\leq& \frac{C}{\rho^n} \rho^{n-1} \sup_{B(x_0,\rho/2)} |u|
\end{aligned}$$

Now in view of the estimates in the step $k = 0$, namely (2.25), we arrive at

$$\sup_{B(x_0,\rho/4)} |\nabla u(x)| \leq \frac{C}{\rho^{n+1}} \|u\|_{L^1(B(x_0,\rho))}.$$

Differentiating the equation again, we find by induction that (the constant changes in each appearance!)

$$|\nabla^k u(x_0)| \leq \sup_{B(x_0,4^{-k}\rho)} |\nabla^k u(x)| \leq \frac{C}{\rho^{n+1}} \|\nabla^{k-1} u\|_{L^1(B(x_0,4^{1-k}\rho))} \leq \ldots \leq \frac{C}{\rho^{n+k}} \|u\|_{L^1(B(x_0,\rho)}.$$

If we want to show the estimate on $\Omega_2 \subset\subset \Omega$ we now pick $\rho < \operatorname{dist}(\Omega_2, \partial\Omega)$ and obtain the claim. $\square$

Proof of Weyl's Lemma: Theorem 2.40. We use a mollification argument, i.e. we approximate u with smooth functions u_ε that also solve (in the classical sense) the Laplace equation.

Let $\eta \in C_c^\infty(B(0,1))$ be another bump function, this time with the condition $\eta(x) = \eta(-x)$, i.e. η is even, $\eta \geq 0$ everywhere, and normalized such that

$$\int_{\mathbb{R}^n} \eta = 1.$$

We rescale η by a factor $\varepsilon > 0$ and set

$$\eta_\varepsilon(x) := \varepsilon^{-n}\eta(x/\varepsilon).$$

Then the convolution[4] is defined as

$$u_\varepsilon(x) := \eta_\varepsilon * u(x) := \int_{\mathbb{R}^n} \eta_\varepsilon(y-x)\, u(y)\, dy$$

Clearly this is not well-defined for all x, if $u \in L^1_{loc}(\Omega)$ only. But it is defined for all $x \in \Omega$ such that $\operatorname{dist}(x, \partial\Omega) > \varepsilon$, since $\operatorname{supp}\eta_\varepsilon(\cdot - x) \subset B(x,\varepsilon)$.

But observe that derivatives on u_ε hit only the kernel η_ε (which is smooth) (there is a dominated convergence to be used to show that, and for this we need L^1_{loc}!)

$$\partial^\gamma u_\varepsilon(x) := \eta_\varepsilon * u(x) := \int_{\mathbb{R}^n} \partial^\gamma \eta_\varepsilon(y-x)\, u(y)\, dy$$

That is $u_\varepsilon \in C^\infty(\Omega_{-\varepsilon})$ where

$$\Omega_{-\varepsilon} = \{x \in \Omega,\ \operatorname{dist}(x, \partial\Omega) > \varepsilon\}$$

The fun part (which we used above already) is that convolutions behave well with differential operators, namely we will show now that $\Delta u_\varepsilon = 0$ in $\Omega_{-\varepsilon}$:

For this let $\psi \in C^\infty_c(\Omega_{-\varepsilon})$ a testfunction, then we have

$$\int_{\Omega_{-\varepsilon}} u^\varepsilon(x)\, \Delta\psi(x)\, dx = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} u(y)\eta_\varepsilon(x-y)\, \Delta\psi(x)\, dy\, dx = \int_{\mathbb{R}^n} u(y) \int_{\mathbb{R}^n} \eta_\varepsilon(x-y)\, \Delta\psi(x)\, dx\, dy$$

Now, by integration by parts (for any fixed $y \in \mathbb{R}^n$)

$$\int_{\mathbb{R}^n} \eta(x-y)\, \Delta\psi(x)\, dx = \int_{\mathbb{R}^n} \Delta_x\eta_\varepsilon(x-y)\, \psi(x)\, dx = \int_{\mathbb{R}^n} \Delta_y\eta_\varepsilon(x-y)\, \psi(x)\, dx = \Delta_y \int_{\mathbb{R}^n} \eta_\varepsilon(x-y)\, \psi(x)\, dx$$

So if we set

$$\varphi(y) := \eta_\varepsilon * \psi(y) \equiv \int_{\mathbb{R}^n} \eta_\varepsilon(x-y)\, \psi(x)\, dx$$

then we have by the support condition on ψ that $\varphi \in C^\infty_c(\Omega)$, and thus

$$\int_{\Omega_{-\varepsilon}} u^\varepsilon(x)\, \Delta\psi(x)\, dx = \int_{\mathbb{R}^n} u(y)\, \Delta\varphi(y)\, dy \overset{(2.22)}{=} 0.$$

This argument works for any $\psi \in C^\infty_c(\Omega_{-\varepsilon})$, that is u^ε is weakly harmonic in $\Omega_{-\varepsilon}$. But since $u_\varepsilon \in C^\infty(\Omega_{-\varepsilon})$ this implies in view of Proposition 2.39 that in the strong sense

$$\Delta u_\varepsilon = 0 \quad \text{in } \Omega_{-\varepsilon}.$$

So now u_ε is a smooth solution to Laplace's equation, so we use the a priori estimates of Lemma 2.41.

Fix $\Omega_2 \subset\subset \Omega$. Between Ω_2 and Ω we can squeeze two more set Ω_3, and Ω_4,

$$\Omega_2 \subset\subset \Omega_3 \subset\subset \Omega_4 \subset\subset \Omega.$$

[4]we have seen this operation for the Fourier Transform argument above after (2.3), there we used a nonsmooth kernel $|\cdot|^{2-n}$ for the convolution

For any ε small enough, namely

$$\varepsilon < \operatorname{dist}(\Omega_3, \partial\Omega_4) \quad \text{and} \quad \varepsilon < \operatorname{dist}(\Omega_3, \partial\Omega_4)$$

we have that $\Delta u^\varepsilon = 0$ in Ω_3, so by the Cauchy estimates, Lemma 2.41, we have for any $k \in \mathbb{N}$

$$\sup_{\Omega_2} |\nabla^k u_\varepsilon| \leq C(k, \Omega_2, \Omega_3) \, \|u_\varepsilon\|_{L^1(\Omega_3)}.$$

Now we estimate, by Fubini,

$$\|u_\varepsilon\|_{L^1(\Omega_3)} \leq \int_{\Omega_3} \int_{\mathbb{R}^n} |\eta_\varepsilon(x-y)| \, |u(y)| \, dy \, dx = \int_{\mathbb{R}^n} |u(y)| \int_{\Omega_3} |\eta_\varepsilon(x-y)| dx \, dy$$

Since ε is small enough we have that

$$\operatorname{supp} \left(\int_{\Omega_3} |\eta_\varepsilon(x - \cdot)| dx \right) \subset \Omega_4.$$

So we get

$$\|u_\varepsilon\|_{L^1(\Omega_3)} \leq \|u\|_{L^1(\Omega_4)} \sup_{y \in \mathbb{R}^n} \int_{\Omega_3} |\eta_\varepsilon(x-y)| dx \leq \|u\|_{L^1(\Omega_4)} \int_{\mathbb{R}^n} |\eta_\varepsilon(z)| dz.$$

Now we use the definition of η_ε to compute via substitution[5]

$$\int_{\mathbb{R}^n} |\eta_\varepsilon(z)| dz = \varepsilon^{-n} \int_{\mathbb{R}^n} |\eta(z/\varepsilon)| dz = \varepsilon^{-n} \int_{\mathbb{R}^n} |\eta(z/\varepsilon)| dz = \int_{\mathbb{R}^n} |\eta(\tilde{z})| d\tilde{z} = 1.$$

The last equality is due to the normalization of η, $\int \eta = 1$.

That is, we have shown that for any $k \in \mathbb{N} \cup \{0\}$

$$\sup_{\Omega_2} |\nabla^k u_\varepsilon| \leq C(k, \Omega_2, \Omega_3) \, \|u\|_{L^1(\Omega_4)},$$

and the right-hand side is finite since $u \in L^1_{loc}(\Omega)$ and $\Omega_4 \subset\subset \Omega$.

This estimate holds for any $\varepsilon > 0$, so u_ε and all its derivative are uniformly equicontinuous (in ε). By Arzela-Ascoli (and a diagonal argument in k) we find a converging subsequence $\varepsilon \to 0$ and a function $u_0 \in C^\infty(\Omega_2)$ such that for any $k \in \mathbb{N} \cup \{0\}$.

$$|\nabla^k u_\varepsilon(x) - \nabla^k u_0(x)| \xrightarrow{\varepsilon \to 0} 0 \quad \text{locally uniformly in } \Omega_2.$$

We claim that $u = u_0$ in almost every point (since u is an L^1_{loc}-function it is actually a the class of maps equal up to almost every point, u_0 is a continuous representative of the class u). Indeed, by the normalization $\int \eta = 1$ which implies $\int \eta_\varepsilon = 1$ we have

$$|u_\varepsilon(x) - u(x)| = \left| \int \eta_\varepsilon(y - x) \left(u(y) - u(x) \right) \, dy \right| \leq C(\eta) \fint_{B(x,\varepsilon)} |u(y) - u(x)| \, dy.$$

So, by the Lebesgue differentiation theorem, we have for almost every $x \in \Omega_2$,

$$\lim_{\varepsilon \to 0} |u_\varepsilon(x) - u(x)| = 0,$$

[5] observe for $\tilde{z} = z/\varepsilon$ we have in n space dimensions $d\tilde{z} = \varepsilon^{-n} dz$

that is

$$u_0 = u \quad \text{a.e. in } \Omega_2.$$

Thus $u \in C^\infty(\Omega_2)$, and $\Delta u = 0$ in classical sense in Ω_2.

Since this holds for any $\Omega_2 \subset \Omega$ we have shown

$u \in C^\infty(\Omega)$, and $\Delta u = 0$ in classical sense in Ω. □

Corollary 2.42 (Liouville)**.** *Let $u \in C^2(\mathbb{R}^n)$ and $\Delta u = 0$ in all of $\mathbb{R}^n$. If u is a bounded function then $u \equiv const$.*

Proof. Fix $x_0 \in \mathbb{R}^n$. In view of Lemma 2.41 we have for such a function u, for any radius $r > 0$,

$$|Du(x_0)| \leq \frac{C}{r^{n+1}} \|u\|_{L^1(B(x_0,r))}$$

If u is bounded,

$$\|u\|_{L^1(B(x_0,r))} \leq C\, r^n \sup_{\mathbb{R}^n} |u| < \infty$$

and thus

$$|Du(x_0)| \leq C r^{-1} \sup_{\mathbb{R}^n} |u|.$$

This holds for any $r > 0$, so if we let $r \to \infty$, we get

$$|Du(x_0)| = 0,$$

which holds for any $x_0 \in \mathbb{R}^n$. That is, $Du \equiv 0$, and by the fundamental theorem of calculus this means u is a constant. □

The following is also often referred to as Cauchy estimates:

Exercise 2.43. *Assume*

$$\Delta h = 0 \quad \textit{in } B(0,R)$$

Show that for any $\rho < R$ we have

$$\int_{B(0,\rho)} |h - \fint_{B(0,\rho)} h|^p \leq C \left(\frac{\rho}{R}\right)^{n+p} \int_{B(0,R)} |h|^p.$$

Here C is a constant only depending on p and n, but not on ρ, R, h.

Hint: First rescale the problem so that w.l.o.g. $R = 1$.

Then show the inequality for $\rho \geq \frac{1}{2}$ (no harmonicity is needed then).

If $\rho < \frac{1}{2}$ us that

$$|h - \fint_{B(0,\rho)} h| \leq |\rho| \|\nabla h\|_{L^\infty(B(0,1/2))}.$$

Then use the Cauchy estimates, Lemma 2.41.

2.10. **Methods from Calculus of Variations – Energy Methods.** As we have seen, comparison principles is a strong tool for uniqueness (and also existence). These arguments also work in some situations of nonlinear pdes, where the theory of distributional solutions does not work, but the theory of Viscosity solutions can be applied, see [Koike, 2004].

On the other hand, the comparison methods are (currently) restricted to first or second-order equations, and to scalar equations. For systems or higher-order PDEs they seem not to be that helpful.

In this section we have a short look on energy methods, which is a basic tool of distributional theory. They do not rely on any comparison principle, and they are often used for higher-order differential equations and systems. On the other hand for some fully nonlinear equations ("non-variational" equations, equtions "not in divergence form") they cannot be well applied.

The ideas should be reminiscent of the arguments we employed for the weak solutions in Theorem 2.40.

Assume that we have

$$\begin{cases} \Delta u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{2.26}$$

We have seen before Theorem 2.40 that this equation is related to the integral equation

$$\int_\Omega Du \cdot D\varphi + f\varphi = 0 \quad \forall \varphi \in C_c^\infty(\Omega).$$

The interesting point is that this expression is a Frechet-Derivative of a function acting on the map u in direction φ.

Indeed one can characterize solutions as minimizers of an energy functional. This is sometimes called the *Dirichlet principle*.

Theorem 2.44 (Energy Minimizers are solutions and vice versa)**.** *Assume $f \in C^0(\overline{\Omega})$.*

Denote the class of permissible functions

$$X := \left\{ u \in C^2(\overline{\Omega}), \quad u = 0 \quad \textit{on } \partial\Omega \right\}$$

and define the energy

$$\mathcal{E}(u) := \int_\Omega \frac{1}{2}|Du|^2 + fu.$$

Let $u \in X$ be a minimizer of $\mathcal{E}$ in X, i.e.

$$\mathcal{E}(u) \leq \mathcal{E}(v) \quad \forall v \in X.$$

Then u solves (2.26).

Conversely, if $u \in X$ solves (2.26), then u is a minimizer of $\mathcal{E}$ in the set X.

Proof. We compute what is called the *Euler-Lagrange equation*(s) of $\mathcal{E}$: Let $\varphi \in C_c^\infty(\Omega)$, then certainly $u + t\varphi \in X$ for all $t \in \mathbb{R}$. That is the minimizing property says that the function

$$E(t) := \mathcal{E}(u + t\varphi)$$

has a minimum in $t = 0$. By Fermats theorem (one checks easily that E is differentiable in t)

$$\frac{d}{dt}\Big|_{t=0} E(t) \equiv E'(0) = 0.$$

Now observe that

$$\frac{d}{dt}\Big|_{t=0} |D(u + t\varphi)|^2 = 2\langle Du, D\varphi\rangle$$

and

$$\frac{d}{dt}\Big|_{t=0} f\,(u + t\varphi) = f\,\varphi.$$

Thus, we arrive at

$$0 = \frac{d}{dt}\Big|_{t=0} E(t) = \int_\Omega Du \cdot D\varphi + f\varphi = 0.$$

That is, u is a *weak solution* of (2.26). But $u \in C^2(\overline{\Omega})$, so we argue similar to the proof of Proposition 2.39:

By an integration by parts (for $\varphi \in C_c^\infty(\Omega)$ there are no boundary terms), we thus have

$$0 = \int_\Omega Du \cdot D\varphi + f\varphi = 0 = -\int_\Omega (\Delta u - f)\varphi.$$

Since $\Delta u - f$ is continuous, and the last estimate holds for any smooth $\varphi \in C_c^\infty(\Omega)$ we get that (as for Proposition 2.39, or otherwise by the fundamental lemma of calculus of variations, Lemma 2.45,

$$\Delta u - f = 0.$$

That is the first claim is proven: minimizers are solutions.

For the converse assume u solves (2.26). Let w be any other map in X. Then we have

$$\int_\Omega (\Delta u - f)(u - w) = 0.$$

Observe that u and w have the same boundary value 0 on $\partial\Omega$. Thus, when we perform the following integration by parts we do not find boundary terms,

$$0 = -\int_\Omega \nabla u \cdot \nabla(u - w) + f(u - w) = 0. \tag{2.27}$$

Now we compute (using Young's inequality or Cauchy-Schwarz $2ab \leq a^2 + b^2$)

$$\begin{aligned}\int_\Omega |\nabla u|^2 + fu \overset{(2.27)}{=} & \int_\Omega \nabla u \cdot \nabla w + fw \\ \leq & \int_\Omega \frac{1}{2}|\nabla u|^2 + \frac{1}{2}|\nabla w|^2 + fw \\ = & \frac{1}{2}\int_\Omega |\nabla u|^2 + \mathcal{E}(w)\end{aligned}$$

Subtracting $\frac{1}{2}\int_\Omega |\nabla u|^2$ from both sides in the estimate above we obtain

$$\mathcal{E}(u) \leq \mathcal{E}(w).$$

That is, we have shown: if u solves the equation, then u is a minimizer. $\square$

Above we have used the following statement for continuous functions. It is worth recording that this works also for locally integrable functions.

Lemma 2.45 (Fundamental Lemma of the Calculus of Variations)**.** *Let $\Omega \subset \mathbb{R}^n$ be any open set and assume $f \in L^1_{loc}(\Omega)$, i.e. for any $\Omega' \subset\subset \Omega$ we have*

$$\int_{\Omega'} |f| < \infty.$$

(1) If

$$\int_\Omega f(x)\,\varphi(x) \geq 0 \quad \text{for all } \varphi \in C^\infty_c(\Omega) \text{ that are nonnegative, } \varphi \geq 0,$$

then

$$f \geq 0 \quad \text{almost everywhere in } \Omega.$$

(2) If

$$\int_\Omega f(x)\,\varphi(x) = 0 \quad \text{for all } \varphi \in C^\infty_c(\Omega) \text{ that are nonnegative, } \varphi \geq 0,$$

then

$$f \equiv 0 \quad \text{almost everywhere in } \Omega.$$

The proof is left as an exercise, it is a combination of convolution arguments as in Theorem 2.40 and the argument used for Proposition 2.39.

Theorem 2.46 (Uniqueness)**.** *Assume $f \in C^0(\overline{\Omega}) \cap L^1(\Omega)$*

Denote the class of permissible functions

$$X := \left\{u \in C^2(\overline{\Omega}), \quad u = 0 \quad \text{on } \partial\Omega\right\}$$

Then there is at most one solution $u \in X$ to (2.26)

Proof. Assume $u, w \in X$ are two solutions, then

$$\Delta(u - w) = 0.$$

Multiplying by $u - w$ and integrating by parts (observe that there are no boundary terms since $u = w$ on $\partial\Omega$, we obtain

$$\int_\Omega |\nabla(u - w)|^2 = 0.$$

But this implies $\nabla(u - w) \equiv 0$, so $u - w \equiv$ *const*. Since $u = w$ on the boundary that constant is zero, and $u \equiv w$. $\square$

Exercise 2.47. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with smooth boundary. Assume $f \in C^0(\overline{\Omega})$ and $A \in C^2(\overline{\Omega}, \mathbb{R}^{n\times n})$, A symmetric, and all eigenvalues strictly positive in $\overline{\Omega}$, and let $c \in C^0(\overline{\Omega})$.*

Denote the class of permissible functions

$$X := \left\{ u \in C^2(\overline{\Omega}),\ u = 0 \text{ on } \partial\Omega \right\}$$

and define the energy

$$\mathcal{E}(u) := \int_\Omega \frac{1}{2} \langle ADu, Du \rangle_{\mathbb{R}^n} + \frac{1}{2} \int c|u|^2 + fu.$$

Let $u \in X$ be a minimizer of E in X, i.e.

$$\mathcal{E}(u) \leq E(v) \quad \forall v \in X.$$

Then u solves

$$\begin{cases} \operatorname{div}(A\nabla u) - cu = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{2.28}$$

Conversely, if $u \in X$ solves (2.28), then u is a minimizer of $\mathcal{E}$ in the set Y.

These methods can be extended, e.g. for higher order differential equations (where no maximum principle holds), e.g. the *Neumann* boundary problem. Let $\nu : \partial\Omega \to \mathbb{R}^n$ be the outwards facing unit normal. The *Neumann problem* is the equation

$$\begin{cases} \Delta u = f & \text{in } \Omega \\ \partial_\nu u = 0 & \text{on } \partial\Omega \end{cases} \tag{2.29}$$

Exercise 2.48. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with smooth boundary. Assume $f \in C^0(\overline{\Omega})$.*

Denote the class of permissible functions

$$Y := \left\{ u \in C^2(\overline{\Omega}) \right\}$$

and define the energy

$$\mathcal{E}(u) := \int_\Omega \frac{1}{2} |Du|^2 + fu.$$

Let $u \in Y$ be a minimizer of $\mathcal{E}$ in Y, i.e.

$$\mathcal{E}(u) \leq \mathcal{E}(v) \quad \forall v \in Y.$$

Then u solves (2.29).

Conversely, if $u \in Y$ solves (2.26), then u is a minimizer of $\mathcal{E}$ in the set Y.

Exercise 2.49 (Uniqueness modulo constants)**.** *Let $\Omega \subset \mathbb{R}^n$ be a connected, bounded, open set with smooth boundary. Assume $f \in C^0(\overline{\Omega})$. Assume $f \in C^0(\overline{\Omega})$*

Denote the class of permissible functions

$$Y := \left\{u \in C^2(\overline{\Omega})\right\}$$

Then any two solutions $u, v \in Y$ to (2.29) must satisfy $u - v \equiv constant$

2.11. **Linear Elliptic equations.** From now on we often use the *Einstein summation convention*, often described as "summing over repeated indices". We write

$$a_{ij}\partial_{ij}u \quad \Leftrightarrow \sum_{i,j} a_{ij}\partial_{ij}u.$$

$$b_i\partial_i u \quad \Leftrightarrow \sum_{i} b_i\partial_i u.$$

bul

$$b_i\partial_j u \quad \not\Leftrightarrow \sum_{i,j} b_i\partial_j u.$$

In particular

$$\Delta u \quad \Leftrightarrow \quad \partial_{ii}u.$$

Second order elliptic equations are a class of equations that in some sense are *governed* by the Laplacian operator.

Definition 2.50 (Linear elliptic equations)**.**

(1) ("non-divergence form") linear second order operators are defined to be operators of the form

$$L := a_{ij}\partial_{ij} + b_i\partial_i + c$$

for coefficents $a_{ij}, b_i, c : \Omega \to \mathbb{R}$. They act as follows on functions $u \in C^2(\Omega)$

$$Lu(x) := a_{ij}(x)\partial_{ij}u(x) + b_i(x)\partial_i u(x) + c(x)\, u(x).$$

L is called a constant coefficient operator, if the coefficients a_{ij}, b_i and c are all constant.

(2) ("*divergence form*") linear second order operators are defined to be operators of the form

$$L := \partial_i\left(a_{ij}\partial_j\right) + b_i\partial_i + c$$

for coefficents $a_{ij}, b_i, c : \Omega \to \mathbb{R}$. They act as follows on functions $u \in C^2(\Omega)$

$$Lu(x) := \partial_i\left(a_{ij}(x)\partial_j u(x)\right) + b_i(x)\partial_i u(x) + c(x)\, u(x).$$

(3) Clearly, divergence on non-divergence form are very similar if a_{ij} is smooth enough, but they are different if a is not smooth (or, has happens often in applications: a depends on u).

(4) (divergence-form or non-divergence form) operators L are called *elliptic* (also often called uniformly elliptic and bounded) if there exists an *ellipticity constants* $\Lambda > 0$ such that

$$\xi^T A\xi \equiv \xi^i a_{ij}\xi^j \geq \frac{1}{\Lambda}$$

and

$$\sup_\Omega |a_{ij}|, |b_i|, |c| < \infty.$$

For simplicity, although this is not strictly necessary we will below always assume A is symmetric.

Example 2.51.
- The operator Δ is clearly elliptic in the above sense, with

$$a_{ij} = \delta_{ij} := \begin{cases} 1 & \text{if } i = j \\ 0 & \text{else} \end{cases}$$

- Operators like $\operatorname{div}(|\nabla u|^{p-2}\nabla u)$ are not (uniformly elliptic), since $|\nabla u| = 0$ cannot be excluded. These operators are called degenerate elliptic.

Definition 2.52. $u \in C^2(\Omega)$ is called a *subsolution* of $-Lu = f$ for an elliptic operator L, if

$$-Lu \leq 0 \quad \text{in } \Omega$$

and a *supersolution* if

$$-Lu \geq 0 \quad \text{in } \Omega.$$

$u \in C^2(\Omega)$ is called a solution if it is both sub- and supersolution.

In the following we will restrict ourselves to elliptic non-divergence operators!

2.12. **Maximum principles for linear elliptic equations.** The first result is a generalization of the weak maximum principle for Δ, Corollary 2.18.

Theorem 2.53 (Weak maximum principle for $c = 0$)**.** *Let $\Omega \subset\subset \mathbb{R}^n$, $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ be an L-subsolution, i.e.*

$$-Lu \leq 0 \quad \text{in } \Omega \tag{2.30}$$

If L is (non-divergence form) linear elliptic operator with $c \equiv 0$, then

$$\sup_\Omega u = \sup_{\partial\Omega} u.$$

If instead of (2.30) we have

$$-Lu \geq 0 \quad \text{in } \Omega$$

then

$$\inf_\Omega u = \inf_{\partial\Omega} u.$$

Proof. First we assume instead of (2.30)

$$-Lu > 0 \quad \text{in } \Omega \tag{2.31}$$

Clearly, by continuity of u in $\overline{\Omega}$,

$$\sup_{\Omega} u \geq \sup_{\partial\Omega} u$$

If we had

$$\sup_{\Omega} u > \sup_{\partial\Omega} u,$$

then we would find the global (and thus a local) maximum $x_0 \in \Omega$, at which we have $Du(x_0) = 0$ and $D^2u(x_0) \leq 0$. But this implies (recall $c \equiv 0$)

$$Lu(x_0) = a_{ij}(x_0)\partial_{ij}u(x_0) + b_i(x_0)\underbrace{\partial_i u(x_0)}_{=0}$$

Since $a_{ij}(x_0)$ is elliptic, and $\partial_{ij}u(x_0) \geq 0$ we have

$$a_{ij}(x_0)\partial_{ij}u(x_0) \geq 0.$$

(This is a general Linear Algebra fact, if A, B are symmetric, nonnegative matrices, then their Hilbert-Schmidt Scalar product $A : B := a_{ij}b_{ij} \geq 0$, Exercise 2.55.) That is, we have

$$Lu(x_0) \geq 0$$

which is a contradiction to (2.31).

We conclude that under the assumption (2.31) we have

$$\sup_{\Omega} u = \sup_{\partial\Omega} u.$$

In order to weak the assumption to (2.31) we consider, for some $\gamma > 0$, $v_\gamma(x) := e^{\gamma x_1}$, where x_1 is the first component of $x = (x_1, \ldots, x_n)$. Observe that

$$Lv_\gamma(x) = \Big(a_{11}(x)\gamma^2 + b_1(x)\gamma\Big)\, e^{\gamma x_1}$$

Since L is elliptic we have $a_{11} \geq \frac{1}{\Lambda}$ and $b_1 \geq -\Lambda$, so

$$Lv_\gamma(x) = a_{11}(x)\gamma^2 + b_1(x)\gamma \geq e^{\gamma x_1}\gamma\left(\frac{1}{\Lambda}\gamma - \Lambda\right).$$

If we choose $\gamma = 3\Lambda$ we thus find

$$Lv_\gamma(x) > 0 \quad \text{in } \Omega.$$

Consequently, under the assumption (2.30) we have for any $\varepsilon > 0$, for $w_\varepsilon := u + \varepsilon v_\gamma$,

$$Lw_\varepsilon(x) > 0 \quad \text{in } \Omega.$$

and thus by the first step

$$\sup_{\Omega} w_\varepsilon = \sup_{\partial\Omega} w_\varepsilon$$

Since $w_\varepsilon = u + \varepsilon v_\gamma$ and v_γ is continuous (and Ω is bounded) we have

$$\left|\sup_{\Omega} u - \sup_{\partial\Omega} u\right| \leq C(\Omega)\varepsilon.$$

Letting $\varepsilon \to 0$ we obtain the claim.

The inf claim follows by taking $-u$ instead of u. □

Exercise 2.54. *Let $A \in \mathbb{R}^{n\times n}$ be a symmetric matrices, i.e. $A^t = A$. Show that the following two conditions are equivalent*

(1) $A \geq 0$ in the sense of matrices, i.e.
$$\xi^t A\xi \geq 0$$
(2) all eigenvalues of A are nonnegative.

Exercise 2.55. *Let $A, B \in \mathbb{R}^{n\times n}$ be two symmetric matrices, i.e. $A^t = A$, $B^t = B$. Assume that $A, B \geq 0$ in the sense of matrices, i.e.*
$$\xi^t A\xi \geq 0, \quad \xi^t B\xi \geq 0 \quad \forall \xi \in \mathbb{R}^n$$
Show that
$$A : B \equiv \sum_{i=1}^n \sum_{j=1}^n A_{ij} B_{ij} \geq 0.$$

Also in the case $c \not\equiv 0$ a type of weak maximum principle holds (essentially mimmicking the above argument):

Theorem 2.56 (Weak maximum principle for $c \leq 0$)**.** *Let $\Omega \subset\subset \mathbb{R}^n$, and consider*
$$L := a_{ij}(x)\partial_{ij} + b_i(x)\partial_i + c(x).$$
where $c \leq 0$ in Ω.

Assume $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$.

(1) If u solves
$$-Lu \leq 0 \quad in\ \Omega$$
Then
$$\sup_\Omega u \leq \sup_{\partial\Omega} u_+,$$
where u_+ denotes the positive part of u, namely
$$u_+ = \max\{0, u\}.$$
(2) If on the other hand u solves
$$-Lu \geq 0 \quad in\ \Omega$$
we have
$$\inf_\Omega u \geq \inf_{\partial\Omega}(-u_-),$$
where u_- denotes the positive part of u, namely
$$u_+ = \max\{0, u\}, \quad u_- = -\min\{0, u\}$$

(3) In particular, if $Lu = 0$ then

$$\sup_\Omega |u| \leq \sup_{\partial\Omega} |u|$$

Proof. Let us assume $-Lu \leq 0$. First we observe that if

$$\sup_\Omega u \leq 0$$

then there is nothing to show, since we have $u_+ \geq 0$ by definition and thus

$$\sup_\Omega u \leq 0 \leq \sup_{\partial\Omega} u_+.$$

So w.l.o.g. we may assume that $\sup_\Omega u > 0$. Set

$$\Omega_+ := \{x \in \Omega : \quad u(x) > 0\} \neq \emptyset$$

Since u is continuous $\Omega_+ = u^{-1}((0,\infty))$ is a nonempty, open set.

Define the elliptic operator L_0 by

$$L_0 u := Lu - cu = a_{ij}\partial_{ij}u + b_i\partial_i u.$$

Since $-Lu \leq 0$ we have $-L_0 u \leq cu \leq 0$ in Ω_+ — since by assumption $c \leq 0$. So, using the weak maxum principle for $c \equiv 0$, Theorem 2.53,

$$\sup_\Omega u \overset{u\leq 0:\, \Omega\backslash\Omega_+}{\leq} \sup_{\Omega_+} u \overset{\text{T. } 2.53}{=} \sup_{\partial\Omega_+} u = \sup_{\partial\Omega_+} u_+ \leq \sup_{\partial\Omega} u_+.$$

In the last step we used that $\partial\Omega_+ \subset \overline{\Omega}$ can be split into two parts: the part $\partial\Omega_+ \subset \Omega$ (on this part we have $u = u_+ = 0$), and the part $\partial\Omega_+ \subset \partial\Omega$ where $u_+ \geq 0$.

This settles the claim for $-Lu \leq 0$.

If we assume $-Lu \geq 0$ then $-u$ satisfies $-L(-u) \geq 0$, and we obtain the claim from the previous case

$$-\inf_\Omega u = \sup_\Omega(-u) \leq \sup_{\partial\Omega}(-u)_+ = \sup_{\partial\Omega} u_- = -\inf_{\partial\Omega}(-u_-)$$

so

$$\inf_\Omega u \geq \inf_{\partial\Omega}(-u_-).$$

For the last case assume that $-Lu = 0$. By the arguments before we have then (observe that $|u| = u_+ + u_-$).

$$\sup_\Omega u \leq \sup_{\partial\Omega} u_+ \leq \sup_{\partial\Omega} |u|.$$

and

$$\inf_\Omega u \geq \inf_{\partial\Omega}(-u_-),$$

which can be rewritten as

$$-\inf_\Omega u \leq -\inf_{\partial\Omega}(-u_-) = \sup_{\partial\Omega}(u_-) \leq \sup_{\partial\Omega} |u|.$$

Now at least one of the following cases holds:

$$\sup_\Omega |u| = \sup_\Omega u, \quad \text{or} \quad \sup_\Omega |u| = -\inf_\Omega u$$

but in both cases the estimates above imply

$$\sup_\Omega |u| \le \sup_{\partial\Omega} |u|$$

□

Exercise 2.57 (Counterexample for $c \ge 0$)**.** *Consider*

$$Lu = \Delta u + 5u$$

for $\Omega = (-1,1) \times (-1,1)$. *Take*

$$u = (1 - x^2) + (1 - y^2) + 1$$

Show that

(1) $-Lu = \le 0$ *in* Ω
(2) $\sup_\Omega u \ge u(0) = 3$
(3) $\sup_{\partial\Omega} u = 2$
(4) Why is this no contradiction to Theorem 2.56?

Corollary 2.58 (Eigenvalues of Δ)**.** Δ *with Dirichlet-boundary has no nonnegative eigenvalues. Namely there is no nontrivial solution* $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ *for* $\lambda \ge 0$ *to*

$$\begin{cases} \Delta u = \lambda u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

(Here, nontrivial means $u \not\equiv 0$*).*

Proof. The above equation is for $L := \Delta - \lambda$ equivalent to

$$\begin{cases} -Lu = 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

Since $\lambda \ge 0$, Theorem 2.56 is applicable, so for any solution to the above equation we'd have

$$\sup_\Omega |u| \le \sup_{\partial\Omega} |u| = 0.$$

Thus $u \equiv 0$, i.e. u is the trivial solution. □

As it was the case for the Δ-operator, Theorem 2.22, the weak maximum principle implies uniqueness results.

Corollary 2.59 (Uniqueness for the Dirichlet problem)**.** *Let L be as above a non-divergence form linear elliptic operator, $\Omega \subset\subset \mathbb{R}^n$ with smooth boundary, $c \leq 0$, $f \in C^0(\Omega)$, $g \in C^0(\partial\Omega)$. Then there exists at most one solution $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ of the Dirichlet boundary problem*

$$\begin{cases} Lu = f & \text{in } \Omega \\ u = g & \text{on } \partial\Omega \end{cases}$$

Exercise 2.60. *Prove Corollary 2.59.*

Corollary 2.61 (Comparison principle)**.** *Let L be a linear elliptic differential operator (non-divergence form), and assume that $c \leq 0$ in $\Omega \subset\subset \mathbb{R}^n$. Let $u, v \in C^2(\Omega) \cap C^0(\overline{\Omega})$ satisfy $-Lu \leq -Lv$ in Ω. Then $u \leq v$ on $\partial\Omega$ implies $u \leq v$ in Ω.*

Exercise 2.62. *Prove Corollary 2.61*

Corollary 2.63 (Continuous dependence on data)**.** *Let L be a linear elliptic differential operator (non-divergence form), and assume that $c \leq 0$ in $\Omega \subset\subset \mathbb{R}^n$.*

Let $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ satisfy

$$\begin{cases} -Lu = f & \text{in } \Omega \\ u = g & \text{on } \partial\Omega \end{cases}$$

where $f \in C^0(\overline{\Omega})$ and $g \in C^0(\partial\Omega)$.

Then for some constant $C = C(a, b, c, \Omega)$ we have

$$\sup_{\Omega} |u| \leq C \left(\sup_{\partial\Omega} |g| + \sup_{\Omega} |f| \right).$$

Exercise 2.64. *Prove Corollary 2.63.*

Hint: Set $v_\lambda := u + \lambda e^{\mu|x-x_0|^2} \sup_\Omega |f|$ where $x_0 \in \mathbb{R}^n \setminus \overline{\Omega}$. Choose $\mu \gg 1$. Then choose λ so that $Lv_\lambda \leq 0$ and use the weak maximum principle. Then choose λ so that $Lv_\lambda \geq 0$, and again use the weak maximum principle.

Our next goal is the the strong maximum principle, for this we use the following result by Hopf:

Lemma 2.65 (Hopf Boundary point Lemma)**.** *Let $B \subset \mathbb{R}^n$ be a ball, and let L be as above. Let $u \in C^2(B) \cap C^0(\overline{B})$ and assume that for $x_0 \in \partial B$ we have*

- *$u(x) < u(x_0)$ for all $x \in B$*
- *$-Lu \leq 0$ in B.*
- *One of the following*
  - *(1) $c \equiv 0$*
  - *(2) $c \leq 0$ and $u(x_0) \geq 0$*
  - *(3) $u(x_0) = 0$*

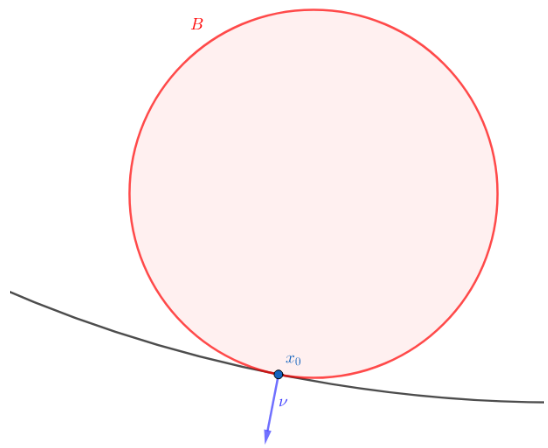


FIGURE 2.4. Illustration of the setup in Hopf's boundary lemma, Lemma 2.65

Then for ν the outwards facing normal of B at x_0 (i.e. if $B = B(y_0, \rho)$ then for $\nu = \frac{y_0 - x_0}{\rho}$

$$\partial_\nu u(x_0) > 0,$$

if that derivative exists.

An illustration of the setup of Lemma 2.65 is in Figure 2.4. Observe that $\partial_\nu u(x_0) \geq 0$ is clear, the Hopf-Lemma says this must be a strict inequality!

Proof. W.l.o.g. we may assume

$$B = B(0,R), \quad c \leq 0, \quad u(x_0) = 0, \quad u < 0 \quad \text{in } B(0,R): \tag{2.32}$$

Indeed, the condition $B = B(0,R)$ can be assumed simply by shifting. As for the other conditions set (recall that $c_+ = \max\{c, 0\}$)

$$\tilde{L} := L - c_+.$$

and

$$\tilde{u} := u - u(x_0).$$

Then in B,

$$-\tilde{L}\tilde{u} = -(L - c_+)(u - u(x_0)) = -Lu + c_+ u + cu(x_0) - c_+ u(x_0) \leq c_+ (u - u(x_0)) + c\, u(x_0)$$

If $c \equiv 0$ then we readily have $-\tilde{L}\tilde{u} \leq 0$.

If $c \leq 0$ we have $c_+ \equiv 0$, and again obtain $-\tilde{L}\tilde{u} \leq 0$.

If $u(x_0) = 0$ then $c_+ u \leq 0$, since $u \leq u(x_0) = 0$ by assumption.

Since $c - c_+ \leq 0$ we observe that $\tilde{L}$ is an operator that satisfies the missing conditions in (2.32). Thus, indeed, (2.32) can be assumed w.l.o.g.

So assume (2.32) from now on.

Set for some $\alpha > 0$
$$v_\alpha(x) := e^{-\alpha|x|^2} - e^{-\alpha R^2}.$$
Clearly $0 \leq v_\alpha \leq 1$ in $B = B(0,R)$. Moreover
$$v_\alpha \equiv 0 \quad \text{on } \partial B(0,R).$$
For $\rho \in (0,R)$ denote by $A(\rho,R)$ the annulus $B(0,R)\backslash B(0,\rho)$. We will show next

$$\text{For any } \rho \in (0,R) \text{ there exists } \alpha > 0 \text{ such that} \quad -Lv_\alpha < 0 \quad \text{in } A(\rho,R) \tag{2.33}$$

For this we first compute
$$\partial_i v_\alpha(x) = -2\alpha x_i e^{-\alpha|x|^2}. \tag{2.34}$$
Next we compute
$$\partial_{ij} v_\alpha(x) = \left(-2\alpha\delta_{ij} + 4\alpha^2 x_i x_j\right) e^{-\alpha|x|^2}$$
so (using the ellipticity conditions, $a_{ij}x_ix_j \geq \lambda|x|^2$, and $|a|, |b|, |c| \leq \Lambda$,
$$\begin{aligned}
-Lv(x) =& - a_{ij}\partial_{ij}v - b_i\partial_i v - cv\\
=& - a_{ij}\left(-2\alpha\delta_{ij} + 4\alpha^2 x_i x_j\right) e^{-\alpha|x|^2} - b_i\left(-2\alpha x_i e^{-\alpha|x|^2}\right) - ce^{-\alpha|x|^2} + \underbrace{ce^{-\alpha R^2}}_{\leq 0}\\
\leq& \left(2\alpha\Lambda - 4\alpha^2\lambda|x|^2 + 2\alpha\Lambda|x| + \Lambda\right) e^{-\alpha|x|^2}.
\end{aligned}$$
That is, for $x \in A(\rho,R)$,
$$-Lv(x) \leq \underbrace{\left(-4\lambda\alpha^2\rho^2 + 2\alpha\Lambda + 2\alpha\Lambda R + \Lambda\right)}_{\leq 0 \text{ for } \alpha \gg 1} e^{-\alpha|x|^2}$$

If we take α large, the (negative) α^2-term dominates, that is for $\alpha \gg 1$ (depending on $\rho > 0$, Λ, λ and R) we have (2.33).

Next, we consider the equation for $u + \varepsilon v$, which in view of (2.33) becomes
$$-L(u + \varepsilon v) < 0 \quad \text{in } A(\rho,R).$$
The weak maximum principle, Theorem 2.56, implies
$$\sup_{A(\rho,R)} u + \varepsilon v \leq \sup_{\partial A(\rho,R)} (u + \varepsilon v)_+. \tag{2.35}$$
The boundary $\partial A(\rho,R)$ is the union of $\partial B(0,R)$ and $\partial B(0,\rho)$.

On $\partial B(0,R)$ we know $v \equiv 0$ and since u is continuous and $u < 0$ in $B(0,R)$ we have $u \leq 0$ on $\partial B(0,R)$. That is $(u + \varepsilon v)_+ = 0$ on $\partial B(0,R)$.

On $\partial B(0,\rho)$, since $u < 0$ on $B(0,R)$ we have $\sup_{\partial B(0,\rho)} u < 0$, and consequently, since $v \leq 1$ we have for all $0 < \varepsilon < \varepsilon_0 := -\sup_{\partial B(0,\rho)} u$
$$u + \varepsilon v < 0 \quad \text{on } \partial B(0,\rho)$$

That is (2.35) implies

$$u + \varepsilon v \leq 0 \quad \text{in } A(\rho, R). \tag{2.36}$$

Now fix $\rho \in (0, R)$, choose ε, α so that the above is true.

Denote $\nu := \frac{x_0}{|x_0|}$ the outwards unit normal to ∂B at $x_0 \in \partial B$. Observe that for all small $0 < t \ll 1$ (depending on ρ) we have $x_0 - t\nu \in A(\rho, R)$.

Recall that by assumption $u(x_0) = 0$, then (2.36) implies for any small $t > 0$,

$$u(x_0 - t\nu) + \varepsilon v(x_0 - t\nu) \overset{(2.36)}{\leq} 0 = u(x_0) + \varepsilon v(x_0).$$

This leads to (again: for all $0 < t \ll 1$)

$$\frac{u(x_0 - t\nu) - u(x_0)}{t} \leq -\varepsilon \frac{v(x_0 - t\nu) - v(x_0)}{t}$$

Letting $t \to 0^+$ on both sides we obtain

$$-\partial_\nu u(x_0) \leq \varepsilon \partial_\nu v(x_0). \tag{2.37}$$

Observe that (2.34) implies

$$\partial_\nu v(x_0) = \partial_i v(x_0) \frac{(x_0)_i}{R} = -2\alpha \frac{|x_0|^2}{R} e^{-\alpha R^2} < 0$$

That is (2.37) implies

$$-\partial_\nu u(x_0) < 0$$

which implies the claim. $\square$

The Hopf Lemma, Lemma 2.65 implies the strong maximum principle.

Corollary 2.66 (Strong maximum principle)**.** *Let $\Omega \subset \mathbb{R}^n$ be an open and connected set, (but Ω may be unbounded). Let $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ satisfy*

$$-Lu \leq 0 \quad \textit{in } \Omega.$$

Assume either

- $c \equiv 0$, *or*
- $c \leq 0$ *and* $\sup_\Omega u \geq 0$.

Then we have the following: If there exists $x_0 \in \Omega$ such that

$$u(x_0) = \sup_\Omega u$$

then $u \equiv u(x_0)$ in Ω.

Proof. Assume the claim is false. Via the modification as in the proof of Lemma 2.65, we may assume w.l.o.g. $u \leq 0$ in Ω and $u(x_0) = 0$ for some $x_0 \in \Omega$, but $u \not\equiv 0$.

Let
$$\Omega_- := \{x \in \Omega : \quad u(x) < 0\}.$$
Observe that Ω_- is open (u is continuous) and $\Omega_- \neq \emptyset$ (because $u \leq 0$ and $u \not\equiv 0$).

Since $x_0 \in \Omega$ and $u(x_0) = 0$, the boundary of Ω_- cannot be contained in $\partial\Omega$, i.e. we have
$$\partial\Omega_- \cap \Omega \neq \emptyset.$$

Indeed, this follows from connectedness: Let $\gamma \subset \Omega$ be a continuous path from x_0 to a point in Ω_-. Then there has to be a point on γ where γ leaves Ω_-. This point lies in $\partial\Omega_-$ and in Ω.

This means we can find a point $x_1 \in \Omega_-$ which is close to $\partial\Omega_-$ but not close to $\partial\Omega$, i.e.
$$x_1 \in \Omega_-, \quad \rho := \mathrm{dist}\,(x_1, \partial\Omega_-) < 10\mathrm{dist}\,(x_1, \partial\Omega).$$
By definition of the distance
$$B(x_1, \rho) \subset \Omega_-, \quad \overline{B(x_1, \rho)} \backslash \Omega_- \neq \emptyset.$$
Let $x_2 \in \partial B(x_1, \rho) \backslash \Omega_-$. Since by construction $x_2 \in \partial\Omega_- \cap \Omega$ we have $u(x_2) = 0$ by continuity. Moreover $u < 0$ in $B(x_1, \rho) \subset \Omega_-$.

Since everything takes place well within Ω, the conditions of the Hopf Lemma, Lemma 2.65, are satisfied and thus for ν the outwards facing normal at x_2 to $\partial B(x_1, \rho)$
$$\partial_\nu u(x_2) > 0.$$
But on the other hand $x_2 \in \Omega$ is a local maximum for u, so $Du(x_2) = 0$, which is a contradiction. The claim is then proven. $\square$

A consequence of the Hopf Lemma, Lemma 2.65, and the strong maximum principle, Corollary 2.66, is the uniqueness for the Neumann problem.

Corollary 2.67 (Uniqueness for Neumann-boundary problem)**.** *Let $\Omega \subset\subset \mathbb{R}^n$ be open and connected. Moreover we assume a boundary regularity of $\partial\Omega$, the interior sphere condition[6]:*

Assume that for any $x_0 \in \partial\Omega$ there exists a ball $B \subset \Omega$ such that $x_0 \in \overline{B}$.

Then the following holds for any elliptic operator as above with $c \equiv 0$: For any given $f \in C^0(\Omega)$ and any $g \in C^0(\partial\Omega)$ there is at most one solution $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$ of the Neumann boundary problem
$$\begin{cases} -Lu = f & \text{in } \Omega \\ \partial_\nu u = g & \text{on } \partial\Omega, \end{cases}$$

[6]This condition does not allow for outwards facing cusps. One can show that every set Ω whose boundary $\partial\Omega$ is a sufficiently smooth manifold satisfies the interior sphere condition

up to constant functions. That means, the difference of two solutions u, v is constant, $u - v \equiv c$.

Proof. The difference of two solutions u, v, $w := u - v$ satisfies[7]

$$\begin{cases} -Lw \leq 0 & \text{in } \Omega \\ \partial_\nu w = 0 & \text{on } \partial\Omega, \end{cases}$$

Firstly, assume that there exists $x_0 \in \Omega$ such that $\sup_\Omega w = w(x_0)$. Then, by the strong maximum principle, Corollary 2.66, we have $w \equiv w(x_0)$ and the claim is proven. If this is not the case, then there must be $x_0 \in \partial\Omega$ with $w(x_0) > w(x)$ for all $x \in \Omega$. If we take a ball from the interior sphere condition of $\partial\Omega$ at x_0 then on this ball B we can apply Hopf Lemma, Lemma 2.65, which leads to $\partial_\nu w(x_0) > 0$, which is ruled out by the Neumann boundary assumption $\partial_\nu w = 0$. □

3. Heat equation

3.1. **Again, sort of a physical motivation.** This is somewhat similar to Section 2.1.

The Laplacian $\Delta u(x)$ describes the difference between the average value of a function around a point x and the value at the point x (cf. the mean value formula)

$$\Delta u \approx \fint_{\partial B(x,r)} u - u(x).$$

If we think of u as a temperature, then $\Delta u(x) > 0$ means that the material surrounding x is hotter than u, and $\Delta u(x) < 0$ means the surroundings are colder than u. Heat will flow from the hotter areas to the lower areas, and the speed of this propagation is proportional to the difference in temperature (second law of thermodynamics). That is,

$$\partial_t u = c\Delta u$$

could describe the change in heat distribution over time (where c is a material property like conductivity). So if we solve

$$\begin{cases} \partial_t u - \Delta u = 0 & \text{in} \quad \Omega \times (0,T) \\ u(0,\cdot) = u_0 & \text{on}\Omega \\ u(x,t) = g(x,t) & \text{on}\partial\Omega \times (0,T) \end{cases}$$

then $u(x,t)$ describes the heat of the body at time t at the point x in the body Ω, of a system that started with the heat distribution u_0 and heat source at $\partial\Omega$ which is $g(x,t)$.

The equation is thus called the heat equation, or it is said that u solves the heat flow.

[7]actually we have $=$ in the equation below, but the argument works for $\leq$ as well

We can believe that as time passes, there will be less and less change in the energy, so at $T = \infty$ maybe we have that $\partial_t u = 0$. That is at $T = \infty$ the solution $u(\infty, x)$ solves

$$-\Delta u = 0$$

that is stationary solutions (could be, this is not always true) appear as $\lim_{t\to\infty}$ of flows.

3.2. Sort of an optimization motivation.

We have discussed in Section 2.10 that we can solve the equation

$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

by minimizing the energy

$$E(u) := \frac{1}{2}\int_\Omega |\nabla u|^2 - uf$$

among functions with $u = 0$ on $\partial\Omega$ (to make this precise we need Sobolev spaces).

So, in some sense ∇E (which we usually write as the *variation* δE corresponds to $\Delta u + f$. ($\delta E = 0$ means that we have found a minimizer of this convex functional.

What is the relation to

$$\begin{cases} \partial_t u - \Delta u = f & \text{in } \Omega \times (0,T)? \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

Well, this is

$$\partial_t u = -\delta E(u).$$

If u was a finite dimensional vector, then

$$\partial_t u = -\nabla E(u)$$

would be that u follows the steepest gradient descent.

3.3. Fundamental solution and Representation.

We consider

$$\begin{aligned} \partial_t u - \Delta u &= f \quad \text{in} \quad \mathbb{R}^{n+1}_+ \\ u(0,\cdot) &= g \quad \text{on} \quad \mathbb{R}^n. \end{aligned} \tag{3.1}$$

If $f = 0$. then (3.1) is called *homogeneous heat equation*. For $f \neq 0$ it is called *inhomogeneous*.

Trivial solutions of the homogeneous equation constant maps $u(x,t) \equiv c$, or (not completely trivial) time-independent harmonic functions $u(x,t) := v(x)$ with $\Delta v = 0$ (these are called *stationary* solutions).

For elliptic equations we had the notion of a fundamental solution, Section 2.3; There exists a similar concept for the heat equation, the *heat kernel*, which we will (formally) derive now.

If we fix $x \in \mathbb{R}^n$ and look at (3.1) as an equation in time t then it looks like an ODE, and naively the solution should be (Duhamel principle!)

$$u(x,t) = e^{t\Delta}u(x,0) + \int_0^t e^{(t-s)\Delta} f(x,s)\, ds.$$

Of course, $e^{t\Delta}$ does not make any sense for now (it can be defined via *semi-group theory*).

To make (still formally, but more precise) sense of the "ODE argument", we use the Fourier-transform (with respect to the variables $x \in \mathbb{R}^n$):

Let u be a solution of $\partial_t u = \Delta u$. Taking the Fourier transform (in x) on both sides we find

$$\begin{aligned}\frac{d}{dt}\hat{u}(\xi,t) = \widehat{\partial_t u}(\xi,t) &= \widehat{\Delta u}(\xi,t)\\ &= -|\xi|^2 \hat{u}(\xi,t).\end{aligned}$$

(There should be a constant c in front of $-|\xi|^2$, but we ignore that for now)

Let ξ be fixed and let

$$v(t) = \hat{u}(\xi,t).$$

Then the above reads as

$$\frac{d}{dt}v(t) = -|\xi|^2 \hat{v}(t).$$

There is one solution to this ODE (starting from a given value $v(0)$):

$$v(t) = e^{-t|\xi|^2} v(0).$$

Observe that in particular $v(\infty) = 0$, $\partial_t v(\infty) = 0$, etc. (i.e. we have strong "decay at infinity").

Ansatz: $v(0) = 1$, resp. $u(0) = \delta_0$. This means

$$\hat{u}(\xi,t) = e^{-t|\xi|^2}.$$

In this case we have

$$u(x,t) = \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{-\frac{|x|^2}{4t}},$$

which seems to be a special solution.

Definition 3.1.

$$\Phi(x,t) = \begin{cases} \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{-\frac{|x|^2}{4t}}, \quad t>0, x \in \mathbb{R}^n \\ 0, \quad t < 0, x \in \mathbb{R}^n. \end{cases}$$

is called *fundamental solution* or *heat kernel*.

One has

$$\partial_t \Phi - \Delta \Phi = 0, \quad \text{for} \quad t > 0$$

and

$$\lim_{t\to 0}\Phi(x_0,t)=\begin{cases}0, & x_0\neq 0\\ \infty, & x_0=0.\end{cases}$$

Lemma 3.2.

$$\forall t>0: \int_{\mathbb{R}^n}\Phi(x,t)\ dx=1.$$

Proof.

$$\int_{\mathbb{R}^n}\Phi(x,t)\ dx=\hat{\Phi}(0,t)=1.$$

□

Analogously to the fundamental solution for the Laplace equation, the heat kernel Φ generates solutions to the heat equation. Indeed, if we set

$$\begin{aligned}u(x,t)&:=\Phi(\cdot,t)*g(x)\\ &=\int_{\mathbb{R}^n}\Phi(x-y,t)g(y)\ dy\end{aligned}$$

Then

$$\hat{u}(\xi,t)=(\widehat{\Phi(\cdot,t)*g})(\xi)=\hat{\Phi}(\xi,t)\hat{g}(\xi).$$

That is,

$$\hat{u}(\xi,0)=\hat{g}(\xi),\quad (\frac{d}{dt}+|\xi|^2)\hat{u}(\xi,t)=0.$$

Revert the Fourier-transformation to obtain

$$\begin{cases}(\partial_t-\Delta)u=0 & \text{in}\quad \mathbb{R}^{n+1}_+\\ u(x,0)=g(x) & x\in\mathbb{R}^n.\end{cases}$$

Motivated by this calculation we set

$$u(x,t)=\int_{\mathbb{R}^n}\Phi(x-y,t)g(y)\ dy.$$

Theorem 3.3 (Potential representation)**.** *Let $g\in C^0(\mathbb{R}^n)\cap L^\infty(\mathbb{R}^n)$. Let u as in (3.3). Then u is defined in $\mathbb{R}^n$ and there holds:*

(i) $u\in C^\infty(\mathbb{R}^{n+1}_+)$,

(ii) $\partial_t u-\Delta u=0$ *in* $\mathbb{R}^{n+1}_+$ *und*

(iii)

$$\forall x_0\in\mathbb{R}^n:\ \lim_{(x,t)\to(x_0,0)}u(x,t)=g(x_0).$$

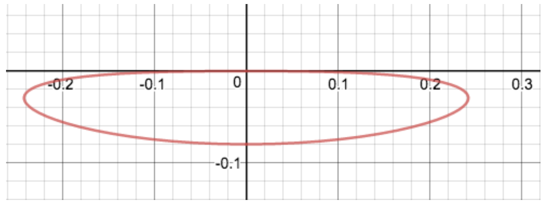


FIGURE 3.1. An illustration of the heatball $E(0,0;1)$ in $n=1$ dimension taken from [user9464,]. Observe that this heatball is "centered" at $(x,t)=(0,0)$, i.e. a heatball always goes backwards in time.

Next we search a potential representation for

$$\begin{aligned}(\partial_t - \Delta)u &= f \quad \text{in } \mathbb{R}^{n+1}_+\\ u(\cdot,0) &= 0 \quad \text{on } \mathbb{R}^n.\end{aligned} \tag{3.2}$$

From the argument in the beginning, using the inverse Fourier transform, and *Duhamel principle*,

$$u(x,t) = \int_0^t \int_{\mathbb{R}^n} \Phi(x-y,t-s)f(y,s)\, dyds. \tag{3.3}$$

Theorem 3.4. *Let $f \in C^2(\mathbb{R}^n \times [0,\infty))$ with compact support and let u as in (3.3). Then*

(i) $u \in C^2(\mathbb{R}^n \times (0,\infty))$,

(ii) $(\partial_t - \Delta)u = f$ *in* $\mathbb{R}^n \times (0,\infty)$

(iii) $\forall x_0 \in \mathbb{R}^n\colon\ \lim_{(x,t)\to(x_0,0)} u(x,t) = 0.$

3.4. Mean-value formula. (cf. [Evans, 2010, Chapter 2.3])

Use the fundamental solution to construct a parabolic ball, or *heat ball*

$$E(x,t;r) \subset \mathbb{R}^{n+1}. \tag{3.4}$$

Definition 3.5 (Heat ball)**.** Let $(x,t) \in \mathbb{R}^{n+1}$. Set

$$E(x,t;r) = \left\{(y,s) \in \mathbb{R}^{n+1}\colon\ s \le t, \Phi(x-y,t-s) \ge \frac{1}{r^n}\right\}. \tag{3.5}$$

Cf. Figure 3.1.

Theorem 3.6 (mean value)**.** *Let $X \subset \mathbb{R}^{n+1}$ be open and $u \in C^2(X)$ solve $(\partial_t - \Delta)u = 0$ in X. Then there holds*

$$u(x,t) = \frac{1}{4r^n} \int_{E(x,t;r)} u(y,s) \frac{|x-y|^2}{(t-s)^2}\, dyds \tag{3.6}$$

for all $E(x,t;r) \subset X$.

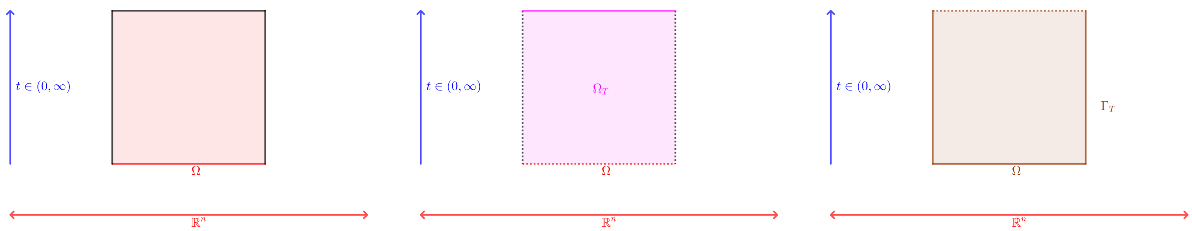


FIGURE 3.2. If Ω is an open set in $\mathbb{R}^n$ then $\Omega_T := \Omega \times (0, T]$ and the parabolic boundary is $\Gamma_T := \partial\Omega \times [0, T) \cup \Omega \times \{0\}$.

3.5. Maximum principle and Uniqueness.

Definition 3.7. Let $\Omega \subset \mathbb{R}^n$ be an open set and denote with $\Omega_T := \Omega \times (0, T]$ for some time $T > 0$. It is important to note that the top $\Omega \times \{T\}$ belongs to Ω_T. The *parabolic boundary* Γ_T of Ω_T is the boundary of Ω_T without the top,

$$\Gamma_T = \overline{\Omega_T} \backslash \Omega_T = \partial\Omega \times [0, T) \cup \Omega \times \{0\}.$$

See Figure 3.2.

Theorem 3.8. *Let Ω be bounded and $u \in C^2(\Omega_T) \cap C^0(\overline{\Omega_T})$ be a solution of $\partial_t u = \Delta u$ in Ω_T. Then there holds*

(1) *the weak maximum principle:*

$$\max_{\overline{\Omega_T}} u = \max_{\Gamma_T} u \tag{3.7}$$

(2) *and the strong maximum principle: If Ω is connected and if there is $(x_0, t_0) \in \Omega_T$ (i.e. $t_0 \in (0, T]$, $x \in \Omega$) with*

$$u(x_0, t_0) = \max_{\overline{\Omega_T}} u, \tag{3.8}$$

then u is constant on all prior times, i.e.

$$u(x, t) = u(x_0, t_0) \quad \forall (x, t) \in \Omega_{t_0}. \tag{3.9}$$

Exercise 3.9. *Show that the strong maximum principle Theorem 3.8(2) implies the weak maximum principle Theorem 3.8(1).*

Proof of Theorem 3.8 (2). Suppose there is $(x_0, t_0) \in \Omega_T$ with

$$u(x_0, t_0) = M = \max_{\overline{\Omega_T}} u. \tag{3.10}$$

Since $t_0 > 0$, there exists a small heat ball $E(x_0, t_0, r_0) \subset \Omega_T$ and we have by Theorem 3.6

$$M = u(x_0, t_0) = \frac{1}{4{r_0}^n} \int_{E(x_0,t_0,r_0)} u(y, s) \frac{|y - x|^2}{(t - s)^2} \, dsdy \leq M. \tag{3.11}$$

Hence $u \equiv M$ in $E(x_0, t_0; r_0)$.

Now we need to show $u = M$ in all of Ω_{t_0}. It suffices to show $u \equiv M$ in any Ω_{t_1} for any $t_1 < t_0$, by continuity $u \equiv M$ in all of Ω_{t_0}. So let $(x_1, t_1) \in \Omega_{t_0}$ and $t_1 < t_0$. Then there exists a continuous path $\gamma\colon [0,1] \to \Omega$ connecting x_0 and x_1. In the spacetime set

$$\Gamma(r) = (\gamma(r), rt_1 + (1-r)t_0). \tag{3.12}$$

Let

$$\rho = \max\{r \in [0,1]\colon u(\Gamma(r)) = M\}. \tag{3.13}$$

Show that $\rho = 1$. Suppose $\rho < 1$. Then we use the proof above to find a heat ball

$$E = E(\Gamma(\rho), r'), \tag{3.14}$$

where $u = M$. Since Γ crosses E (time parameter is decreasing along Γ), we obtain a contradiction to the maximality of ρ. $\square$

Exercise 3.10. *Use Theorem 3.8 to show the following* infinite speed of propagation*:*

Assume $u \in C^2(\overline{\Omega_T})$ satisfies

$$\begin{cases} \partial_t u - \Delta u = 0 & \text{in } \Omega_T \\ u = 0 & \text{on } \partial\Omega \times [0,T] \\ u = g & \text{in } \Omega \times \{0\} \end{cases}$$

1. *Show the following: if $g \geq 0$ in Ω but there exists any $x_0 \in \Omega$ such that $g(x_0) > 0$ then $u(x,t) > 0$ in every point in $(x,t) \in \Omega_T$.*
2. *Think about how this is a non-relativistic behaviour: any at an arbitrary point influences the whole universe instantaneously.*

For general $X \subset \mathbb{R}^{n+1}$ open we have a similar maximum principle:

Exercise 3.11. *In Theorem 3.15 we learned of the strong maximum principle in parabolic Cylinders. Use this to obtain the strong maximum principle in general open sets X:*

let $X \subset \mathbb{R}^{n+1}$ be a bounded, open set. Assume that $u \in C^\infty(\overline{X})$ and

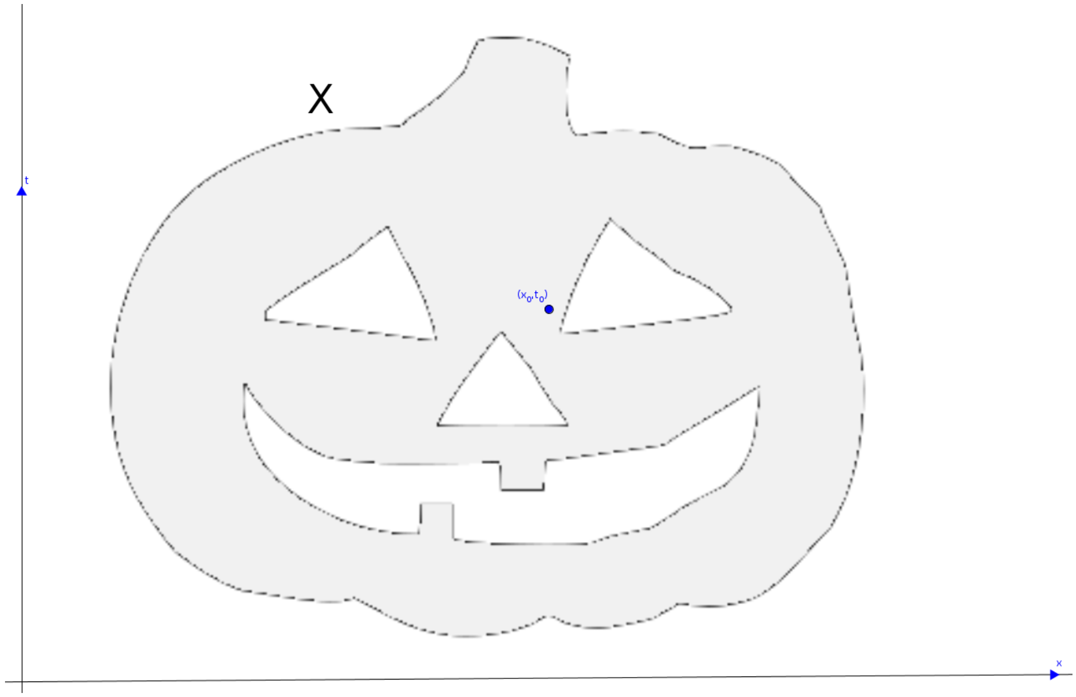
$$\partial_t u - \Delta u \quad \text{in } X.$$

Assume moreover that for some $(x_0, t_0) \in X$ we have

$$M := u(x_0, t_0) = \sup_{(x,t) \in X} u(x,t).$$

1. *Describe (in words) in which set C the function is necessarily constant*

$$C := \{(x,t) \in X : \quad u(x,t) = M\}.$$



(2) Assume the set X (grey) and the point (x_0, t_0) are given in the picture. Draw (in orange) the set C from the question above.

Theorem 3.12 (Uniqueness on bounded domains)**.** *Let $\Omega \Subset \mathbb{R}^n$ bounded and $g \in C^0(\Gamma_T)$, $f \in C^0(\Omega_T)$. Then there is at most one solution $C^2(\Omega_T) \cap C^0(\overline{\Omega_T})$ to*

$$\begin{aligned} \partial_t u - \Delta u &= f \quad \text{in } \Omega_T \\ u &= g \quad \text{on } \Gamma_T. \end{aligned} \tag{3.15}$$

Exercise 3.13. *Prove Theorem 3.12.*

Theorem 3.14. *Let $u \in C^2(\mathbb{R}^n \times (0,T]) \cap C^0(\mathbb{R}^n \times [0,T])$ be a solution of*

$$\begin{aligned} (\partial_t - \Delta) u &= 0 \quad \text{in } \mathbb{R}^n \times (0,T) \\ u &= g \quad \text{on } \mathbb{R}^n \times \{t = 0\} \end{aligned} \tag{3.16}$$

with the growth condition

$$u(x,t) \leq A e^{a|x|^2} \quad \forall (x,t) \in \mathbb{R}^n \times [0,T] \tag{3.17}$$

for some $a, A > 0$. Then there holds

$$\sup_{\mathbb{R}^n \times [0,T]} u \leq \sup_{\mathbb{R}^n} g. \tag{3.18}$$

Proof. It suffices to show this estimate for small times, by splitting up the time interval into many small time steps. For this reason we assume first:

$$4aT < 1. \tag{3.19}$$

For $\varepsilon > 0$ and μ chosen below, let

$$v(x,t) = u(x,t) - \frac{\mu}{(T+\varepsilon-t)^{\frac{n}{2}}} e^{\frac{|x|^2}{4(T+\varepsilon-t)}} \tag{3.20}$$

for some $\mu > 0$. Then $v_t - \Delta v = 0$ in $\mathbb{R}^n \times [0,T]$ (observe that t appears in the negative above). Theorem 3.8 implies

$$\forall \Omega \subset\subset \mathbb{R}^n:\ \max_{\overline{\Omega_T}} v \le \max_{\Gamma_T} v \le \max(\max v(\cdot,0), \max_{\partial\Omega\times[0,T]} v(x,t)). \tag{3.21}$$

We have

$$v(x,0) = g(x) - \frac{\mu}{(T+\varepsilon)^{\frac{n}{2}}} e^{\frac{|x|^2}{4(T+\varepsilon)}} \le \sup_{\mathbb{R}^n} g. \tag{3.22}$$

Let $\Omega = B_R(0)$, then

$$\max_{\bar{B}_R(0)\times[0,T]} v \le \max\left(\sup_{\mathbb{R}^n} g, \max_{|x|=R, t\in[0,T]} v(x,t)\right). \tag{3.23}$$

For $|x| = R$ and $t \in (0,T)$

$$\begin{aligned} v(x,t) &= u(x,t) - \frac{\mu}{(T+\varepsilon-t)^{\frac{n}{2}}} e^{\frac{R^2}{4(T+\varepsilon-t)}} \\ &\le Ae^{aR^2} - \frac{\mu}{(T+\varepsilon-t)^{\frac{n}{2}}} e^{\frac{R^2}{4(T+\varepsilon-t)}} \\ &\le Ae^{aR^2} - \frac{\mu}{(T+\varepsilon)^{\frac{n}{2}}} e^{\frac{R^2}{4(T+\varepsilon)}} \end{aligned}$$

Since $4aT < 1$, there exist $\varepsilon > 0, \gamma > 0$, such that

$$a + \gamma = \frac{1}{4(T+\varepsilon)} \tag{3.24}$$

and hence

$$v(x,t) \le Ae^{aR^2} - \frac{\mu}{(T+\varepsilon)^{\frac{n}{2}}} e^{aR^2+\gamma R^2}. \tag{3.25}$$

In particular, the right term dominates for $R >> 0$: in particular for all large $R > 0$ we have $v(x,t) \le g(0)$. So for large R and $|x| = R$ we have for all $t \in (0,T]$,

$$v(x,t) \le g(0) \le \sup_{\mathbb{R}^n} g \tag{3.26}$$

and so

$$\max_{(x,t)\in \overline{B_R(0)\times(0,T]}} v(x,t) \le \sup_{\mathbb{R}^n} g \quad \forall R >> 1. \tag{3.27}$$

Letting $R \to \infty$ we find that

$$\sup_{\mathbb{R}^n\times[0,T]} v(x,t) \le \sup_{\mathbb{R}^n} g, \tag{3.28}$$

i.e.

$$\sup_{\mathbb{R}^n\times[0,T]}\left(u(x,t)-\frac{\mu}{(T+\varepsilon-t)^{\frac{n}{2}}}e^{\frac{|x|^2}{4(T+\varepsilon-t)}}\right)\leq\sup_{\mathbb{R}^n}g \tag{3.29}$$

This holds for any any $\mu > 0$.

Now fix $\rho > 0$. Then we have in particular

$$\sup_{B(0,\rho)\times[0,T]} v(x,t)\leq\sup_{\mathbb{R}^n}g,$$

and thus

$$\sup_{B(0,\rho)\times[0,T]} u(x,t)-\mu\underbrace{\sup_{B(0,\rho)\times[0,T]}\frac{1}{(T+\varepsilon-t)^{\frac{n}{2}}}e^{\frac{|x|^2}{4(T+\varepsilon-t)}}}_{<\infty}\leq\sup_{\mathbb{R}^n}g.$$

Letting $\mu \to 0$ for fixed[8] ρ

$$\sup_{B(0,\rho)\times[0,T]} u(x,t)\leq\sup_{\mathbb{R}^n}g,$$

Now we let $\rho \to \infty$ to conclude

$$\sup_{\mathbb{R}^n\times[0,T]} u(x,t)\leq\sup_{\mathbb{R}^n}g,$$

i.e. we have the claim under the assumption that $4aT < 1$.

If $4aT \geq 1$, we can slice the time interval $(0,T]$ into parts $(0,T_1]\cup(T_1,T_2]\cup\ldots\cup(T_K,T]$ with $4a(T_{i+1}-T_i) < 1$ for all i. Using the estimate in each of these time intervals we conclude. $\square$

Theorem 3.15. *Let $g \in C^0(\mathbb{R}^n)$, $f \in C^0(\mathbb{R}^n\times[0,T])$. Then there is at most one solution $u \in C^2(\mathbb{R}^n\times(0,T])\cap C^0(\mathbb{R}^n\times[0,T])$ of*

$$\begin{aligned}(\partial_t-\Delta)u&=f\quad\text{in }\mathbb{R}^n\times(0,T)\\ u&=g\quad\text{on }\mathbb{R}^n\times\{0\}\end{aligned} \tag{3.30}$$

with

$$|u(x,t)|\leq Ae^{a|x|^2}\quad\forall(x,t)\in\mathbb{R}^n\times(0,T). \tag{3.31}$$

Exercise 3.16. *Prove Theorem 3.15*

Without the assumption (3.31), Theorem 3.15 may fail. These solutions are sometimes called *non-physical solutions*, since they grow too fast.

[8] this does not work if $\rho = \infty$!

Exercise 3.17. *(cf. [John, 1991]) Define the following Tychonoff-function,*

$$u(x,t) := \sum_{k=0}^{\infty} \frac{g^{(k)}(t)}{(2k)!}\, x^{2k}.$$

Here $g^{(k)}$ denotes the k-th derivative of g, given as

$$g(t) := \begin{cases} e^{(-t^{-\alpha})} & t > 0 \\ 0 & t \leq 0. \end{cases}$$

(1) *Show that $u \in C^2(\mathbb{R}^2_+) \cap C^0(\mathbb{R} \times [0,\infty))$.*
(2) *Show moreover that*

$$\begin{cases} (\partial_t - \Delta)u = 0 & \text{in } \mathbb{R}^n \times (0,T), \\ u(x,0) = 0 & \text{für } x \in \mathbb{R}^n. \end{cases} \tag{3.32}$$

(3) *Find a different solution $v \not\equiv u$ of (3.32).*
(4) *Why (without proof) does this not contradict 3.15?*

3.6. **Harnack's Principle.** In the parabolic setting an "immediate" Harnack principle is not true in general, to compare sup and inf of a function one needs to wait for an (arbitrary short) amount of time.

Theorem 3.18 (Parabolic Harnack inequality)**.** *Assume $u \in C^2(\mathbb{R}^n \times (0,T]) \cap L^\infty(\mathbb{R}^n \times [0,T])$ and solves*

$$\partial_t u - \Delta u = 0 \quad \text{in } \mathbb{R}^n \times (0,T)$$

and

$$u \geq 0 \quad \text{in } \mathbb{R}^n \times (0,T)$$

Then for any compactum $K \subset \mathbb{R}^n$ and any $0 < t_1 < t_2 < T$ there exists a constant C, so that

$$\sup_{x \in K} u(x,t_1) \leq C \inf_{y \in K} u(y,t_2)$$

Proof. By the representation formula, Section 3.3, and uniqueness of the Cauchy problem

$$u(x_2,t_2) = \int_{\mathbb{R}^n} \frac{1}{(4\pi t_2)^{\frac{n}{2}}} e^{-\frac{|x_2-y|^2}{4t_2}}\, u_0(y)\, dy.$$

Now, for $t_1 < t_2$ whenever $|x_1|$, $|x_2| \leq \Lambda < \infty$, there exists a constant $C = C(|t_1 - t_2|, \Lambda)$ so that

$$-\frac{|x_2-y|^2}{4t_2} \geq -\frac{|x_1-y|^2}{4t_1} - C. \quad \forall y \in \mathbb{R}^n$$

See Exercise 3.19.

Consequently,

$$u(x_2,t_2) \geq \left(\frac{t_1}{t_2}\right)^{\frac{n}{2}} e^{-C} \int_{\mathbb{R}^n} \frac{1}{(t_1)^{\frac{n}{2}}} e^{-\frac{|x_1-y|^2}{4t_1}}\ u_0(y)\ dy = \left(\frac{t_1}{t_2}\right)^{\frac{n}{2}} e^{-C} u(x_1,t_1).$$

□

Exercise 3.19. *Show the following estimate, which we used for Harnack-principle, Theorem 3.18:*

If $K \subset \mathbb{R}^n$ is compact and $0 < t_1 < t_2 < \infty$, then there exists a constant $C > 0$ depending on K and $t_2, t_1 > 0$, such that

$$\frac{|x_1-y|^2}{t_2} \leq \frac{|x_2-y|^2}{t_1} + C \quad \forall x_1, x_2 \in K,\ y \in \mathbb{R}^n.$$

Exercise 3.20 (Counterexample Harnack)**.** *(1) Let $u_0 : \mathbb{R}^n \to [0,\infty)$ a smooth function with compact support such that $u_0(0) = 1$. Set*

$$u(x,t) := \int_{\mathbb{R}^n} \Phi(x-y,t)\ u_0(y) \quad t > 0$$

Show that

$$\inf_{x\in\mathbb{R}^n} u(x,t) = 0 \quad \textit{for all } t > 0.$$

However

$$\sup_{x\in\mathbb{R}^n} u(x,t) > 0 \quad \textit{for all } t > 0.$$

Why does this not contradict Harnack's principles, Theorem 3.18?

(2) Let us consider one space-dimension. Let $\xi \in \mathbb{R}$ be given and u defined as

$$u_\xi(x,t) := (t+1)^{-\frac{1}{2}} e^{-\frac{|x+\xi|^2}{4(t+1)}}.$$

Show that u is a solution of $(\partial_t - \Delta)u = 0$ in $\mathbb{R} \times (0,\infty)$.

Moroever show for each fixed $t > 0$ there is no constant $C = C(t) > 0$ such that

$$\sup_{x\in[-1,1]} u_\xi(x,t) \leq C \inf_{y\in[-1,1]} u_\xi(y,t) \quad \forall \xi \in \mathbb{R}^n.$$

Why does this not contradict Harnack's principles, Theorem 3.18?

Hint: Choose $x = -\frac{\xi}{|\xi|}$ and $y = 0$. What happens if $|\xi| \to \infty$?

3.7. **Parabolic scaling.** While we will not use it in this (short) section, let us introduce the notion of parabolic scaling.

Exercise 3.21. *Assume that $\Omega \subset \mathbb{R}^n$ is an open set and $u \in C^2(\Omega)$, $f \in C^0(\Omega)$ solve*

$$\Delta u = f \quad \textit{in } \Omega$$

Let $r > 0$ and set

$$u_r(x) := u(rx), \quad f_r := f(rx).$$

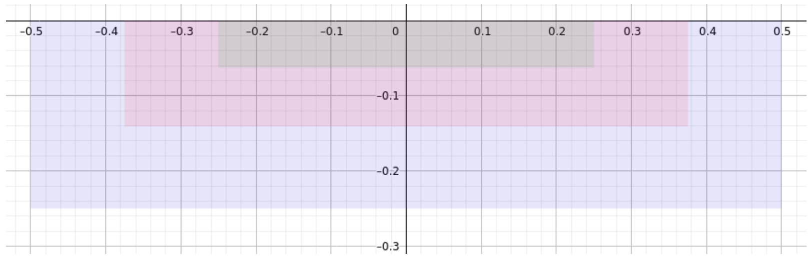


FIGURE 3.3. The sets $C(0,0,r)$, $C(0,0,\frac{3}{4}r)$ and $C(0,0,\frac{1}{2}r)$ with $r=\frac{1}{2}$ from the proof of Theorem 3.23.

Show that

$$\Delta u_r = r^2 f_r \quad \textit{in } \frac{1}{r}\Omega$$

where

$$\frac{1}{r}\Omega = \left\{\frac{1}{r}y : y \in \Omega\right\}.$$

If we try to get the same for the equation $(\partial_t - \Delta)u = f$ we have to use a *parabolic scaling*

Exercise 3.22. *Assume that $\Omega \subset \mathbb{R}^n$, $T > 0$ is an open set and $u \in C^2(\Omega)$ solves*

$$(\partial_t - \Delta)u = f \quad \textit{in } \Omega \times (0,T]$$

Let $r > 0$ and set

$$u_r(x,t) := u(rx, r^2 t), \quad f_r := f(rx, r^2 t).$$

Show that

$$(\partial_t - \Delta)u_r = r^2 f_r \quad \textit{in } \frac{1}{r}\Omega \times (0, \frac{T}{r^2}]$$

where

$$\frac{1}{r}\Omega = \left\{\frac{1}{r}y : y \in \Omega\right\}.$$

3.8. Regularity and Cauchy-estimates.

Theorem 3.23 (Smoothness). *Let $u \in C^2(\Omega_T)$ satisfy*

$$\partial_t u = \Delta u \quad \textit{in } \Omega_T. \tag{3.33}$$

Then $u \in C^\infty(\mathrm{int}(\Omega_T))$.

Proof. The main idea is to transform the equation in Ω_T into an equation in $\mathbb{R}^n \times [0,T]$ and use the representation from Theorem 3.4. This is a very common and very useful technique:

Fix some $(x_0,t_0) \in \Omega_T$, i.e. $x_0 \in \Omega$ and $t \in (0,T]$.

We will use the parabolic regions

$$C(x,t;r) = \{(y,s)\colon |x-y| \le r, t-r^2 \le s \le t\}. \tag{3.34}$$

We set

$$C_1 = C(x_0,t_0;r), \quad C_2 = C\left(x_0,t_0;\frac{3}{4}r\right), \quad C_3 = C\left(x_0,t_0;\frac{r}{2}\right) \tag{3.35}$$

for some suitably small $r > 0$ such that $C_1 \subset \Omega_T$. Cf. Figure 3.3.

We now choose a cut-off function

$$\eta \in C^\infty(\mathbb{R}^n \times [0,t_0]) \tag{3.36}$$

with $0 \le \eta \le 1$, $\eta_{|C_2} \equiv 1$, $\eta \equiv 0$ around $\mathbb{R}^n \times [0,t_0] \backslash C_1$.

Set

$$v(x,t) = \eta(x,t)u(x,t) \quad \forall (x,t) \in \mathbb{R}^n \times (0,t_0]. \tag{3.37}$$

This is well defined for all $(x,t) \in \mathbb{R}^n \times (0,t_0]$ because $\eta \equiv 0$ where u is not defined!

Then in $\mathbb{R}^n \times (0,t_0]$ we have the following equation (using that $(\partial_t - \Delta)u = 0$ in any point where $\eta \neq 0$.

$$\begin{aligned} \partial_t v - \Delta v &= \partial_t u\eta + \partial_t \eta u - \eta\Delta u - u\Delta\eta - 2\langle \nabla u, \nabla\eta\rangle \\ &= \partial_t\eta u - u\Delta\eta - 2\langle\nabla u,\nabla\eta\rangle \\ &=: f(x,t). \end{aligned} \tag{3.38}$$

Observe, $v \in C^2(\mathbb{R}^n \times [0,t_0]$ and $f \in C^1(\mathbb{R}^n \times [0,t_0])$.

By Theorem 3.4

$$\begin{aligned} v(x,t) &= \int_0^t \int_{\mathbb{R}^n} \Phi(x-y,t-s) f(y,s)\ dyds \\ &= \int_0^t \int_{\mathbb{R}^n} \Phi(x-y,t-s) \Big(u(y,s)\partial_t\eta(y,s) - u(y,s)\Delta\eta(y,s) \\ &\quad - 2\langle \nabla u(y,s), \nabla\eta(y,s)\rangle \Big)\ dyds \end{aligned} \tag{3.39}$$

If we assume $(x,t) \in C_3$ we see that $\partial_t\eta(y,s), \Delta\eta(y,s), \nabla\eta(y,s) \equiv 0$ around $y = x$ and $s = t$.

That is for any $(x,t) \in C_3$ we have

$$v(x,t) = \int_{\mathbb{R}^n} K(x,y,s,t)\, u(y,s)\ dyds \tag{3.40}$$

where $K(x,\cdot,s,t)$ has uniformly compact support in $\mathbb{R}^n$ and is $K(\cdot)$ is smooth in all variables.

Thus v is smooth and so is $u \equiv v$ around (x_0, t_0). $\square$

We can make the above more precise

Theorem 3.24 (Cauchy estimates)**.** *For all $k, l \in \mathbb{N}$ there exists $C > 0$ such that for all $u \in C^{2,1}(\Omega_T)$ ($u \in L^1_{\text{loc}}$ will be sufficient), solving*

$$(\partial_t - \Delta)\, u = 0, \tag{3.41}$$

there holds

$$\max_{C(x_0,t_0;\frac{r}{2})} |D_x^k \partial_t^l u| \leq \frac{C}{r^{k+2l+n+2}} \|u\|_{L^1(C(x_0,t_0;r))} \tag{3.42}$$

for all $C(x_0, t_0; r) \subset \Omega_T$.

Proof. Suppose first $(x_0, t_0) = (0, 0)$ and $r = 1$. Set

$$C(1) = C(0, 0; 1). \tag{3.43}$$

Then as in the proof of Theorem 3.23 we have

$$u(x,t) = \int_{C(1)} K(x,t,y,s) u(y,s)\ dyds \quad \forall (x,t) \in C\left(\frac{1}{2}\right). \tag{3.44}$$

Then

$$D_x^k \partial_t^l u(x,t) = \int_{C(1)} \left(D_x^k \partial_t^l K(x,t,y,s)\right) u(y,s)\ dyds \tag{3.45}$$

and hence

$$|D_x^k \partial_t^l u(x,t)| \leq C_{k,l} \|u\|_{L^1(C(1))} \quad \forall (x,t) \in C\left(\frac{1}{2}\right). \tag{3.46}$$

Thus the claim is proven for $r = 1$. For $r > 0$ and $(x_0, t_0) \in \mathbb{R}^{n+1}$ set

$$v(x,t) = u(x_0 + rx, t_0 + r^2 t). \tag{3.47}$$

Then

$$\max_{C(\frac{1}{2})} |D_x^k \partial_t^l v| \leq C_{k,l} \|v\|_{L^1(C(1))}. \tag{3.48}$$

Hence

$$\max_{C(x_0,r_0;\frac{r}{2})} |D_x^k \partial_t^l u| r^{k+2l} \leq C_{k,l} r^{-(n+2)} \|u\|_{L^1(C(1))}. \tag{3.49}$$

$\square$

3.9. **Variational Methods.** Consider

$$\begin{cases} \partial_t u - \Delta u = f & \text{in } \Omega \in (0,T) \\ u = g & \text{on } \Omega \times \{0\},\ \partial\Omega \times [0,T), \end{cases}$$

and we want to discuss uniqueness – but for some reason we dont want to use maximum principles.

Assume there is anothter solution of the same problem, lets call it v. Then set $w := u - v$, then that would solve

$$\begin{cases} \partial_t w - \Delta w = 0 & \text{in } \Omega \in (0,T) \\ w = 0 & \text{on } \Omega \times \{0\},\ \partial\Omega \times [0,T), \end{cases}$$

As in the Laplace equation case, we multiply this equation by w, and we find

$$\partial_t \int_\Omega |w|^2 = -2 \int_\Omega |Dw|^2$$

Observe the right-hand side is negative (unless w is constant, then it is zero – this is called the *energy decay*). Anyways, integrating this equation we obtain

$$\int_\Omega |w(t)|^2 - \int_\Omega |w(0)|^2 = -2 \int \int_\Omega |Dw|^2.$$

That implies that if $w(0) = 0$ (which it is by assumption), then $w(T) = 0$. That is $w(t) \equiv 0$, i.e. $u = v$.

4. Wave Equation

The wave equation is written as

$$\partial_{tt} u - \Delta u = 0.$$

Alternatively we can think of it as

$$\partial_{tt} u = \Delta u.$$

In this form, we can consider it as Newton's law: Force equals mass times acceleration. The mass is set to 1. If we think about $u(x,t)$ as the dilation of a surface from an equilibrium state (if x is one dimensional, then height of string) then $\Delta u(x,t)$ is proportional to the stress that this dilation exacts on the surface, i.e. the force. By Newton's law, this force Δu is equal to the acceleration $\partial_{tt} u$ – and this is the wave equation.

In one space dimension

$$\partial_{tt} u - \partial_{xx} u = (\partial_t - \partial_x)(\partial_t + \partial_x) u.$$

So we could hope by solving the one-dimension wave equation by considering solutions of

$$\partial_t u \pm u_x = 0.$$

This is a transport equation which could be solved via the method of characteristics.

In more than one space dimension this is more complicated, because Du is a vector, so

$$\partial_{tt}u - \Delta u = (\partial_t - D)(\partial_t + D)$$

does not really make sense. What would make sense it so

$$\partial_{tt}u - \Delta u = \left(\partial_t - \mathbf{i}\sqrt{-\Delta}\right)\left(\partial_t + \mathbf{i}\sqrt{-\Delta}\right) u$$

if only we understood $\sqrt{-\Delta}$ (we can e.g. via Fourier transform). This is called the halfwave decomposition.

4.1. **Global Solution via Fourier transform.** We want to consider the wave equation

$$\begin{cases}(\partial_{tt} - \Delta_x)u = 0 \quad \text{in } \mathbb{R}^n \times \mathbb{R}\end{cases}$$

If we again take the point of view that this is an ODE in time then this is a second order ODE, so the initial value problem should depend on $u(0)$ and $\partial_t u(0)$.

$$\begin{cases}(\partial_{tt} - \Delta_x)u = 0 & \text{in } \mathbb{R}^n \times \mathbb{R}\\ u(0,x) = u_0(x) \quad \text{in } \mathbb{R}^n \\ \partial_t u(0,x) = v_0(x) \quad \text{in } \mathbb{R}^n\end{cases}$$

Let us take the Fourier transform in space, then the above becomes

$$\begin{cases}\partial_{tt}u(\hat{\xi},t) + c|\xi|^2 u(\hat{\xi},t) = 0, & \text{in } \mathbb{R}^n \times \mathbb{R}\\ u(0,x) = u_0(x) & \text{in } \mathbb{R}^n \\ \partial_t u(0,x) = v_0(x) & \text{in } \mathbb{R}^n\end{cases}$$

This is an equation of the type

$$g''(t) = -cg(t)$$

Fundamental solutions to this equation are $\sin(\sqrt{c}t)$ and $\cos(\sqrt{c}t)$ – which gets messy. It is more convenient to use complex notation: For some $A \in \mathbb{C}$,

$$g(t) = A\, e^{\mathbf{i}\sqrt{c}|\xi|t} + B\, e^{-\mathbf{i}\sqrt{c}|\xi|t}$$

and we must choose $A, B \in \mathbb{C}$ so that

$$\hat{u}_0(\xi) = g(0) = A + B$$

and

$$\hat{v_0}(\xi) = g'(0) = \mathbf{i}\sqrt{c}\,|\xi|\,(A - B)\,.$$

or equivalently (unless $|\xi| = 0$)

$$\frac{\hat{v_0}(\xi)}{\mathbf{i}\sqrt{c}|\xi|} = (A - B)\,.$$

We add the equation for $A + B$ to the equation for $A - B$ and find

$$A = \frac{1}{2}\hat{u}_0(\xi) + \frac{1}{2\mathbf{i}\sqrt{c}|\xi|}\,\hat{v_0}(\xi)$$

and subtracting the equation for $A-B$ from the equation for $A+B$ we have

$$B = \frac{1}{2}\hat{u}_0(\xi) - \frac{1}{\mathbf{i}\sqrt{c}|\xi|}\hat{v_0}(\xi)$$

Together we have found that

$$g(t) = \hat{u}_0(\xi)\left(\frac{1}{2}e^{\mathbf{i}\sqrt{c}|\xi|t} + \frac{1}{2}e^{-\mathbf{i}\sqrt{c}|\xi|t}\right) + \frac{1}{2\mathbf{i}\sqrt{c}|\xi|}\hat{v_0}(\xi)\left(e^{\mathbf{i}\sqrt{c}|\xi|t} - e^{-\mathbf{i}\sqrt{c}|\xi|t}\right)$$

If we call suggestively

$$e^{\mathbf{i}t\sqrt{-\Delta}}f := \mathcal{F}^{-1}(e^{\mathbf{i}t\sqrt{c}|\xi|}\mathcal{F}f)$$

we have the semigroup representation

$$u(x,t) = \frac{e^{\mathbf{i}t\sqrt{-\Delta}} + e^{-\mathbf{i}t\sqrt{-\Delta}}}{2}u_0(x) + \frac{e^{\mathbf{i}t\sqrt{-\Delta}} - e^{-\mathbf{i}t\sqrt{-\Delta}}}{2\mathbf{i}}\sqrt{-\Delta}^{-1}v_0(x)$$

We next discuss the *Duhamel principle*:

If we want to consider

$$\begin{cases} (\partial_{tt} - \Delta_x)u = f & \text{in } \mathbb{R}^n \times \mathbb{R} \\ u(0,x) = u_0(x) \quad \text{in } \mathbb{R}^n \\ \partial_t u(0,x) = v_0(x) \quad \text{in } \mathbb{R}^n \end{cases}$$

$$\begin{aligned} u(x,t) =& \frac{e^{\mathbf{i}t\sqrt{-\Delta}} + e^{-\mathbf{i}t\sqrt{-\Delta}}}{2}u_0(x) + \frac{e^{\mathbf{i}t\sqrt{-\Delta}} - e^{-\mathbf{i}t\sqrt{-\Delta}}}{2\mathbf{i}}\sqrt{-\Delta}^{-1}v_0(x) \\ &+ \int_0^t \frac{e^{\mathbf{i}(t-s)\sqrt{-\Delta}} - e^{-\mathbf{i}(t-s)\sqrt{-\Delta}}}{2\mathbf{i}}\sqrt{-\Delta}^{-1}f(x,s)ds. \end{aligned}$$

Indeed,

$$\int_0^t \frac{e^{\mathbf{i}(t-s)\sqrt{-\Delta}} - e^{-\mathbf{i}(t-s)\sqrt{-\Delta}}}{2\mathbf{i}}\sqrt{-\Delta}^{-1}f(x,s)ds\Big|_{t=0} = 0$$

$$\partial_t\Big|_{t=0}\int_0^t \frac{e^{\mathbf{i}(t-s)\sqrt{-\Delta}} - e^{-\mathbf{i}(t-s)\sqrt{-\Delta}}}{2\mathbf{i}}\sqrt{-\Delta}^{-1}f(x,s)ds = 0$$

and

$$
\begin{aligned}
&\partial_{tt}\int_0^t \frac{e^{\mathbf{i}(t-s)\sqrt{-\Delta}}-e^{-\mathbf{i}(t-s)\sqrt{-\Delta}}}{2\mathbf{i}}\sqrt{-\Delta}^{-1}f(x,s)ds\\
=&\partial_t\left(\frac{e^{\mathbf{i}0\sqrt{-\Delta}}-e^{-\mathbf{i}0\sqrt{-\Delta}}}{2\mathbf{i}}\sqrt{-\Delta}^{-1}f(x,t)+\int_0^t\partial_t\frac{e^{\mathbf{i}(t-s)\sqrt{-\Delta}}-e^{-\mathbf{i}(t-s)\sqrt{-\Delta}}}{2\mathbf{i}}\sqrt{-\Delta}^{-1}f(x,s)ds\right)\\
=&\partial_t\left(\int_0^t\partial_t\frac{e^{\mathbf{i}(t-s)\sqrt{-\Delta}}-e^{-\mathbf{i}(t-s)\sqrt{-\Delta}}}{2\mathbf{i}}\sqrt{-\Delta}^{-1}f(x,s)ds\right)\\
=&\partial_t\left(\int_0^t\frac{e^{\mathbf{i}(t-s)\sqrt{-\Delta}}\mathbf{i}\sqrt{-\Delta}+\mathbf{i}\sqrt{-\Delta}e^{-\mathbf{i}(t-s)\sqrt{-\Delta}}}{2\mathbf{i}}\sqrt{-\Delta}^{-1}f(x,s)ds\right)\\
=&\frac{e^{\mathbf{i}0\sqrt{-\Delta}}\mathbf{i}\sqrt{-\Delta}+\mathbf{i}\sqrt{-\Delta}e^{-\mathbf{i}0\sqrt{-\Delta}}}{2\mathbf{i}}\sqrt{-\Delta}^{-1}f(x,t)\\
&+\int_0^t\partial_t\frac{e^{\mathbf{i}(t-s)\sqrt{-\Delta}}\mathbf{i}\sqrt{-\Delta}+\mathbf{i}\sqrt{-\Delta}e^{-\mathbf{i}(t-s)\sqrt{-\Delta}}}{2\mathbf{i}}\sqrt{-\Delta}^{-1}f(x,s)ds\\
=&\sqrt{-\Delta}\sqrt{-\Delta}^{-1}f(x,t)+\int_0^t\frac{-e^{\mathbf{i}(t-s)\sqrt{-\Delta}}\sqrt{-\Delta}^2+\sqrt{-\Delta}^2e^{-\mathbf{i}(t-s)\sqrt{-\Delta}}}{2\mathbf{i}}\sqrt{-\Delta}^{-1}f(x,s)ds\\
=&f(x,t)-\sqrt{-\Delta}^2\int_0^t\frac{e^{\mathbf{i}(t-s)\sqrt{-\Delta}}-e^{-\mathbf{i}(t-s)\sqrt{-\Delta}}}{2\mathbf{i}}\sqrt{-\Delta}^{-1}f(x,s)ds\\
=&f(x,t)+\Delta\int_0^t\frac{e^{\mathbf{i}(t-s)\sqrt{-\Delta}}-e^{-\mathbf{i}(t-s)\sqrt{-\Delta}}}{2\mathbf{i}}\sqrt{-\Delta}^{-1}f(x,s)ds
\end{aligned}
$$

or, in other words,

$$
\begin{aligned}
&(\partial_{tt}-\Delta)\int_0^t\frac{e^{\mathbf{i}(t-s)\sqrt{-\Delta}}-e^{-\mathbf{i}(t-s)\sqrt{-\Delta}}}{2\mathbf{i}}\sqrt{-\Delta}^{-1}f(x,s)ds\\
=&f(x,t)
\end{aligned}
$$

4.2. **Energy methods.** Cf. [Evans, 2010, 2.4.3].

Consider solutions to the inhomogeneous wave equation.

$$
\begin{cases}(\partial_{tt}-\Delta)u=f & \text{in } \Omega\times(0,T)\\ u=g & \text{on } \Omega\times\{0\}\cup\partial\Omega\times(0,T)\\ \partial_t u=h & \text{on } \Omega\times\{0\}\end{cases}\tag{4.1}
$$

Theorem 4.1 (Uniqueness)**.** *There exist at most one function $u\in C^2(\overline{\Omega}\times[0,T))$ which solves (4.1).*

Proof. Assume there are two solutions $u, v \in C^2(\overline{\Omega} \times [0,T))$. Then we can consider $w := u - v$ which solves

$$\begin{cases} (\partial_{tt} - \Delta) w = 0 & \text{in } \Omega \times (0,T) \\ w = 0 & \text{on } \Omega \times \{0\} \cup \partial\Omega \times (0,T) \\ \partial_t w = 0 & \text{on } \Omega \times \{0\} \end{cases} \tag{4.2}$$

For $t \in [0,T)$ define

$$E(t) := \frac{1}{2} \int_\Omega |\partial_t w(x,t)|^2 \, dx + \int_\Omega |Dw(x,t)|^2 \, dx.$$

We compute the derivative of E (which we can do since $w \in C^2$,

$$\begin{aligned} \dot{E}(t) &= \int_\Omega \partial_t w(x,t) \, \partial_{tt} w(x,t) \, dx + \int_\Omega Dw(x,t) D\partial_t w(x,t) \, dx \\ &= \int_\Omega \partial_t w(x,t) \, \partial_{tt} w(x,t) \, dx - \int_\Omega \operatorname{div}(Dw(x,t)) \partial_t w(x,t) \, dx \\ &= \int_\Omega \partial_t w(x,t) \, (\partial_{tt} - \Delta) w(x,t) \, dx \\ &\overset{(4.2)}{=} \int_\Omega \partial_t w(x,t) \, 0 \, dx = 0. \end{aligned}$$

That is we have $\dot{E}(t) = 0$ for all $t \in (0,T)$

$$E(t) = E(0) \overset{(4.2)}{=} 0$$

In particular $Dw \equiv 0$, so w is constant, and because of the boundary conditions in (4.2) we conclude $w \equiv 0$. Thus $u \equiv v$. ☐

5. Black Box – Sobolev Spaces

A remark on literature: A standard reference for Sobolev spaces is [Adams and Fournier, 2003]. Very readable is also [Evans and Gariepy, 2015]. Popular is also the introduction to Sobolev spaces in [Evans, 2010]. A classical reference Sobolev spaces in PDEs is [Gilbarg and Trudinger, 2001]. Also [Ziemer, 1989]. For very delicate problems one might also consult [Maz'ya, 2011].

Definition 5.1. (1) Let $1 \le p \le \infty$, $k \in \mathbb{N}$ and $\Omega \subset \mathbb{R}^n$ open, nonempty. The *Sobolev space* $W^{k,p}(\Omega)$ is the set of functions

$$u \in L^p(\Omega)$$

such that for any multiinidex γ, $|\gamma| \le k$ we find a function (the *distributional γ-derivative* or *weak γ-derivative*) *"$\partial^\gamma u$"*$\in L^p(\Omega)$ such that

$$\int_\Omega u \, \partial^\gamma \varphi = (-1)^{|\gamma|} \int_\Omega \text{“}\partial^\gamma u\text{”} \, \varphi \quad \forall \varphi \in C_c^\infty(\Omega).$$

Such u are also sometimes called Sobolev-functions.

(2) For simplicity we write $W^{0,p} = L^p$.

(3) The norm of the Sobolev space $W^{k,p}(\Omega)$ is given as

$$\|u\|_{W^{k,p}(\Omega)} = \sum_{|\gamma| \le k} \|\partial^\gamma u\|_{L^p(\mathbb{R}^n)}$$

or equivalently (exercise!)

$$\|u\|_{W^{k,p}(\Omega)} = \left(\sum_{|\gamma| \le k} \|\partial^\gamma u\|^p_{L^p(\mathbb{R}^n)} \right)^{\frac{1}{p}}.$$

(4) We define another Sobolev space $H^{k,p}(\Omega)$ as follows

$$H^{k,p}(\Omega) = \overline{C^\infty(\overline{\Omega})}^{\|\cdot\|_{W^{k,p}(\Omega)}}.$$

that is the (metric) closure or completion of the space $(C^\infty(\Omega), \|\cdot\|_{W^{k,p}(\Omega)})$. In yet other words, $H^{k,p}(\Omega)$ consists of such functions $u \in L^p(\Omega)$ such that there exist approximations $u_k \in C^\infty(\overline{\Omega})$ with

$$\|u_k - u\|_{W^{k,p}(\Omega)} \xrightarrow{k \to \infty} 0.$$

We will later see that $H^{k,p}$ is the same as $W^{k,p}$ locally, or for nice enough domains; and use the notation H or W interchangeably.

(5) Now we introduce the Sobolev space $H^{k,p}_0(\Omega)$

$$H^{k,p}_0(\Omega) = \overline{C^\infty_c(\overline{\Omega})}^{\|\cdot\|_{W^{k,p}(\Omega)}}.$$

We will later see that this space consists of all maps $u \in H^{k,p}(\Omega)$ that satisfy $u, \nabla u, \ldots \nabla^{k-1} u \equiv 0$ on $\partial\Omega$ in a suitable sense (the *trace sense*, for a precise formulation see Theorem 5.28). – Again, later we see that $H = W$ and thus, $W^{k,p}_0(\Omega) = H^{k,p}_0(\Omega)$ for nice sets Ω.

Observe that $L^p(\Omega) = W^{0,p}(\Omega) = W^{0,p}_0(\Omega)$.

(6) The local space $W^{k,p}_{loc}(\Omega)$ is similarly defined as $L^p_{loc}(\Omega)$. A map belongs to $u \in W^{k,p}_{loc}(\Omega)$ if for any $\Omega' \subset\subset \Omega$ we have $u \in W^{k,p}(\Omega')$.

Remark 5.2. Some people write $H^{k,p}(\Omega)$ instead of $W^{k,p}(\Omega)$. Other people use $H^k(\Omega)$ for $H^{k,2}$ – notation is inconsistent...

Some people claim that W stand for Weyl, and H for Hardy or Hilbert.

Example 5.3. For $s > 0$ let

$$f(x) := |x|^{-s}.$$

Observe that f is only defined for $x \neq 0$, but since measurable functions need only be defined outside of a null-set this is still a reasonable function.

We have already seen, when working with fundamental solutions, that $f \in L^p_{loc}(\mathbb{R}^n)$ for any $1 \le p < \frac{n}{s}$.

We can compute for $x \neq 0$ that

$$\partial_i f(x) = -s\,|x|^{-s-2} x^i \tag{5.1}$$

and by the same argument as above we could conjecture that $\partial_i f \in L^q_{loc}(\mathbb{R}^n)$ for any $1 \leq q < \frac{n}{s+1}$.

Exercise 5.4. *Show that*

(1) (5.1) *holds in the distributional sense, i.e. that if $n \geq 2$ and $0 < s < n-1$ then for any $\varphi \in C_c^\infty(\mathbb{R}^n)$,*

$$\int_{\mathbb{R}^n} f(x)\,\partial_i\varphi(x)\,dx = \int_{\mathbb{R}^n} s\,|x|^{-s-2}x^i\varphi(x)\,dx.$$

(2) *to conclude that $f \in W^{1,q}_{loc}(\mathbb{R}^n)$ for any $1 \leq q < \frac{n}{s+1}$.*

However pointwise a.e. derivatives and the Sobolev/distributional derivative does not necessarily coincide always:

Exercise 5.5. *Let $\Omega = (-1,1)$ and consider the* Heaviside function

$$f(x) = \begin{cases} -0 & x < 0 \\ 1 & x \geq 0 \end{cases}$$

Show that

(1)
$$f'(x) = 0 \quad \text{for a.e. } x \in (-1,1)$$

(2) *$f' \notin L^1((-1,1))$ in the sense of Sobolev spaces – i.e. $f \notin W^{1,1}((-1,1))$.*
Hint: *You can first show*

$$\int_\Omega f\varphi'(x) = \varphi(0).$$

Exercise 5.6. *Let*

$$f(x) := \log|x|.$$

One can show that $f \in L^p_{loc}(\mathbb{R}^n)$ for any $1 \leq p < \infty$, and $f \in W^{1,p}_{loc}(\mathbb{R}^n)$ for all $p \in [1,n)$, if $n \geq 2$.

Exercise 5.7. *Let*

$$f(x) := \log\log\frac{2}{|x|} \quad \text{in } B(0,1)$$

One can show that for $n \geq 2$, $f \in W^{1,n}(B(0,1))$.

Moreover, for $n = 2$, in distributional sense

$$-\Delta f = |Df|^2$$

Observe that this serves as an example for solutions to nice differential equations that are not continuous!

Proposition 5.8 (Basic properties of weak derivatives)**.** *Let $u, v \in W^{k,p}(\Omega)$ and $|\gamma| \leq k$. Then*

(1) $\partial^\gamma u \in W^{k-|\gamma|,p}(\Omega)$.
(2) *Moreover* $\partial^\alpha \partial^\beta u = \partial^\beta \partial^\alpha u = \partial^{\alpha+\beta} u$ *if* $|\alpha| + |\beta| \leq k$.
(3) *For each* $\lambda, \mu \in \mathbb{R}$ *we have* $\lambda u + \mu v \in W^{k,p}(\Omega)$ *and*

$$\partial^\alpha(\lambda u + \mu v) = \lambda \partial^\alpha u + \mu \partial^\alpha v$$

(4) *If* $\Omega' \subset \Omega$ *is open then* $u \in W^{k,p}(\Omega')$
(5) *For any* $\eta \in C_c^\infty(\Omega)$, $\eta u \in W^{k,p}$ *and (if* $k \geq 1$*), and we have the Leibniz formula (aka product rule)*

$$\partial_i(\eta u) = \partial_i \eta\, u + \eta \partial_i u.$$

(6) *if* $f : \mathbb{R} \to \mathbb{R}$ *is Lipschitz and bounded, and* $u \in W^{1,p}(\Omega)$ *then* $f(u) \in W^{1,p}(\Omega)$*, and*

$$Df(u) = Df(u) \cdot Du$$

Proposition 5.9. $(W^{k,p}(\Omega), \|\cdot\|_{W^{k,p}(\Omega)})$, $(H^{k,p}(\Omega), \|\cdot\|_{W^{k,p}(\Omega)})$, $(H_0^{k,p}(\Omega), \|\cdot\|_{W^{k,p}(\Omega)})$ *are all Banach spaces.*

For $p = 2$ *they are Hilbert spaces, with inner product*

$$\langle u, v \rangle = \sum_{|\gamma| \leq k} \int \partial^\gamma u\, \partial^\gamma v.$$

For $p \in (1, \infty)$ (not $p = 1$ and not $p = \infty$), $W^{k,p}(\Omega)$ and $W_0^{k,p}(\Omega)$ are *reflexive*. In particular we have the following consequence of Banach-Alaoglu:

Theorem 5.10 (Weak compactness)**.** *Let* $1 < p < \infty$, $k \in \mathbb{N}$, $\Omega \subset \mathbb{R}^n$ *open. Assume that* $(f_i)_{i \in \mathbb{N}}$ *is a bounded sequence in* $W^{k,p}(\Omega)$*, that is*

$$\sup_{i \in \mathbb{N}} \|f_i\|_{W^{k,p}(\Omega)} < \infty.$$

Then there exists a function $f \in W^{k,p}(\Omega)$ *and a subsequence* f_{i_j} *such that* f_{i_j} *weakly* $W^{k,p}$*-converges to* f*, that is for any* $|\gamma| \leq k$ *and any* $g \in L^{p'}(\Omega)$*, where* $p' = \frac{p}{p-1}$ *is the Hölder dual of* p*, we have*

$$\int_\Omega \partial^\gamma f_{i_j}\, g \xrightarrow{i \to \infty} \int_\Omega \partial^\gamma f\, g.$$

We have lower semicontinuity *of the norm,*

$$\|f\|_{W^{k,p}(\Omega)} \leq \liminf_{i \to \infty} \|f_i\|_{W^{k,p}(\Omega)}.$$

The same statement holds when we replace $W^{k,p}(\Omega)$ *with* $W_0^{k,p}(\Omega)$.

5.1. **Approximation by smooth functions.** It is often ok to think of Sobolev maps as (essentially) *smooth* functions with bounded $W^{k,p}$-norm. The reason is approximation:

Proposition 5.11 (Local approximation by smooth functions)**.** *Let* Ω *be open,* $u \in W^{k,p}(\Omega)$, $1 \leq p < \infty$. *Set*

$$u_\varepsilon(x) := \eta_\varepsilon * u(x) = \int_{\mathbb{R}^n} \eta_\varepsilon(y - x)\, u(y)\, dy \quad x \in \Omega_{-\varepsilon}.$$

Here $\eta_\varepsilon(z) = \varepsilon^{-n}\eta(z/\varepsilon)$ *for the usual bump function* $\eta \in C_c^\infty(B(0,1),[0,1])$, $\int_{B(0,1)} \eta = 1$. *Then*

(1) $u_\varepsilon \in C^\infty(\Omega_{-\varepsilon})$, *where as before*
$$\Omega_{-\varepsilon} := \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) > \varepsilon\}$$
for each $\varepsilon > 0$ *such that* $\Omega_{-\varepsilon} \neq \emptyset$.

(2) Moreoever for any $\Omega' \subset\subset \Omega$,
$$\|u_\varepsilon - u\|_{W^{k,p}(\Omega')} \xrightarrow{\varepsilon\to 0} 0.$$
We call this $W^{k,p}_{loc}$*-approximation.*

If we want to approximate $W^{k,p}(\Omega)$ with functions $u \in C^\infty(\overline{\Omega})$ we need regularity of Ω.

Theorem 5.12 (Smooth approximation for Sobolev functions)**.** *Let* $\Omega \subset \mathbb{R}^n$ *be open and bounded, and* $\partial\Omega \in C^1$. *For any* $u \in W^{k,p}(\Omega)$ *there exist a smooth approximating sequence* $u_i \in C^\infty(\overline{\Omega})$ *such that*
$$\|u_i - u\|_{W^{k,p}(\Omega)} \xrightarrow{i\to\infty} 0.$$

On $\mathbb{R}^n$ approximation is much easier, indeed we can approximate with respect to the $W^{k,p}$-norm any $u \in W^{k,p}(\mathbb{R}^n)$ by functions $u_k \in C_c^\infty(\mathbb{R}^n)$. That is, $W^{k,p}(\mathbb{R}^n) = W_0^{k,p}(\mathbb{R}^n)$. We could describe this as "$u \in W^{k,p}(\mathbb{R}^n)$ implies that u and $k-1$-derivatives of u all vanish at infinity".

Proposition 5.13. *(1) Let* $u \in W^{k,p}(\Omega)$, $p \in [1,\infty)$. *If* $\operatorname{supp} u \subset\subset \mathbb{R}^n$ *then there exists* $u_k \in C_c^\infty(\Omega)$ *such that*
$$\|u - u_k\|_{W^{k,p}(\Omega)} \xrightarrow{k\to\infty} 0.$$

(2) Let $u \in W^{k,p}(\mathbb{R}^n)$, $p \in [1,\infty)$. *Then there exists* $u_k \in C_c^\infty(\mathbb{R}^n)$ *such that*
$$\|u - u_k\|_{W^{k,p}(\mathbb{R}^n)} \xrightarrow{k\to\infty} 0.$$
That is $W^{k,p}(\mathbb{R}^n) = W_0^{k,p}(\mathbb{R}^n)$

(3) Let $u \in W^{k,p}(\mathbb{R}^n_+) = \mathbb{R}^{n-1} \times (0,\infty))$. *Then there exists* $u \in C_c^\infty(\mathbb{R}^{n-1} \times [0,\infty)$ *(i.e.* u *may not be zero on* $(x',0)$ *for small* x'*) such that*
$$\|u - u_k\|_{W^{k,p}(\mathbb{R}^n_+)} \xrightarrow{k\to\infty} 0.$$

5.2. **Difference Quotients.** We used above, e.g. for the Cauchy estimates, Proof of Lemma 2.41 the method of differentiating the equation (e.g. that if $\Delta u = 0$ then also for $v := \partial_i u$ we have $\Delta v = 0$ – so we can easier estimates for $\partial_i u$). In the Sobolev space category this is also a useful technique. Sometimes, the "first assume that everything is smooth, then use mollification"-type argument as for Lemma 2.41 is difficult to put into practice. In this case, a technique developed by Nirenberg, is discretely differentiating the equation (which does not require the function to be a priori differentiable):

$$\Delta u = 0 \Rightarrow v(x) := (\Delta_h^{e_i} u)(x) := \frac{u(x+he_i)-u(x)}{h}: \quad \Delta v = 0$$

For this to work, we need some good estimates. Recall that (by the fundamental theorem of calculus), for C^1-functions u,

$$\|\Delta_h^{e_\ell} u\|_{L^\infty} \le \|\partial_\ell u\|_{L^\infty}.$$

This also holds in L^p for $W^{1,p}$-functions u, which is a result attributed to Nirenberg, see Proposition 5.15.

One important ingredient is that the fundamental theorem of calculus holds for Sobolev functions:

Lemma 5.14. *Let $u \in W^{1,1}_{loc}(\Omega)$. Fix $v \in \mathbb{R}^n$. Then for almost every $x \in \Omega$ such that the path $[x, x+v] \subset \Omega$ we have*

$$u(x+v) - u(x) = \int_0^1 \partial_\alpha u(x+tv) v^\alpha \, dt.$$

Proposition 5.15. *(1) Let $k \in \mathbb{N}$, (i.e. $k \neq 0$), and $1 \le p < \infty$. Assume that $\Omega' \subset\subset \Omega$ are two open (nonempty) sets, and let $0 < |h| < \operatorname{dist}(\Omega', \partial\Omega)$. For $u \in W^{k,p}(\Omega)$ we have*

$$\|\Delta_h^{e_\ell} u\|_{W^{k-1,p}(\Omega')} \le \|\partial_\ell u\|_{W^{k-1,p}(\Omega)}.$$

Moreover we have

$$\|\Delta_h^{e_\ell} u - \partial_\ell u\|_{W^{k-1,p}(\Omega')} \xrightarrow{h\to 0} 0.$$

(2) Let $u \in W^{k-1,p}(\Omega)$, $1 < p \le \infty$. Assume that for any $\Omega' \subset\subset \Omega$ and any $\ell = 1, \ldots, n$ there exists a constant $C(\Omega')$ such that

$$\sup_{|h| < \operatorname{dist}(\Omega', \partial\Omega)} \|\Delta_h^{e_\ell} u\|_{W^{k-1,p}(\Omega')} \le C(\Omega', \ell)$$

Then we $u \in W^{k,p}_{loc}(\Omega)$, and for any $\Omega' \subset \Omega$ we have

$$\|\partial_\ell u\|_{W^{k-1,p}(\Omega')} \le \sup_{|h| < \operatorname{dist}(\Omega', \partial\Omega)} \|\Delta_h^{e_\ell} u\|_{W^{k-1,p}(\Omega')}. \tag{5.2}$$

If $p = \infty$ we even have $u \in W^{k,\infty}(\Omega)$ with the estimate

$$\|\partial_\ell u\|_{W^{k-1,\infty}(\Omega)} \le \sup_{\Omega' \subset\subset \Omega} \sup_{|h| < \operatorname{dist}(\Omega', \partial\Omega)} \|\Delta_h^{e_\ell} u\|_{W^{k-1,\infty}(\Omega')}. \tag{5.3}$$

5.3. $W^{1,\infty}$ **is Lipschitz.** Let Ω be a bounded set with smooth boundary $\partial\Omega \in C^\infty$ (this is not optimal).

From our definition we have $f \in W^{1,\infty}(\Omega)$ if and only if $f \in L^\infty(\Omega)$ and $Df \in L^\infty(\Omega, \mathbb{R}^n)$.

Assume that f is Lipschitz, i.e. f is continuous (so $f \in L^\infty(\Omega)$) and we have

$$|f(x) - f(y)| \leq \Lambda|x-y| \quad x, y \in \Omega.$$

From Proposition 5.15, we conclude that $f \in W^{1,\infty}(\Omega)$.

The other direction is also true. Assume that $f \in W^{1,\infty}(\Omega)$. Then in particular $f \in W^{1,2}(\Omega)$, and thus by mollification we can approximate

$$f(x) = \lim_{\varepsilon \to 0} f * \eta_\varepsilon(x) \quad \text{in } \Omega' \subset\subset \Omega.$$

The right-hand side is smooth, and we observe

$$\sup_\varepsilon \|D\left(f * \eta_\varepsilon\right)\|_{L^\infty(\Omega')} \lesssim \|Df\|_{L^\infty(\Omega)}.$$

That is, $f * \eta_\varepsilon$ all have a Lipschitz constant which is uniformly bounded. By Arzela Ascoli we get that (up to subsequence) $f * \eta_\varepsilon \xrightarrow{\varepsilon \to 0} g$ uniformly in Ω'. Thus g is Lipschitz. Is $g = f$? Yes, but only almost everywhere!

Since $f * \eta_\varepsilon$ converges to f in $L^2(\mathbb{R}^d)$, we have (up to subsequence) that $f * \eta_\varepsilon$ converges to f almost everywhere. Thus $f = g$ a.e.

So what we have is the following

Theorem 5.16. *Let $\Omega \subset \mathbb{R}^n$ be an open, bounded set with smooth boundary $\partial\Omega \in C^\infty$.*

Then the following are equivalent

(1) *$u \in W^{1,\infty}(\Omega)$*
(2) *There exists a representative $\tilde{u} \in C^{0,1}(\overline{\Omega})$ such that $u = \tilde{u}$ almost everywhere.*

One can also show now that $D\tilde{u}$ (from Rademeacher's theorem) equals a.e. Du (from Sobolev space definition) – in particular we can *prove* Rademacher's theorem like this: any Lipschitz function has a.e. a derivative.

In the same vein we can identify $W^{k,\infty}(\Omega)$ with $C^{k-1,1}(\overline{\Omega})$.

5.4. **Composition.** We have, cf. [Ziemer, 1989, Theorem 2.1.11].

Theorem 5.17 (Composition with Lipschitz functions)**.** *Let $u \in W^{1,p}(\Omega)$ and $f : \mathbb{R} \to \mathbb{R}$ be Lipschitz continuous, $f \in C^{0,1}$ then $f \circ u \in W^{1,p}(\Omega)$.*

Moreover if $f' \in L^\infty(\mathbb{R})$ denotes the distributitonal derivative (since f is Lipschitz, $f \in W^{1,\infty}$, so f' makes sense), then

$$\partial_\alpha u = f' \circ u \; \partial_\alpha u \quad \textit{a.e. in } \Omega$$

ALthough it is not extremely clear, but a special case implies the following, cf. [Evans and Gariepy, 2015, §4.2.2., Theorem 4]

Lemma 5.18. *Let $u \in W^{1,1}(\Omega)$, then we have $Du = 0$ almost everywhere in $\{u(x) = 0\}$*[9].

5.5. Embedding Theorems.

Theorem 5.19 (Rellich-Kondrachov)**.** *Let $\Omega \subset\subset \mathbb{R}^n$, $\partial\Omega \subset C^{0,1}$, $1 \leq p \leq \infty$. Assume that $(u_k)_{k\in\mathbb{N}} \in W^{1,p}(\Omega)$ is bounded, i.e.*

$$\sup_{k\in\mathbb{N}} \|u_k\|_{W^{1,p}(\Omega)} < \infty.$$

Then there exists a subsequence $k_i \to \infty$ and $u \in L^p(\Omega)$ such that u_{k_i} is (strongly) convergent in $L^p(\Omega)$, moreover the convergence is pointwise a.e..

Theorem 5.20 (Poincaré)**.** *Let $\Omega \subset\subset \mathbb{R}^n$ be open and connected, $\partial\Omega \in C^{0,1}$, $1 \leq p \leq \infty$.*

Let $K \subset W^{1,p}(\Omega)$ be a closed (with respect to the $W^{1,p}$-norm) cone with only one constant function $u \equiv 0$. That is, let $K \subset W^{1,p}(\Omega)$ be a closed set such that

(1) $u \in K$ implies $\lambda u \in K$ for any $\lambda \geq 0$.
(2) if $u \in K$ and $u \equiv const$ then $u \equiv 0$.

Then there exists a constant $C = C(K, \Omega)$ such that

$$\|u\|_{L^p(\Omega)} \leq C\|\nabla u\|_{L^p(\Omega)} \quad \forall u \in K. \tag{5.4}$$

In one dimension this is an easy consequence of the fundamental theorem, and sometimes called *Wirtinger's inequality*. Denote $(u) := \frac{1}{2}\int_{-1}^{1} u$ the mean value. Then

$$\begin{aligned}
\|u-(u)\|^p_{L^p((-1,1))} = \int_{(-1,1)} |u(x)-(u)|^p \leq& \frac{1}{2}\int_{(-1,1)}\int_{(-1,1)} |u(x)-u(z)|^p\,dx\,dz\\
\leq& \frac{1}{2}\int_{(-1,1)}\int_{(-1,1)} |\int_x^z u'(y)dy|^p dxdz\\
\leq& 2^{p-2}\int_{(-1,1)}\int_{(-1,1)}\int_{-1}^{1} |u'(y)|^p dydxdz\\
=& 2^p \int_{-1}^{1} |u'(y)|^p\,dy.
\end{aligned}$$

Corollary 5.21 (Poincaré type lemma)**.** *Let $\Omega \subset\subset \mathbb{R}^n$ be open, connected, and $\partial\Omega \in C^{0,1}$.*

(1) There exists $C = C(\Omega)$ such that for all $u \in W^{1,p}(\Omega)$ we have

$$\|u-(u)_\Omega\|_{L^p(\Omega)} \leq C(\Omega)\|\nabla u\|_{L^p(\Omega)}$$

[9] Check this for smooth functions: Either $\{u(x) = 0\}$ is a zeroset. On the other hand, on the "substantial" parts of $\{u(x) = 0\}$ we should think of u as constant

(2) For any $\Omega' \subset\subset \Omega$ open and nonempty there exists $C = C(\Omega, \Omega')$ such that for all $u \in W^{1,p}(\Omega)$ we have

$$\|u - (u)_{\Omega'}\|_{L^p(\Omega)} \leq C(\Omega, \Omega')\|\nabla u\|_{L^p(\Omega)}$$

(3) There exists $C = C(\Omega)$ such that for all $u \in W_0^{1,p}(\Omega)$

$$\|u\|_{L^p(\Omega)} \leq C(\Omega)\|\nabla u\|_{L^p(\Omega)}$$

If $\Omega = B(x, r)$ (and in the second claim $\Omega' B(x, \lambda r)$) then $C(\Omega) = C(B(0, 1))\, r$ (and for the second claim: $C(\Omega, \Omega') = C(B(0, 1), B(0, \lambda))\, r$).

Exercise 5.22. *Let $B(x_0, r) \subset \mathbb{R}^n$ a ball. Show that there exists a constant $C > 0$ independent of r and x_0 such that the following holds*

(1) $\|u - (u)_{B(x_0,r)}\|_{L^p(B(x_0,r))} \leq C\, r\|\nabla u\|_{L^p(B(x_0,r))}$ for all $u \in W^{1,p}(B(x_0, r))$. (Here, as before $(u)_{B(x_0,r)} = |B(x_0, r)|^{-1} \int_{B(x_0,r)} u$).

(2) $\|u\|_{L^p(B(x_0,r))} \leq C\, r\|\nabla u\|_{L^p(B(x_0,r))}$ for all $u \in W_0^{1,p}(B(x_0, r))$.

Theorem 5.23 (Sobolev inequality)**.** *Let $p \in [1, \infty)$ such that $p^* := \frac{np}{n-p} \in (1, \infty)$ (equivalently: $p \in [1, n)$). Then $W^{1,p}(\mathbb{R}^n)$ embedds into $L^{p^*}(\mathbb{R}^n)$. That is, if $u \in W^{1,p}(\mathbb{R}^n)$ then $u \in L^{p^*}(\mathbb{R}^n)$ and we have*[10]

$$\|u\|_{L^{p^*}(\mathbb{R}^n)} \leq C(p, n)\ \|Du\|_{L^p(\mathbb{R}^n)}.$$

Corollary 5.24 (Sobolev-Poincaré embedding)**.** *Let $u \in W^{1,p}(\mathbb{R}^n)$, $1 \leq p < n$. For any $q \in [p, p^*]$ we have $u \in L^q(\mathbb{R}^n)$ with the estimate*

$$\|f\|^q_{L^q(\mathbb{R}^n)} \leq C(q, n)\ \left(\|f\|^p_{L^p(\mathbb{R}^N)} + \|Df\|^{p^*}_{L^p(\mathbb{R}^n)}\right).$$

Corollary 5.25 (Sobolev-Poincaré embedding on domains)**.** *Let $\Omega \subset \mathbb{R}^n$ and $\partial\Omega$ be C^1. For $1 \leq p < n$ we have for any $u \in W^{1,p}(\Omega)$,*

$$\|u\|_{L^{p^*}(\Omega)} \leq C(\Omega) \left(\|u\|_{L^p(\Omega)} + \|Du\|_{L^p(\Omega)}\right)$$

Also, for any $q \in [p, p^*]$ [11]

$$\|u\|_{L^q(\Omega)} \leq C(\Omega, q, \|u\|_{W^{1,p}(\Omega)}).$$

If moreover $\Omega \subset\subset \mathbb{R}^n$ and $u \in W_0^{1,p}(\Omega)$ then

$$\|u\|_{L^{p^*}(\Omega)} \leq C(\Omega)\, \|Du\|_{L^p(\Omega)}.$$

[10] The optimal constant $C(p, n)$ has actually a geometric meaning, and is related to the isoperimetric inequality, cf. [Talenti, 1976]

[11] This means the following: For any $\Lambda > 0$ there exists a constant $C(\Omega, q, \Lambda)$ such that

$$\|u\|_{L^q(\Omega)} \leq C(\Omega, q, \Lambda) \quad \forall u :\ \|u\|_{W^{1,p}(\Omega)} \leq \Lambda.$$

Lastly, if $1 \leq p < \infty$ and $\Omega \subset\subset \mathbb{R}^n$, $u \in W^{1,p}(\Omega)$ then for any $q \in [1, p^*]$ (if $p < n$) or for any $q \in [1, \infty)$ (if $p \geq n$)

$$\|u\|_{L^q(\Omega)} \leq C(\Omega, q, p, n)\ \|u\|_{W^{1,p}(\Omega)}.$$

Theorem 5.26 (Sobolev Embedding)**.** *Let $\Omega \subset\subset \mathbb{R}^n$ be open, $\partial\Omega \in C^{0,1}$, $k \geq \ell$ for $k, \ell \in \mathbb{N} \cup \{0\}$, and $1 \leq p, q < \infty$ such that*

$$k - \frac{n}{p} \geq \ell - \frac{n}{q}. \tag{5.5}$$

Then the identity is a continuous embedding $W^{k,p}(\Omega) \hookrightarrow W^{\ell,q}(\Omega)$. That is,

$$\|u\|_{W^{\ell,q}(\Omega)} \leq C(\|u\|_{W^{k,p}(\Omega)}) \tag{5.6}$$

If $k > \ell$ and we have the strict inequality

$$k - \frac{n}{p} > \ell - \frac{n}{q}, \tag{5.7}$$

then the embedding above is compact. That is, whenever $(u_i)_{i \in \mathbb{N}} \subset W^{k,p}(\Omega)$ such that

$$\sup_i \|u_i\|_{W^{k,p}(\Omega)} < \infty$$

then there exists a subsequence $(u_{i_j})_{j \in \mathbb{N}}$ such that $(u_{i_j})_{j \in \mathbb{N}}$ is convergent in $W^{\ell,q}(\Omega)$.

Theorem 5.27 (Morrey Embedding)**.** *Let $\Omega \subset\subset \mathbb{R}^n$ with $\partial\Omega \in C^k$, $k \in \mathbb{N}$. Assume that for $p \in (1, \infty)$, $\alpha \in (0, 1)$ and $\ell < k$ we have*

$$k - \frac{n}{p} \geq \ell + \alpha.$$

Then the embedding $W^{k,p}(\Omega) \hookrightarrow C^{\ell,\alpha}(\overline{\Omega})$ is continuous.

If $k - \frac{n}{p} > \ell + \alpha$ then the embedding is compact.

5.6. **Trace Theorems.** Let Ω be a smoothly bounded domain, i.e. $\partial\Omega \subset \mathbb{R}^n$ is a smooth (compact) manifold.

For $s \in (0, 1)$, $p \in [1, \infty)$ and for $u \in C^\infty(\partial\Omega)$ we set (one of) the *fractional Sobolev space* norm, often called *Gagliardo-seminorm* or *Sobolev-Slobodeckij*-norm as

$$[u]_{W^{s,p}(\partial\Omega)} := \left(\int_{\partial\Omega} \int_{\partial\Omega} \left(\frac{|u(x) - u(y)|}{|x - y|^s} \right)^p \frac{dx dy}{|x - y|^{n-1}} \right)^{\frac{1}{p}}.$$

We say $u \in W^{s,p}(\partial\Omega)$ if $u \in L^p(\partial\Omega)$ and $[u]_{W^{s,p}(\partial\Omega)} < \infty$.

These spaces are sometimes called *trace space*, because of the following property: they describe the trace of $W^{1,p}$-function

Theorem 5.28 (Trace theorem)**.** *Let Ω be open, bounded domain with smooth boundary $\partial\Omega$ and $p \in (1, \infty)$. Then*

- *$W^{1,p}(\Omega) \hookrightarrow W^{1-\frac{1}{p},p}(\partial\Omega)$ in the following sense.*
  For every $u \in W^{1,p}(\Omega)$, if we restrict $u\Big|_{\partial\Omega}$ (in the right way), then
  $$[u\Big|_{\partial\Omega}]_{W^{1-\frac{1}{p},p}(\partial\Omega)} \precsim \|\nabla u\|_{L^p(\Omega)}$$
  and
  $$\|u\Big|_{\partial\Omega}\|_{L^p(\partial\Omega)} + [u\Big|_{\partial\Omega}]_{W^{1-\frac{1}{p},p}(\partial\Omega)} \precsim \|u\|_{L^p(\Omega)} + \|\nabla u\|_{L^p(\Omega)}.$$
  "In the right way" means that the restriction operator $T : u \in C^\infty(\overline{\Omega}) \to C^\infty(\partial\Omega)$, $u \mapsto u\Big|_{\partial\Omega}$ has the above estimates. By density this operator than is defined for any $u \in W^{1,p}(\Omega)$.
- *For any $u \in W^{1-\frac{1}{p},p}(\partial\Omega)$ there exists $U \in W^{1,p}(\Omega)$ and*
  $$\|\nabla U\|_{L^p(\Omega)} \precsim_{p,\Omega} [u]_{W^{1-\frac{1}{p},p}(\partial\Omega)}$$
  and
  $$\|U\|_{L^p(\Omega)} + \|\nabla U\|_{L^p(\Omega)} \precsim \|u\|_{L^p(\partial\Omega)} + [u]_{W^{1-\frac{1}{p},p}(\partial\Omega)}$$
  ‘

The statement above holds also for $p = \infty$ (if we recall that $W^{1,\infty}$ are simply Lipschitz maps. For $p = 1$ there are versions in the spirit of the above trace (observe $1 - 1/1 = 0$)

6. Existence and L^2-regularity theory for Laplace Equation

In this section we want to discuss the basic existence and regularity theory, namely we want to show existence and regularity for the following model equation for an *elliptic equation*. We will later extend this to more complicated linear equations Section 6.6.

Let $\Omega \subset \mathbb{R}^n$ be an open set (and we shall for now always assume Ω to be bounded and $\partial\Omega \in C^\infty$). We want to find a solution to

$$\begin{cases} -\mathrm{div}\,(A\nabla u) = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{6.1}$$

which we equivalently write as

$$\begin{cases} \sum_{\alpha,\beta=1}^n \partial_\alpha\left(A_{\alpha\beta}\partial_\beta u\right) = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

and using the *Einstein summation convention*

$$\begin{cases} \partial_\alpha\left(A_{\alpha\beta}\partial_\beta u\right) = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

Here f is a given datum (we discuss below what assumptions we need). $A \in C^\infty(\overline{\Omega}, \mathbb{R}^{n\times n})$ are symmetric matrices which are (uniformly) *elliptic* and bounded. It is important that *each and any* of the previous assumptions can be relaxed, and leads to interesting, and possibly very challenging theories – we focus on a simple, basic model case.

Uniform boundedness means that there exists $\Lambda > 0$ such that

$$\sup_{x\in\overline{\Omega}} |A| \leq \Lambda < \infty.$$

Uniform ellipticity means that all eigenvalues of $A(x)$ (which is symmetric) are positive and bounded from below. Equivalently, there exits $\lambda > 0$ such that

$$\inf_{|v|=1, x\in\overline{\Omega}} \langle A(x)v, v\rangle \geq \lambda > 0.$$

The above will be our standing assumptions below (again, some assumptions can be weakened).

The following theorems describe a very typical way of action in order of finding solutions to the above equation:

Theorem 6.1 (Existence in $W^{1,2}_0$)**.** *Let $f \in \left(W^{1,2}_0(\Omega)\right)^*$, that is f is a linear bounded functional on $W^{1,2}_0(\Omega)$ such that*

$$|f[\varphi]| \lesssim \|f\|_{\left(W^{1,2}_0(\Omega)\right)^*} \|\varphi\|_{W^{1,2}(\Omega)} \quad \forall \varphi \in C^\infty_c(\Omega).$$

Then there exists a solution $u \in W^{1,2}(\Omega)$ to

$$\begin{cases} -\mathrm{div}\,(A\nabla u) \equiv \sum_{\alpha\beta} \partial_\alpha(A_{\alpha\beta}\partial_\beta u) = f & \text{in } \Omega\\ u = 0 & \text{on } \partial\Omega,\end{cases}$$

in the sense that $u \in W^{1,2}_0(\Omega)$ and

$$\int_\Omega A_{\alpha\beta}\,\partial_\alpha u\,\partial_\beta \varphi = f[\varphi] \quad \forall \varphi \in C^\infty_c(\Omega).$$

Our particular solution u satisfies

$$\|u\|_{W^{1,2}(\Omega)} \leq C(\Omega)\|f\|_{(W^{1,2}(\Omega))^*}.$$

Exercise 6.2. *Show that*

(1) if $f \in L^2(\Omega)$ then $f \in \left(W^{1,2}_0(\Omega)\right)^*$, via the identification

$$f[\varphi] := \int_\Omega f\varphi, \quad \varphi \in W^{1,2}_0(\Omega).$$

Show that

$$\|f\|_{(W^{1,2}(\Omega))^*} \leq C(\Omega)\|f\|_{L^2(\Omega)}.$$

Hint: Poincarè inequality.

(2) *For some* $g \in L^2(\Omega)$ *and* $\alpha \in \{1, \ldots, n\}$ *define*

$$f[\varphi] := \int_\Omega g \partial_\alpha \varphi$$

Show that $f \in (W^{1,2}(\Omega))^*$ *and*

$$\|f\|_{(W^{1,2}(\Omega))^*} \le \|g\|_{L^2(\Omega)}.$$

Theorem 6.3 (Uniqueness in $W_0^{1,2}$). *For fixed* $f \in (W_0^{1,2}(\Omega))^*$ *there is at most one solution* $u \in W^{1,2}(\Omega)$ *that solves in the above sense*

$$\begin{cases} -\mathrm{div}\,(A\nabla u) \equiv \sum_{\alpha\beta} \partial_\alpha(A_{\alpha\beta}\partial_\beta u) = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

That is, if u *and* v *are both solutions, then* $u = v$ *a.e.*

Theorem 6.4 (Interior L^2-regularity). *Let* $f \in L^2(\Omega)$ *and assume* $u \in W^{1,2}(\Omega)$ *solves*

$$\begin{cases} -\mathrm{div}\,(A\nabla u) = f & \text{in } \Omega \end{cases}$$

in the above sense. (Observe we assume nothing on the boundary).

Then $u \in W^{2,2}_{loc}(\Omega)$, *and we have for any* $\Omega' \subset\subset \Omega$

$$\|D^2 u\|_{L^2(\Omega')} \le C(A, \Omega', \Omega) \left(\|u\|_{W^{1,2}(\Omega)} + \|f\|_{L^2(\Omega)} \right).$$

More generally, if $f \in W^{k,2}(\Omega)$ *then* $u \in W^{k+2,2}_{loc}(\Omega)$ *and for any* $\Omega' \subset\subset \Omega$ *we have*

$$\|D^{k+2} u\|_{L^2(\Omega')} \le C(A, k, \Omega', \Omega) \left(\|u\|_{W^{1,2}(\Omega)} + \|f\|_{L^2(\Omega)} \right).$$

In particular if $f \in C^\infty$ *then* $u \in C^\infty$ *– and the equation holds in the classical sense*[12].

Remark 6.5. In the estimate above one can replace the $\|u\|_{W^{1,2}(\Omega)}$-term on the right-hand side with $\|u\|_{L^2(\Omega)}$, cf. Exercise 6.7, e.g. get an estimate of the form

$$\|D^{k+2} u\|_{L^2(\Omega')} \le C(\Omega', \Omega) \left(\|u\|_{L^2(\Omega)} + \|f\|_{L^2(\Omega)} \right).$$

Theorem 6.6 (Global L^2-regularity). *Assume* Ω *is a bounded open set with smooth boundary* $\partial\Omega \in C^\infty$. *Let* $f \in L^2(\Omega)$ *and assume* $u \in W_0^{1,2}(\Omega)$ *solves*

$$\begin{cases} -\mathrm{div}\,(A\nabla u) = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

in the above sense.

Then $u \in W^{2,2}(\Omega)$, *and we have*

$$\|D^2 u\|_{L^2(\Omega)} \le C(\Omega) \left(\|u\|_{W^{1,2}(\Omega)} + \|f\|_{L^2(\Omega)} \right).$$

[12]this last part follows from the Morrey embedding, Theorem 5.27, and then by convolution

As for regularity theory: there is also L^p-theory, but this is way more involved, and called *Calderón-Zygmund theory*. We can also obtain regularity theory in the sense of Hölder spaces, which is called *Schauder theory*, Section 8.

6.1. **Existence: Proof of Theorem 6.1.** The PDE is in divergence form, which means its variational – which essentially means that the direct method of Calculus of Variations works: [13]

Proof of Theorem 6.1. We use what is called the direct method of Calculus of Variations: Set

$$\mathcal{E}(u) := \frac{1}{2} \int_\Omega A_{\alpha\beta} \partial_\alpha u \partial_\beta u + f[u].$$

As in Section 2.10 one can check that there is at most one minimizer in $W_0^{1,2}(\Omega)$ of this functional, that any minimizer is a solution to (6.1) and that any solution is a minimizer.

So all that is needed to show is the existence of a minimizer with the claimed estimate.

Let $u_k \in W_0^{1,2}(\Omega)$ be a sequence that approximates $\inf \mathcal{E}$ (this exists by the very definition of inf),

$$\lim_{k\to\infty} \mathcal{E}(u_k) = \inf_{W_0^{1,2}(\Omega)} \mathcal{E}.$$

In particular, we can assume that $\mathcal{E}(u_k) \le \mathcal{E}(0) = 0$ for all $k \in \mathbb{N}$. Now observe that by ellipticity[14],

$$\lambda |Du|^2 \le A_{\alpha\beta} \partial_\alpha u \, \partial_\beta u.$$

Thus,

$$\frac{\lambda}{2} \|Du_k\|^2_{L^2(\Omega)} = \mathcal{E}(u_k) - f[u_k] \le \mathcal{E}(0) + \|f\|_{(W^{1,2}(\Omega))^*} \, \|u_k\|_{W^{1,2}(\Omega)}.$$

That is, by Poincaré inequality, Corollary 5.21,

$$\|u_k\|^2_{W^{1,2}(\Omega)} \le C \|f\|_{(W^{1,2}(\Omega))^*} \, \|u_k\|_{W^{1,2}(\Omega)}.$$

Dividing both sides by $\|u_k\|_{W^{1,2}(\Omega)}$ we get

$$\sup_k \|u_k\|_{W^{1,2}(\Omega)} \le C \|f\|_{W^{1,2}(\Omega)}.$$

That is u_k is uniformly bounded in $W^{1,2}(\Omega)$.

The property we have just shown (of the energy $\mathcal{E}$ and the space $W_0^{1,2}(\Omega)$) is called *coercivity*: sequences $u_k \in W_0^{1,2}(\Omega)$ with bounded $\sup_k \mathcal{E}(u_k) < \infty$ must satisfy $\sup_k \|u_k\|_{W^{1,2}} < \infty$.

[13] we did something very similar in the variational methods section, Section 2.10, but we did not have the tools to show existence of a minimizer

[14] Einstein summation!!!

By the weak compactness theorem, Theorem 5.10, we can thus (up to taking a subsequence) assume u_k weakly converging to $u \in W_0^{1,2}(\Omega)$, which in particular implies

$$f[u_k] \xrightarrow{k\to\infty} f[u].$$

We also have by symmetry of A[15]

$$\begin{aligned}0 &\leq \int_\Omega A_{\alpha\beta}\partial_\alpha(u-u_k)\,\partial_\beta(u-u_k)\\ &= -\int_\Omega A_{\alpha\beta}\partial_\alpha u\,\partial_\beta u + \int_\Omega A_{\alpha\beta}\partial_\alpha u_k\,\partial_\beta u_k - 2\int_\Omega A_{\alpha\beta}\partial_\alpha(u_k-u)\,\partial_\beta u\end{aligned}$$

Thus,

$$\int_\Omega A_{\alpha\beta}\partial_\alpha u\,\partial_\beta u \leq \int_\Omega A_{\alpha\beta}\partial_\alpha u_k\,\partial_\beta u_k - 2\int_\Omega A_{\alpha\beta}\partial_\alpha(u_k-u)\,\partial_\beta u.$$

This holds for any $k \in \mathbb{N}$, so taking the liminf on both sides we have, using weak convergence,

$$\int_\Omega A_{\alpha\beta}\partial_\alpha u\,\partial_\beta u \leq \liminf_{k\to\infty}\int_\Omega A_{\alpha\beta}\partial_\alpha u_k\,\partial_\beta u_k + \underbrace{2\liminf_{k\to\infty}\left|\int_\Omega A_{\alpha\beta}\partial_\alpha(u_k-u)\,\partial_\beta u\right|}_{=0 \text{ since } u_k-u\rightharpoonup 0}$$

Thus, we conclude

$$E(u) \leq \liminf_{k\to\infty} E(u_k) = \inf_{W_0^{1,2}(\Omega)} E.$$

This property (again of the *energy* $\mathcal{E}$ and the *topology of weak convergence* $W^{1,2}(\Omega)$) is called *lower semicontinuity*: if u_k converges to u w.r.t. *weak convergence* $W^{1,2}(\Omega)$ then $E(u) \leq \liminf_{k\to\infty} E(u_k)$.

We can now conclude: since $u \in W_0^{1,2}(\Omega)$ we also have

$$E(u) \geq \inf_{W_0^{1,2}(\Omega)} E,$$

and thus

$$E(u) = \inf_{W_0^{1,2}(\Omega)} E.$$

That is we have found a minimizer of E. $\square$

Remark: The above is a *variational technique* (Direct Method of Calculus of Variations). Other possible techniques are: *Lax-Milgram*. More advanced methods are fixed point theorems (Banach, or Leray-Schauder/Schaefer). Fredholm-alternative, Closed Range theorem.

[15] we could also use more abstractly that norms are weakly lower semicontinuous

6.2. **Uniqueness: Proof of Theorem 6.3.** There are two proofs of uniqueness which are both very useful (here both work – in general this might not be the case):

Proof of Theorem 6.3 by convexity. Any solution to

$$\begin{cases} -\operatorname{div}(A\nabla u) \equiv \sum_{\alpha\beta} \partial_\alpha(A_{\alpha\beta}\partial_\beta u) = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

is a minimizer in $W_0^{1,2}(\Omega)$ of

$$\mathcal{E}(u) := \frac{1}{2}\int_\Omega \langle A_{\alpha\beta}\partial_\alpha u \partial_\beta u + f[u].$$

We have already discussed one direction: if u is a minimizer then u solves the PDE.

For the other direction assume that u solves the PDE and v is any other map in $W_0^{1,2}(\Omega)$. Then we have (recall that A is symmetric)

$$\begin{aligned} &A_{\alpha\beta}\partial_\alpha v \partial_\beta v - A_{\alpha\beta}\partial_\alpha u \partial_\beta u \\ =&A_{\alpha\beta}\partial_\alpha(v-u)\partial_\beta u + A_{\alpha\beta}\partial_\alpha u \partial_\beta(v-u) + A_{\alpha\beta}\partial_\alpha(v-u)\partial_\beta(v-u) \\ =&2A_{\alpha\beta}\partial_\alpha u \partial_\beta(v-u) + A_{\alpha\beta}\partial_\alpha(v-u)\partial_\beta(v-u) \\ \geq&2A_{\alpha\beta}\partial_\alpha u \partial_\beta(v-u) + \lambda|D(v-u)|^2 \\ \geq&2A_{\alpha\beta}\partial_\alpha u \partial_\beta(v-u). \end{aligned}$$

Thus,

$$\mathcal{E}(v) - \mathcal{E}(u) \geq \frac{2}{2}\int_\Omega \langle A_{\alpha\beta}\partial_\alpha u \partial_\beta(v-u) + f[v-u] = 0,$$

by the PDE. That is u is a minimizer.

Now we observe that $u \mapsto \mathcal{E}(u)$ is *strictly convex*. Indeed, let $u \neq v$ and $\mu \in (0,1)$ then

$$\begin{aligned} &A_{\alpha\beta}\partial_\alpha(\mu u + (1-\mu)v)\,\partial_\beta(\mu u + (1-\mu)v) \\ =&\mu^2 A_{\alpha\beta}\partial_\alpha u\,\partial_\beta u + (1-\mu)^2 A_{\alpha\beta}\partial_\alpha v\,\partial_\beta v + 2\mu(1-\mu)A_{\alpha\beta}\partial_\alpha u\,\partial_\beta v \\ =&\mu A_{\alpha\beta}\partial_\alpha u\,\partial_\beta u - \mu(1-\mu)A_{\alpha\beta}\partial_\alpha u\,\partial_\beta u \\ &+ (1-\mu)A_{\alpha\beta}\partial_\alpha v\,\partial_\beta v - (1-\mu)\mu A_{\alpha\beta}\partial_\alpha v\,\partial_\beta v \\ &+ 2\mu(1-\mu)A_{\alpha\beta}\partial_\alpha u\,\partial_\beta v \\ =&\mu A_{\alpha\beta}\partial_\alpha u\,\partial_\beta u + (1-\mu)A_{\alpha\beta}\partial_\alpha v\,\partial_\beta v \\ &- \mu(1-\mu)A_{\alpha\beta}\left(\partial_\alpha u\,\partial_\beta u + \partial_\alpha v\,\partial_\beta v - 2\partial_\alpha u\,\partial_\beta v\right) \\ =&\mu A_{\alpha\beta}\partial_\alpha u\,\partial_\beta u + (1-\mu)A_{\alpha\beta}\partial_\alpha v\,\partial_\beta v \\ &- \mu(1-\mu)A_{\alpha\beta}\left(\partial_\alpha(u-v)\,\partial_\beta(u-v)\right) \\ \leq&\mu A_{\alpha\beta}\partial_\alpha u\,\partial_\beta u + (1-\mu)A_{\alpha\beta}\partial_\alpha v\,\partial_\beta v - \lambda\mu(1-\mu)|D(u-v)|^2. \end{aligned}$$

That is, whenever $D(u-v) \not\equiv 0$ (i.e. $u-v$ not constant, which by the same boundary data means $u \not\equiv v$), and $\mu \in (0,1)$ we have

$$\mathcal{E}(\mu u+(1-\mu)v) \leq \mu\mathcal{E}(u)+(1-\mu)\mathcal{E}(v)-\lambda\mu(1-\mu)\|D(u-v)\|^2_{L^2(\Omega)} < \mu\mathcal{E}(u)+(1-\mu)\mathcal{E}(v),$$

which is strict convexity.

Now, strictly convex functions have at most one global minimizer. Indeed assume that u, v are both global minimizer (thus $\mathcal{E}(u) = \mathcal{E}(v) \leq \mathcal{E}(w)$ for any competitor w). We set $w := \frac{1}{2}(u+v)$. Unless $u \equiv v$ we'd then have

$$\mathcal{E}(w) < \frac{1}{2}\mathcal{E}(u) + \frac{1}{2}\mathcal{E}(v) = \mathcal{E}(u) \leq \mathcal{E}(w),$$

a contradiction. □

Uniqueness by testing. Assume we have two solutions u and v then (by linearity) $w := u-v$ solves the equation

$$\begin{cases} -\mathrm{div}\,(A\nabla w) = 0 & \text{in } \Omega \\ w = 0 & \text{on } \partial\Omega, \end{cases}$$

We can use w as the test function of this PDE and have

$$\lambda\|Dw\|^2_{L^2} \leq \int_\Omega A_{\alpha\beta}\partial_\alpha w \partial_\beta w = 0.$$

Thus $\|Dw\|_{L^2} = 0$ and since $w \in W^{1,2}_0(\Omega)$ we have $w \equiv 0$. □

6.3. **Interior regularity theory: Proof of Theorem 6.4.** Many techniques in regularity theory of PDE are based on using (a version of) the solution u as a test function, or colloquiually "multiply by u and integrate by parts".

Let us illustrate this (without getting any good estimate). Assume u solves

$$-\mathrm{div}\,(A\nabla u) = f \quad \text{in } \Omega$$

The basic idea by using u as a test function, we obtain good estimates.

Formally, we could the equation with u and (if we *ignore* the boundary data) integrating we obtain

$$\lambda\int_\Omega |\nabla u|^2 \leq \int_\Omega A_{\alpha\beta}\partial_\alpha u \partial_\beta u = \int fu \precsim \|f\|_{L^2}\|u\|_{L^2}.$$

Now we have an estimate of u in terms of the $W^{1,2}$-function of u (assuming that $f \in L^2$!).

But we made a mistake! u is not a permissible test-function, since u is not zero at the boundary (and we cannot ignore the boundary data – which we actually do not know).

But don't despair. Pick any $\Omega' \subset\subset \Omega$. We can find a cutoff function $\eta \in C^\infty_c(\Omega, [0,1])$ such that $\eta \equiv 1$ in Ω'. It is enough to find a good estimate for ηu. So let us compute the equation of ηu.

$$-\mathrm{div}\,(A\nabla(\eta u)) = -\mathrm{div}\,(\eta A\nabla u) - \mathrm{div}\,(A(\nabla\eta)u) = \eta f - (\partial_\beta\eta)\, A_{\alpha\beta}\partial_\beta u - \mathrm{div}\,(A(\nabla\eta)u).$$

Observe that even for $u \in L^2$, the right-hand side belongs to $(W_0^{1,2})^*$ in a nice way, e.g.

$$\int (\partial_\beta \eta)\, A_{\alpha\beta} \partial_\beta u \varphi = -\int u \partial_\beta \left((\partial_\beta \eta)\, A_{\alpha\beta} \partial_\beta \varphi\right) \lesssim \|u\|_{L^2} \|\varphi\|_{W^{1,2}}.$$

So, we can use existence, Theorem 6.1 to show that there exists $v \in W_0^{1,2}(\Omega)$

$$-\mathrm{div}\,(A\nabla v) = \eta f - (\partial_\beta \eta)\, A_{\alpha\beta} \partial_\beta u - \mathrm{div}\,(A(\nabla \eta) u).$$

and we have

$$\|v\|_{W^{1,2}} \lesssim \|\eta f - (\partial_\beta \eta)\, A_{\alpha\beta} \partial_\beta u - \mathrm{div}\,(A(\nabla\eta)u)\|_{\left(W_0^{1,2}(\Omega)\right)^*} \lesssim \|f\|_{L^2} + \|u\|_{L^2}. \tag{6.2}$$

But on the other hand we have uniqueness, Theorem 6.3, so $\eta u = v$ and we have found an estimate, cf. Exercise 6.7.

Exercise 6.7. *Let $u \in W^{1,2}(\Omega)$ solves the equation*

$$\mathrm{div}\,(A\nabla u) = f$$

Show that for any $\Omega_1 \subset\subset \Omega$

$$\|u\|_{W^{1,2}(\Omega_1)} \lesssim \|f\|_{\left(W_0^{1,2}(\Omega)\right)^*} + \|u\|_{L^2(\Omega)},$$

where the constants in $\lesssim$ depends on A and Ω_1.

Hint: Use the argument that lead to Equation (6.2) and Theorem 6.3.

We want to apply this idea to the derivative, i.e. we compute the PDE for $\eta \partial_\gamma u$.

$$\begin{aligned}
-\mathrm{div}\,(A\nabla(\eta\partial_\gamma u)) &= -\,\mathrm{div}\,(A\nabla\left(\partial_\gamma(\eta u)\right)) + \mathrm{div}\,(A\nabla\left((\partial_\gamma \eta)u\right))\\
&= -\,\partial_\gamma \mathrm{div}\,(A\nabla\left((\eta u)\right)) + \mathrm{div}\,(\partial_\gamma A\nabla\left((\eta u)\right)) + \mathrm{div}\,(A\nabla\left((\partial_\gamma \eta)u\right))\\
&= -\,\partial_\gamma \mathrm{div}\,(\eta A\nabla\left(u\right)) + \partial_\gamma \mathrm{div}\,(A\nabla \eta\ u) + \mathrm{div}\,(\partial_\gamma A\nabla\left((\eta u)\right)) + \mathrm{div}\,(A\nabla\left((\partial_\gamma \eta)u\right))\\
&= -\,\partial_\gamma\left(\eta \mathrm{div}\,(A\nabla\left(u\right))\right) + \partial_\gamma \partial_{\alpha\beta}\eta A_{\alpha\beta}\partial_\beta u + \partial_\gamma \mathrm{div}\,(A\nabla \eta\ u) + \mathrm{div}\,(\partial_\gamma A\nabla\left((\eta u)\right)) + \mathrm{div}\,(A\nabla
\\
&= -\,\partial_\gamma(\eta f) + \partial_\gamma\left(\partial_{\alpha\beta}\eta A_{\alpha\beta}\partial_\beta u\right) + \partial_\gamma \mathrm{div}\,(A\nabla \eta\ u) + \mathrm{div}\,(\partial_\gamma A\nabla\left((\eta u)\right)) + \mathrm{div}\,(A\nabla\left((\partial_\gamma \eta)u\right
\end{aligned}$$

Now as above[16] we have that this implies

$$\|-\mathrm{div}\,(A\nabla(\eta\partial_\gamma u))\|_{\left(W_0^{1,2}(\Omega)\right)^*} \lesssim \|u\|_{W^{1,2}(\Omega)}.$$

So, again using existence, Theorem 6.1, to find $v \in W_0^{1,2}(\Omega)$ such that

$$-\mathrm{div}\,(A\nabla(\eta\partial_\gamma u)) = -\mathrm{div}\,(A\nabla v) \quad \text{in } \Omega$$

v is is unique – but here is the *problem*: uniqueness, Theorem 6.3, is in $W_0^{1,2}$! and $\eta\partial_\gamma u \in L^2(\Omega) \supsetneq W_0^{1,2}(\Omega)$. So all we get are *a priori estimates*

[16] We will do this computation in details below, for the discrete differentiation

Lemma 6.8 (A priori estimates). *Assume that $u \in W^{2,2}(\Omega)$ solves*

$$-\mathrm{div}\,(A\nabla u) = f \quad \text{in } \Omega$$

Then in any open $\Omega' \subset\subset \Omega$ we have the estimate

$$\int_{\Omega'} |D^2u|^2 \leq C(\lambda, \Lambda, \Omega, \Omega')\Big(\|f\|^2_{L^2(\Omega)} + \|u\|^2_{W^{1,2}(\Omega)}\Big)$$

Proof. Fix $\gamma \in \{1, \ldots, n\}$, $\Omega' \subset\subset \Omega$ and let $\eta \in C_c^\infty(\Omega)$ such that $\eta \equiv 1$ in Ω'. Take $v \in W_0^{1,2}(\Omega)$ solving

$$-\mathrm{div}\,(A\nabla(\eta\partial_\gamma u)) = -\mathrm{div}\,(A\nabla v) \quad \text{in } \Omega,$$

which by the above argument satisfies

$$\|v\|_{W^{1,2}(\Omega)} \precsim \| - \mathrm{div}\,(A\nabla(\eta\partial_\gamma u))\|_{\left(W_0^{1,2}(\Omega)\right)^*} \precsim \|f\|_{L^2(\Omega)} + \|u\|_{W^{1,2}(\Omega)}.$$

Since $u \in W^{2,2}(\Omega)$ we have that $\eta\partial_\gamma u \in W_0^{1,2}(\Omega)$, so we have uniqueness, Theorem 6.3, which implies that $v = \eta\partial_\gamma u$. Thus we have

$$\|\eta\partial_\gamma u\|_{W^{1,2}(\Omega)} \precsim \|f\|_{L^2(\Omega)} + \|u\|_{W^{1,2}(\Omega)}.$$

Since $\eta \equiv 1$ in Ω,

$$\|\partial_\gamma u\|_{W^{1,2}(\Omega')} \leq \|\eta\partial_\gamma u\|_{W^{1,2}(\Omega)} \precsim \|f\|_{L^2(\Omega)} + \|u\|_{W^{1,2}(\Omega)}.$$

Since this holds for any $\gamma \in \{1, \ldots, n\}$ we have

$$\|u\|_{W^{2,2}(\Omega')} \leq \|\eta\partial_\gamma u\|_{W^{1,2}(\Omega)} \precsim \|f\|_{L^2(\Omega)} + \|u\|_{W^{1,2}(\Omega)}.$$

We can conclude.

□

So how do we show that $u \in W^{2,2}$? There are different ways to do this, the one we illustrate here is using difference quotients (i.e. discrete differentiation of the PDE):

Proposition 6.9. *Assume that $u \in W^{1,2}(\Omega)$ solves*

$$-\mathrm{div}\,(A\nabla u) = f \quad \text{in } \Omega$$

Then $u \in W^{2,2}_{loc}(\Omega)$ (and thus we have the estimate from the previous theorem).

We will use *discrete differentiation* (which in view of Proposition 5.15 we can relate do Sobolev space estimates) for which we will use some properties

Exercise 6.10 (Discrete differentiation). *For $h \in \mathbb{R}^n \setminus \{0\}$ denote by $\delta_h f(x) := f(x+h) - f(x)$.*

(1) Show the discrete product rule: $\delta_h(fg)(x) = \delta_h f(x)g(x) + f(x)\delta_h g(x) + \delta_h f(x)\delta_h g(x)$
(2) Let Ω be an open set and $\Omega_2 \subset\subset \Omega$. Assume $|h| \leq \frac{1}{2}\mathrm{dist}\,(\Omega_2, \partial\Omega)$. Show the discrete integration by parts formula

$$\int_\Omega f\delta_h g = \int_\Omega \delta_{-h} f g \quad \text{if } \mathrm{supp}\, f \subset \Omega_2 \text{ or } \mathrm{supp}\, g \subset \Omega_2.$$

(3) Use the discrete integration by parts formula to show the usual integration by parts formula

$$\int_\Omega f\partial_\gamma g = -\int_\Omega \partial_\gamma f g$$

for all smooth functions f and g which vanish in a neighborhood of $\partial\Omega$, where $\gamma \in \{1,\dots,n\}$.

Proof of Proposition 6.9. Fix $\Omega' \subset\subset \Omega_2 \subset\subset \Omega$. Let $\eta \in C_c^\infty(\Omega_2,[0,1])$ with $\eta \equiv 1$ in Ω'.

Fix $h \in \mathbb{R}^n$ with $|h| \leq \frac{1}{100}\mathrm{dist}\,(\mathrm{supp}\,\eta, \partial\Omega_2)$.

Denote by $\delta_h u(x) := u(x+h) - u(x)$. We now do what we did above, but with ∂_γ replaced by δ_h.

We observe that we have a discrete analogue of the product rule and the discrete integration by parts, see Exercise 6.10.

So let us compute

$$g := -\mathrm{div}\,(A\nabla\,(\eta\delta_h u)) \quad \text{in } \Omega.$$

and show that

$$\|g\|_{(W_0^{1,2})^*(\Omega)} \lesssim |h|\|u\|_{W^{1,2}(\Omega)}. \tag{6.3}$$

The computation is quite involved and long.

We have

$$-\mathrm{div}\,(A\nabla\,(\eta\delta_h u)) = -\mathrm{div}\,(A\nabla\delta_h\,(\eta u)) + \mathrm{div}\,(A\nabla\,(\delta_h\eta\; u)) - \mathrm{div}\,(A\nabla\,(\delta_h\eta\delta_h u))$$

Set

$$\Gamma_1 := \mathrm{div}\,(A\nabla\,(\delta_h\eta\; u)), \quad \Gamma_2 := -\mathrm{div}\,(A\nabla\,(\delta_h\eta\delta_h u))$$

Both Γ_1 and Γ_2 satisfy the estimate in (6.3). Indeed, for $\varphi \in C_c^\infty(\Omega)$ we have

$$\begin{aligned}
|\Gamma_1[\varphi]| =& \left|\int_\Omega A_{\alpha\beta}\partial_\beta\,(\delta_h\eta\; u)\;\partial_\alpha\varphi\right| \\
\lesssim& \|A\|_{L^\infty}\|D\,(\delta_h\eta\; u)\,\|_{L^2(\Omega)}\|D\varphi\|_{L^2(\Omega)} \\
\lesssim& \|A\|_{L^\infty}\left(\|\delta_h(D\eta)\; u\|_{L^2(\Omega)} + \|\,(\delta_h\eta\; Du)\,\|_{L^2(\Omega)}\right)\|D\varphi\|_{L^2(\Omega)} \\
\lesssim& \|A\|_{L^\infty}\left(|h|\|D^2\eta\|_{L^\infty}\|u\|_{L^2(\Omega_2)} + |h|\|D\eta\|_{L^\infty}\|Du\|_{L^2(\Omega_2)}\right)\|D\varphi\|_{L^2(\Omega)} \\
\leq& C(\Lambda,\eta)|h|\|u\|_{W^{1,2}(\Omega)}\|\varphi\|_{W^{1,2}(\Omega_2)}.
\end{aligned}$$

Thus,

$$\|\Gamma_1\|_{(W_0^{1,2})^*(\Omega)} \leq C(\Lambda,\eta)|h|\|u\|_{W^{1,2}(\Omega_2)} \leq C(\Lambda,\eta)|h|\|u\|_{W^{1,2}(\Omega)}.$$

We argue the same way for Γ_2 and have

$$\begin{aligned}\|\Gamma_2\|_{(W_0^{1,2})^*(\Omega)} \leq& C(\Lambda,\eta)|h|\|\delta_h u\|_{W^{1,2}(\Omega_2)}\\ \lesssim& C(\Lambda,\eta)|h| \left(\|u\|_{W^{1,2}(\Omega_2)} + \|u\|_{W^{1,2}(\Omega_2+h)}\right)\\ \lesssim& 2C(\Lambda,\eta)|h|\,\|u\|_{W^{1,2}(\Omega)}\end{aligned}$$

So, we have shown

$$-\mathrm{div}\,(A\nabla\,(\eta\delta_h u)) = -\mathrm{div}\,(A\nabla\delta_h\,(\eta u)) + \Gamma_1 + \Gamma_2$$

where Γ_1 and Γ_2 satisfy the estimate we want, (6.3). Since there will be many Γ_i we are going to call Γ any "good term" that satisfies the estimate (6.3), i.e. whenever

$$\|\Gamma\|_{(W_0^{1,2})^*(\Omega)} \lesssim |h|\|u\|_{W^{1,2}(\Omega)}. \tag{Γ}$$

– and Γ will change from line to line. For now we have

$$\begin{aligned}-\mathrm{div}\,(A\nabla\,(\eta\delta_h u)) =& -\mathrm{div}\,(A\nabla\delta_h\,(\eta u)) + \Gamma\\ =& -\delta_h\mathrm{div}\,(A\nabla\,(\eta u)) + \mathrm{div}\,(\delta_h A\,\nabla\,(\eta u)) + \mathrm{div}\,(\delta_h A\,\delta_h\nabla\,(\eta u)) + \Gamma\end{aligned}$$

Set

$$\Gamma_3 := \mathrm{div}\,(\delta_h A\,\nabla\,(\eta u)), \quad \Gamma_4 := \mathrm{div}\,(\delta_h A\,\delta_h\nabla\,(\eta u)).$$

We check that Γ_1 and Γ_3 are of the type Γ: For $\varphi \in C_c^\infty(\Omega)$ we have

$$\begin{aligned}\mathrm{div}\,(\delta_h A\,\nabla\,(\eta u))[\varphi] =& -\int_\Omega \delta_h A_{\alpha\beta}\,\partial_\beta\,(\eta u)\,\partial_\alpha\varphi\\ \lesssim& \|\delta_h A_{\alpha\beta}\|_{L^\infty}\|\partial_\beta\,(\eta u)\,\|_{L^2}\|D\varphi\|_{L^2}\\ \lesssim& |h|\|DA\|_{L^\infty}C(\eta)\|u\|_{W^{1,2}(\Omega_2)}\|\varphi\|_{W^{1,2}(\Omega)}\end{aligned}$$

The estimate for Γ_4 is similar, with the same adaptations as for Γ_2 (using that $\eta \in C_c^\infty(\Omega_2)$ *localizes* everything to Ω_2). Thus, we have shown (for a new Γ but still satisfying (6.3))

$$\begin{aligned}-\mathrm{div}\,(A\nabla\,(\eta\delta_h u)) =& -\delta_h\mathrm{div}\,(A\nabla\,(\eta u)) + \Gamma\\ =& -\delta_h\mathrm{div}\,(A\,(\eta\nabla u)) - \delta_h\mathrm{div}\,(A\,(\nabla\eta\; u)) + \Gamma\end{aligned}$$

We show that

$$\Gamma_5 := -\delta_h\mathrm{div}\,(A\,(\nabla\eta u))$$

is of type Γ. Let $\varphi \in C_c^\infty(\Omega)$ then (using discrete integration by parts, again η localizes everything to Ω_2)

$$\begin{aligned}
\Gamma_5[\varphi] =& -\int_\Omega \mathrm{div}\,(A\,(\nabla\eta\; u))\delta_{-h}\varphi\\
\le& \|\mathrm{div}\,(A\,(\nabla\eta\; u))\|_{L^2(\Omega)}\|\delta_{-h}\varphi\|_{L^2(\Omega_2)}\\
\overset{\text{P 5.15}}{\lesssim}& |h|\,\|\mathrm{div}\,(A\,(\nabla\eta\; u))\|_{L^2(\Omega)}\|D\varphi\|_{L^2(\Omega)}\\
\lesssim& |h|\left(\|DA\|_{L^\infty}\,\|D\eta\|_{L^\infty}\|u\|_{L^2(\Omega)} + \|A\|_{L^\infty}\|D\,(\nabla\eta\; u))\|_{L^2(\Omega)}+\right)\|\varphi\|_{W^{1,2}(\Omega)}\\
\lesssim& |h|\left(\|DA\|_{L^\infty}\,\|D\eta\|_{L^\infty}\|u\|_{L^2(\Omega)} + \|A\|_{L^\infty}\left(\|D^2\eta\|_{L^\infty}\|u\|_{L^2(\Omega)} + \|D\eta\|_{L^\infty}\|Du\|_{L^2(\Omega)}\right)\right)\|\varphi\|_{W^{1,2}(\Omega)}\\
\lesssim& C(A,\eta)\,|h|\,\|u\|_{W^{1,2}(\Omega)}\,\|\varphi\|_{W^{1,2}(\Omega)}.
\end{aligned}$$

Thus, Γ_5 is of type (Γ), and we have

$$\begin{aligned}
-\mathrm{div}\,(A\nabla\,(\eta\delta_h u)) =& -\delta_h \mathrm{div}\,(A\,(\eta\nabla u)) + \Gamma\\
=& -\delta_h\,(\eta\mathrm{div}\,(A\nabla u)) - \delta_h\,(\partial_\alpha)\,\eta\, A_{\alpha\beta}\partial_\beta u) + \Gamma\\
=& -\delta_h\,(\eta f) - \delta_h\,(\partial_\alpha\eta\, A_{\alpha\beta}\partial_\beta u) + \Gamma
\end{aligned}$$

So we finally set

$$\Upsilon := -\delta_h\,(\eta f)\,,\quad \Gamma_6 := -\delta_h\,(\partial_\alpha\eta\, A_{\alpha\beta}\partial_\beta u)\,.$$

We first show that Γ_6 is of type Γ. Let $\varphi \in C_c^\infty(\Omega)$, then by an discrete integration by parts (again: the integral is actually in a strict subset of Ω because of $\eta \in C_c^\infty(\Omega_2)$ and $|h| \ll 1$),

$$\begin{aligned}
\Gamma_6[\varphi] =& \int_\Omega -\delta_h\,(\partial_\alpha\eta\, A_{\alpha\beta}\partial_\beta u)\,\varphi\\
=& -\int_\Omega (\partial_\alpha\eta\, A_{\alpha\beta}\partial_\beta u)\,\delta_{-h}\varphi\\
\lesssim& \|D\eta\|_{L^\infty}\|A\|_{L^\infty}\|Du\|_{L^2(\Omega)}\|\delta_{-h}\varphi\|_{L^2(\Omega_2)}\\
\overset{\text{P 5.15}}{\lesssim}& C(\eta, A)\|u\|_{W^{1,2}(\Omega)}\,\|\varphi\|_{W^{1,2}(\Omega)}.
\end{aligned}$$

So Γ_6 is of type Γ.

Lastly we need to show an estimate for Υ – and here is the first and only time we use that $f \in L^2(\Omega)$: Let $\varphi \in C_c^\infty(\Omega)$

$$\Upsilon[\varphi] = -\int_\Omega \eta f\,\delta_{-h}\varphi \lesssim \|f\|_{L^2(\Omega)}\,\|\delta_{-h}\varphi\|_{L^2(\Omega_2)} \overset{\text{P 5.15}}{\lesssim} |h|\,\|f\|_{L^2(\Omega)}\,\|\varphi\|_{W^{1,2}(\Omega)}.$$

In conclusion, we have shown

$$-\mathrm{div}\,(A\nabla\,(\eta\delta_h u)) = \Upsilon + \Gamma,$$

and we have

$$\|\Upsilon\|_{\left(W_0^{1,2}(\Omega)\right)^*} \lesssim |h|\|f\|_{L^2(\Omega)}$$

and

$$\|\Gamma\|_{\left(W_0^{1,2}(\Omega)\right)^*} \precsim |h|\, \|u\|_{W^{1,2}(\Omega)}$$

On the other hand from Theorem 6.1 there exists some $v = v_h \in W_0^{1,2}(\Omega)$ such that

$$-\mathrm{div}\,(A\nabla v) = \Upsilon + \Gamma,$$

and v comes with the estimate

$$\|v\|_{W^{1,2}(\Omega)} \precsim \|\Upsilon\|_{\left(W_0^{1,2}(\Omega)\right)^*} + \|\Gamma\|_{\left(W_0^{1,2}(\Omega)\right)^*} \precsim |h| \left(\|f\|_{L^2(\Omega)} + \|u\|_{W^{1,2}(\Omega)}\right).$$

By Theorem 6.3 v is unique in $W_0^{1,2}$ – and we observe that $(\eta\delta_h u) \in W_0^{1,2}(\Omega)$. So we actually have $v = (\eta\delta_h u)$ and thus

$$\|\eta\delta_h u\|_{W^{1,2}(\Omega)} \precsim |h| \left(\|f\|_{L^2(\Omega)} + \|u\|_{W^{1,2}(\Omega)}\right).$$

Since $\eta \equiv 1$ in Ω' we conclude that

$$|h|^{-1}\|\delta_h u\|_{W^{1,2}(\Omega')} \precsim \left(\|f\|_{L^2(\Omega)} + \|u\|_{W^{1,2}(\Omega)}\right).$$

This holds for any $|h| \ll 1$. So by Proposition 5.15 we conclude that

$$\|u\|_{W^{2,2}(\Omega')} \leq C(A, \Omega', \Omega) \left(\|f\|_{L^2(\Omega)} + \|u\|_{W^{1,2}(\Omega)}\right).$$

This argument works for any $\Omega' \subset\subset \Omega$, so we conclude that $u \in W^{2,2}_{loc}(\Omega)$. □

We are essentially done, now we argue by induction to get

Proof of Theorem 6.4. We claim that for all $k \in \mathbb{N} \cup \{0\}$, whenever $v \in W^{1,2}_{loc}(U)$ and $g \in W^{k,2}(\Omega)$ solves

$$\mathrm{div}\,(A\nabla v) = g \quad \text{in } U$$

then for any $U_1 \subset\subset U_2 \subset\subset U$ we have

$$\|v\|_{W^{k+2,2}(U_1)} \precsim C(U_1, U_2, A) \left(\|g\|_{W^{k,2}(\Omega_2)} + \|v\|_{W^{1,2}(U_2)}\right).$$

For $k = 0$ this is proven in Proposition 6.9 (nevermind the "loc" in $v \in W^{1,2}_{loc}(U)$ – the equation is then satisfied in U_2 and we have $u \in W^{1,2}(U_2)$).

Now fix $k \geq 1$ and assume the claim is shown already for $k-1$.

Let $u \in W^{1,2}_{loc}(\Omega)$ and assume $f \in W^{k,2}(\Omega)$ solves

$$\mathrm{div}\,(A\nabla u) = f \quad \text{in } \Omega.$$

Then from the induction hypothesis

$$\|u\|_{W^{k+1,2}(\Omega_1)} \precsim C(\Omega_1, \Omega_2, A) \left(\|f\|_{W^{k-1,2}(\Omega_2)} + \|u\|_{W^{1,2}(\Omega_2)}\right).$$

Since $k \geq 1$ we thus already know $u \in W^{2,2}_{loc}(\Omega)$ so $\partial_\gamma u \in W^{1,2}_{loc}(\Omega)$ for each fixed $\gamma \in \{1, \ldots, n\}$. We then can differentiate the equation

$$-\mathrm{div}\,(A\nabla\partial_\gamma u) = \partial_\gamma f + \mathrm{div}\,(\partial_\gamma A\nabla u)$$

Applying the induction hypothesis to $v := \partial_\gamma u \in W^{k,2}_{loc}(\Omega) \subset W^{1,2}_{loc}(\Omega)$ we find for any $\Omega_1 \subset\subset \Omega_2 \subset\subset \Omega_3 \subset\subset \Omega$

$$\begin{aligned}
&\|\partial_\gamma u\|_{W^{k+1,2}(\Omega_1)}\\
\precsim & C(\Omega_1,\Omega_2,A)\left(\|\partial_\gamma f\|_{W^{k-1,2}(\Omega_2)} + \|\mathrm{div}\,(\partial_\gamma A\nabla u)\|_{W^{k-1,2}(\Omega_2)} + \|\partial_\gamma u\|_{W^{1,2}(\Omega_2)}\right)\\
\precsim & C(\Omega_1,\Omega_2,A)\left(\|f\|_{W^{k,2}(\Omega_2)} + C(D^2A)\|\nabla u\|_{W^{k-1,2}(\Omega_2)} + C(DA)\|D^2 u\|_{W^{k-1,2}(\Omega_2)} + \|u\|_{W^{2,2}(\Omega_2)}\right)\\
\precsim & C(\Omega_1,\Omega_2,A)\left(\|f\|_{W^{k,2}(\Omega_2)} + \|u\|_{W^{k,2}(\Omega_2)} + \|u\|_{W^{k+1,2}(\Omega_2)} + \|u\|_{W^{2,2}(\Omega_2)}\right)\\
\precsim & C(\Omega_1,\Omega_2,A)\left(\|f\|_{W^{k,2}(\Omega_2)} + \|u\|_{W^{k+1,2}(\Omega_2)}\right)
\end{aligned}$$

This holds for any $\gamma \in \{1,\ldots,n\}$ so we actually find

$$\begin{aligned}
\|u\|_{W^{k+2,2}(\Omega_1)} \precsim & \max\{\gamma \in \{1,\ldots,n\}\}\|\partial_\gamma u\|_{W^{k+1,2}(\Omega_1)}\\
\precsim & C(\Omega_1,\Omega_2,A)\left(\|f\|_{W^{k,2}(\Omega_2)} + \|u\|_{W^{k+1,2}(\Omega_2)}\right)
\end{aligned}$$

Applying once more the induction hypothesis (to u this time) we have

$$\begin{aligned}
\|u\|_{W^{k+1,2}(\Omega_2)} \precsim & C(\Omega_2,\Omega_3,A)\left(\|f\|_{W^{k-1,2}(\Omega_3)} + \|u\|_{W^{1,2}(\Omega_3)}\right)\\
\precsim & C(\Omega_2,\Omega_3,A)\left(\|f\|_{W^{k,2}(\Omega_3)} + \|u\|_{W^{1,2}(\Omega_3)}\right)
\end{aligned}$$

So, we have shown

$$\begin{aligned}
\|u\|_{W^{k+2,2}(\Omega_1)} \precsim & \max\{\gamma \in \{1,\ldots,n\}\}\|\partial_\gamma u\|_{W^{k+1,2}(\Omega_1)}\\
\precsim & C(\Omega_1,\Omega_2,\Omega_3,A)\left(\|f\|_{W^{k,2}(\Omega_3)} + \|u\|_{W^{1,2}(\Omega_3)}\right).
\end{aligned}$$

This holds for any $\Omega_1 \subset\subset \Omega_3$ (we can always find a suitable Ω_2), so we have shown the induction step and can conclude.

□

Exercise 6.11. *Prove the statement in Remark 6.5.*

6.4. **Global/Boundary regularity theory: Proof of Theorem 6.6.** Assume u is a solution to

$$\begin{cases} -\mathrm{div}\,(A\nabla u) = f & \text{in } \Omega\\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

Since $\partial\Omega$ is a smooth manifold, for any point $x_0 \in \partial\Omega$ there exists a small radius $r(x_0) > 0$ and a diffeomorphism

$$\Phi : B(0,1) \to \mathbb{R}^n$$

with $\Phi(0) = x_0$, $\Phi(B(0,1)\cap\mathbb{R}^n_+) \subset \Omega$, $\Phi(B(0,1)\cap\mathbb{R}^n_-) \subset \mathbb{R}^n\setminus\Omega$ and $\Phi(B(0,1)) \supset B(x_0,r)$.

Take $\eta \in C^\infty_c(B(x_0,r))$ such that $\eta \equiv 1$ in $B(x_0,r/2)$. It suffices to show that $\eta u \in W^{2,2}(\Omega)$ and

$$\|\eta u\|_{W^{2,2}(\Omega)} \precsim \|u\|_{W^{1,2}(\Omega)} + \|f\|_{L^2(\Omega)}. \tag{6.4}$$

Indeed, if we can do that we can find finitely many balls $(B(x_i, r_i))_{i=1}^N$ such that $B(x_i, r_i/2)$ covers a neighborhood of $\partial\Omega$ (because $\partial\Omega$ is compact). Set $\Omega_0 := \Omega \setminus \overline{\bigcup_i B(x_i, r_i/4)} \subset\subset \Omega$. Then we take a partition of unity η_i of Ω: $\eta_i \in C_c^\infty(B(x_i, r_i))$, $\eta_i \equiv 1$ in $B(x_i, r_i/2)$, and $\eta_0 \in C_c^\infty(\Omega_0)$ such that

$$\sum_{i=0}^N \eta_i \equiv 1 \quad \text{in } \Omega.$$

From the interior theory we have already the estimate

$$\|\eta_0 u\|_{W^{2,2}(\Omega)} \lesssim \|u\|_{W^{1,2}(\Omega)} + \|f\|_{L^2(\Omega)}.$$

If we have (6.4) then we'd get

$$\|u\|_{W^{2,2}(\Omega)} \lesssim \sum_{i=0}^N \sum \|\eta_i u\|_{W^{2,2}(\Omega)} \lesssim N \left(\|u\|_{W^{1,2}(\Omega)} + \|f\|_{L^2(\Omega)} \right).$$

Observe that the choice of η_i and $B(x_i, r_i)$ and N only depends on the set Ω.

As before,

$$\operatorname{div}(A\nabla(\eta u)) = \eta f + \operatorname{div}(A(\nabla\eta)u) + \partial_\alpha \eta A_{\alpha\beta} \nabla u \quad \text{in } \Omega$$

If we set

$$g := \eta f + \operatorname{div}(A(\nabla\eta)u) + \partial_\alpha \eta A_{\alpha\beta} \nabla u,$$

we see that

$$\|g\|_{L^2(\Omega)} \lesssim C(A, \eta) \left(\|f\|_{L^2(\Omega)} + \|u\|_{W^{1,2}(\Omega)} \right).$$

So if we set $\tilde{u} := (\eta u)$ we actually have to consider $W^{2,2}$ estimates the equation

$$\begin{cases} -\operatorname{div}(A\nabla\tilde{u}) = g & \text{in } B(x_0, r) \cap \Omega \\ \tilde{u} = 0 & \text{on } \partial(B(x_0, r) \cap \Omega) \end{cases}$$

Now we proceed with a method called *flattening the boundary*.

We set $B(0,1)^+ := B(0,1) \cap \mathbb{R}^n_+$ and define

$$v(x) := \tilde{u} \circ \Phi \in W^{1,2}_0(B(0,1) \cap \mathbb{R}^n_+).$$

We then have (also in the distributional sense which is proven by approximation)

$$\partial_\alpha v(x) = (\partial_\gamma \tilde{u})(x) \partial_\alpha \Phi^\gamma(x).$$

Denote the matrix $(D\Phi(x))_{\alpha\beta} := \partial_\alpha \Phi^\gamma(x)$. This matrix is invertible, and we call this inverse $(D\Phi)^{-1}(x)$. Observe that by chain rule

$$(D\Phi)^{-1}(x) = D(\Phi^{-1})(\Phi(x)).$$

Set

$$\tilde{A}(x) := \left(|\det(D\Phi)| \left(D\Phi^{-1} \right)^t A\, D\Phi^{-1} \right) (\Phi(x)) : B(0,1)^+ \to \mathbb{R}^{n \times n}.$$

This is a smooth, symmetric, *elliptic* matrix-valued map (using heavily that Φ is a diffeomorphism!). Moreover we have for any $\varphi \in C_c^\infty(B(0,1)^+)$, setting $\psi := \varphi \circ \Phi^{-1} \in C_c^\infty(B(x_0,r) \cap \Omega)$

$$
\begin{aligned}
&\int_{B(0,1)^+} \partial_\beta \tilde{v}(x) \tilde{A}_{\alpha\beta}(x) \partial_\alpha \varphi(x)\, dx\\
=&\int_{B(0,1)^+} \partial_\beta \left(\tilde{u} \circ \Phi\right)(x) \tilde{A}_{\alpha\beta}(x) \partial_\alpha \left(\psi(\Phi)\right)(x)\, dx\\
=&\int_{\Omega \cap B(x_0,r)} \partial_\beta \left(\tilde{u} \circ \Phi\right)(\Phi^{-1}(z)) \tilde{A}_{\alpha\beta}(\Phi^{-1}(z)) \partial_\alpha \left(\psi(\Phi)\right)(\Phi^{-1}(z))\, |\det(D\Phi^{-1}(z))|\, dz\\
=&\int_{\Omega \cap B(x_0,r)} \partial_\beta \tilde{u}(z) \, \left(|\det(D\Phi)|^{-1} D\Phi^t \tilde{A} D\Phi\right)_{\alpha\beta} (\Phi^{-1}(z))\, \partial_\alpha \psi(z)\, dz\\
=&\int_{\Omega \cap B(x_0,r)} \partial_\beta \tilde{u}(z)\, A(z)\, \partial_\alpha \psi(z)\ dz\\
=&\int_{\Omega \cap B(x_0,r)} g\psi\, dz\\
=&\int_{B(0,1)^+} g \circ \Phi\, |\det(D\Phi)|\, \varphi
\end{aligned}
$$

We have reduced Theorem 6.6 to the following

Proposition 6.12. *Let $u \in W_0^{1,2}(\mathbb{R}^n_+)$ solve the equation*

$$-\mathrm{div}\,(A\nabla u) = f \quad \text{in } \mathbb{R}^n_+.$$

Also assume that $u \equiv 0$ in $\mathbb{R}^n_+ \setminus B(0,1)$.

If $f \in L^2(\mathbb{R}^n_+)$ then $u \in W^{2,2}(\mathbb{R}^n_+)$ with the estimate

$$\|u\|_{W^{2,2}(\mathbb{R}^n_+)} \precsim \|f\|_{L^2(\mathbb{R}^n_+)} + \|u\|_{W^{1,2}(\mathbb{R}^n_+)}.$$

We sketch the idea of the proof.

Sketch of the proof of Proposition 6.12. The idea is – yet again – differentiation of the equation. If we consider $i \in \{1,\ldots,n-1\}$ then

$$-\mathrm{div}\,(A\nabla \partial_{x_i} u) = \partial_{x_i} f - \mathrm{div}\,(\partial_{x_i} A\, \nabla u) \quad \text{in } \mathbb{R}^n_+.$$

Observe that since $i \neq n$ we still believe we could have $\partial_{x_i} u = 0$ on $\partial\mathbb{R}^n_+ = \mathbb{R}^{n-1} \times \{0\}$. As before the right-hand side belongs to $\left(W_0^{1,2}(\mathbb{R}^n_+)\right)^*$,

$$\|\partial_{x_i} f - \mathrm{div}\,(\partial_{x_i} A\, \nabla u)\|_{\left(W_0^{1,2}(\mathbb{R}^n_+)\right)^*} \precsim \|f\|_{L^2(\mathbb{R}^n_+)} + \|u\|_{W^{1,2}(\mathbb{R}^n_+)}.$$

So it sounds believable that by the same strategy as before (testing with $\partial_{x_i} u$) we would get

$$\|\partial_{x_i} u\|_{W^{1,2}(\mathbb{R}^n_+)} \precsim \|f\|_{L^2(\mathbb{R}^n_+)} + \|u\|_{W^{1,2}(\mathbb{R}^n_+)}.$$

This holds for any $i \in \{1,\ldots,n-1\}$. We call this a *tangential estimate*. But it does not work for $i = n$, since $\partial_n u$ is possibly nonzero on $\partial\mathbb{R}^n_+ = \mathbb{R}^{n-1} \times \{0\}$!

So what we get is

$$\|\partial_{\alpha\beta}u\|_{L^2(\mathbb{R}^n_+)} \precsim \|f\|_{L^2(\mathbb{R}^n_+)} + \|u\|_{W^{1,2}(\mathbb{R}^n_+)} \quad \forall \alpha,\beta \in \{1,\ldots,n\}: \quad (\alpha,\beta) \neq (n,n). \tag{6.5}$$

How do we get information on $\partial_{nn}u$? We use that $\Delta = \partial_{nn} + \sum_{i=1}^n \partial_{x_ix_i}$!

Namely observe that

$$\partial_n(A_{nn}\partial_n u) = f - \sum_{(\alpha,\beta)\neq(n,n)} \partial_\alpha(A_{\alpha\beta}\partial_\beta u)$$

Since $(\alpha,\beta) \neq (n,n)$ we have from the previous estimate (6.5)

$$\|\partial_\alpha(A_{\alpha\beta}\partial_\beta u)\|_{L^2(\mathbb{R}^n_+)} \precsim \|u\|_{W^{1,2}(\mathbb{R}^n_+)} + \max_{i\in 1,\ldots,n-1} \|\partial_{x_i}u\|_{W^{1,2}(\mathbb{R}^n_+)} \overset{(6.5)}{\precsim} \|f\|_{L^2(\mathbb{R}^n_+)} + \|u\|_{W^{1,2}(\mathbb{R}^n_+)}.$$

and we conclude that

$$\|\partial_n(A_{nn}\partial_n u)\|_{L^2(\mathbb{R}^n_+)} \precsim \|f\|_{L^2(\mathbb{R}^n_+)} + \|u\|_{W^{1,2}(\mathbb{R}^n_+)}.$$

Thus, since $A_{nn}\partial_{nn}u = \partial_n(A_{nn}\partial_n u) - \partial_n A_{nn}\,\partial_n u$,

$$\|A_{nn}\partial_{nn}u\|_{L^2(\mathbb{R}^n_+)} \precsim \|\partial_n(A_{nn}\partial_n u)\|_{L^2(\mathbb{R}^n_+)} + \|u\|_{W^{1,2}(\mathbb{R}^n_+)} \precsim \|f\|_{L^2(\mathbb{R}^n_+)} + \|u\|_{W^{1,2}(\mathbb{R}^n_+)}.$$

And now we use yet again ellipticity: $A_{nn} = \langle e_n, Ae_n\rangle \geq \lambda$, so we have finally shown

$$\lambda\|\partial_{nn}u\|_{L^2(\mathbb{R}^n_+)} \precsim \|A_{nn}\partial_{nn}u\|_{L^2(\mathbb{R}^n_+)} \precsim \|f\|_{L^2(\mathbb{R}^n_+)} + \|u\|_{W^{1,2}(\mathbb{R}^n_+)}.$$

Now the above argument only delivers an *a priori* argument, since we needed assumed that $\partial_{x_i}u \in W^{1,2}_0(\mathbb{R}^n_+)$. The precise argument goes as follows: Let $h \in \mathbb{R}^{n-1}\times\{0\}$ and consider $\delta_h u \in W^{1,2}_0(\mathbb{R}^n_+)$. Then

$$-\operatorname{div}(A\nabla\delta_h u) = g_h \quad \text{in } \mathbb{R}^n_+,$$

where we can estimate

$$\|g_h\|_{\left(W^{1,2}_0(\mathbb{R}^n+)\right)^*} \precsim |h|\left(\|u\|_{W^{1,2}(\mathbb{R}^n_+)} + \|f\|_{L^2(\mathbb{R}^n_+)}\right)$$

Testing the equation with $\delta_h u$ we then obtain

$$\sup_{h\in\mathbb{R}^{n-1}\times\{0\}} |h|^{-1}\|\delta_h u\|_{W^{1,2}(\mathbb{R}^n_+)} \precsim \|u\|_{W^{1,2}(\mathbb{R}^n_+)} + \|f\|_{L^2(\mathbb{R}^n_+)}.$$

Suitably adapting the argument of Proposition 5.15 we find as desired that

$$\|\partial_{x_i}u\|_{W^{1,2}(\mathbb{R}^n_+)} \precsim \|u\|_{W^{1,2}(\mathbb{R}^n_+)} + \|f\|_{L^2(\mathbb{R}^n_+)} \quad \forall i = 1,\ldots,n-1.$$

Now let $\varphi \in C^\infty_c(\mathbb{R}^n_+)$ then we have

$$\int_{\mathbb{R}^n_+} A_{nn}\,\partial_n u\partial_n\varphi = \int f\varphi - \int_{\mathbb{R}^n_+} \sum_{(\alpha,\beta)\neq(n,n)} \underbrace{\partial_\beta\left(A_{\alpha\beta}\,\partial_\alpha u\right)}_{\in L^2}\varphi$$

From the previous estimates we conclude that

$$|\int_{\mathbb{R}^n_+} A_{nn}\,\partial_n u\partial_n\varphi| \precsim \|\varphi\|_{L^2(\mathbb{R}^n_+)}\left(\|u\|_{W^{1,2}(\Omega)} + \|f\|_{L^2}\right).$$

In particular we have

$$|\int_{\mathbb{R}^n_+} \partial_n u \partial_n \left(A_{nn}\varphi\right)| \precsim \|\varphi\|_{L^2(\mathbb{R}^n_+)} \left(\|u\|_{W^{1,2}(\Omega)} + \|f\|_{L^2}\right).$$

Take now $\psi \in C_c^\infty(\mathbb{R}^n_+)$ then $\varphi := (A_{nn})^{-1}\psi \in C_c^\infty(\mathbb{R}^n_+)$ (using ellipticity), so we have

$$\left|\int_{\mathbb{R}^n} \partial_n u\, \partial_n \psi\right| = \left|\int_{\mathbb{R}^n_+} \partial_n u \partial_n \psi\right| \precsim \|\psi\|_{L^2(\mathbb{R}^n_+)} \left(\|u\|_{W^{1,2}(\Omega)} + \|f\|_{L^2}\right).$$

But this implies by the definition of distributional derivative (and Riesz representation theorem) that $\partial_{nn} u \in L^2(\mathbb{R}^n_+)$ awith the estimate

$$\|\partial_{nn}u\|_{L^2(\mathbb{R}^n_+)} \precsim \|u\|_{W^{1,2}(\Omega)} + \|f\|_{L^2}.$$

Thus we have shown

$$\|u\|_{W^{2,2}(\mathbb{R}^n_+)} \precsim \max_{i=1,\ldots,n-1} \|\partial_{x_i} u\|_{W^{1,2}(\mathbb{R}^n_+)} + \|\partial_{nn}u\|_{L^2(\mathbb{R}^n_+)} \precsim \|u\|_{W^{1,2}(\Omega)} + \|f\|_{L^2}.$$

□

It is now not difficult (but very cumbersome) to prove the following generalization

Theorem 6.13 (Global $W^{k,2}$-regularity). *Assume Ω is a bounded open set with smooth boundary $\partial\Omega \in C^\infty$. Let $f \in W^{k,2}(\Omega)$ and assume $u \in W^{1,2}_0(\Omega)$ solves*

$$\begin{cases} -\operatorname{div}(A\nabla u) = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

in the above sense.

Then $u \in W^{k+2,2}(\Omega)$, and we have

$$\|D^{k+2}u\|_{L^2(\Omega)} \leq C(\Omega) \left(\|u\|_{W^{1,2}(\Omega)} + \|f\|_{L^2(\Omega)}\right).$$

Exercise 6.14. *Think about how (formally) you could prove now Theorem 6.13*

We can also treat more generic boundary data:

Theorem 6.15 (Global $W^{k,2}$-regularity). *Assume Ω is a bounded open set with smooth boundary $\partial\Omega \in C^\infty$. Let $f \in W^{k,2}(\Omega)$ and $g \in W^{k+2,2}(\Omega)$ and assume $u \in W^{1,2}(\Omega)$ solves*

$$\begin{cases} -\operatorname{div}(A\nabla u) = f & \text{in } \Omega \\ u = g & \text{on } \partial\Omega, \end{cases}$$

in the weak sense (where $u = g$ on $\partial\Omega$ simply means $u - g \in W^{1,2}_0(\Omega)$).

Then $u \in W^{k+2,2}(\Omega)$, and we have

$$\|D^{k+2}u\|_{L^2(\Omega)} \leq C(\Omega) \left(\|u\|_{W^{1,2}(\Omega)} + \|f\|_{L^2(\Omega)} + \|g\|_{W^{k+2,2}(\Omega)}\right).$$

Exercise 6.16. *Think about how (formally) you could prove now Theorem 6.15.*

Hint: consider $v := u - g$.

6.5. **An alterative approach to boundary regularity theory: reflection.** There is another (in general quite delicate) argument for boundary regularity that we want to very briefly discuss.

Assume that we have the equation

$$\begin{cases} \Delta u = f & \text{in } \mathbb{R}^n_+ \\ u = 0 & \text{in } \mathbb{R}^{n-1} \times \{0\}. \end{cases}$$

If we want to find regularity at the boundary, we could use the following reflection argument.

For $x = (x', x_n) \in \mathbb{R}^n_+$ set

$$\tilde{u}(x', x_n) := u(x', |x_n|).$$

Now observe that this is a Lipschitz operation and for continuous functions u we have that $\tilde{u}$ is also continuous (thanks to the boundary data being zero). So if $u \in W^{1,2}_0$ then it seems believable (a proof is needed however) that $\tilde{u} \in W^{1,2}(\mathbb{R}^n_+)$. The formal computation goes like this: Clearly

$$\partial_n \tilde{u}(x', x_n) = \begin{cases} \partial_n u(x', x_n) & x_n > 0 \\ -\partial_n u(x', -x_n) & x_n < 0 \end{cases}$$

Observe (think about the Heaviside function) that this does not imply that $\partial_n u$ exists in distributional sense)! But (formally) we now can show for any $\varphi \in C_c^\infty(\mathbb{R}^n)$

$$\begin{aligned}
\int_{\mathbb{R}^n} \tilde{u} \partial_n \varphi &= \int_{\mathbb{R}^n_+} u(x', x_n) \partial_n \varphi(x) - \int_{\mathbb{R}^n_-} u(x', -x_n) \partial_n \varphi(x) \\
&\overset{\text{substitution}}{=} \int_{\mathbb{R}^n_+} u(x', x_n) \partial_n \varphi - \int_{\mathbb{R}^n_+} u(x', +x_n) \left(\partial_n \varphi\right)(x', -x_n) \\
&= \int_{\mathbb{R}^n_+} u(x', x_n) \partial_n \varphi + \int_{\mathbb{R}^n_+} u(x', x_n) \partial_n \left(\varphi(x', -x_n)\right) \\
&\overset{\text{P.I.}}{=} - \int_{\mathbb{R}^n_+} \partial_n u(x', x_n) \varphi(x', x_n) - \int_{\mathbb{R}^n_+} \partial_n u(x', x_n) \, \varphi(x', -x_n) \\
&\quad - \int_{\mathbb{R}^{n-1} \times \{0\}} \underbrace{u(x', 0)}_{=0} \varphi(x', 0) dx' - \int_{\mathbb{R}^{n-1} \times \{0\}} \underbrace{u(x', 0)}_{=0} \varphi(x', 0) dx' \\
&\overset{\text{substitution}}{=} - \int_{\mathbb{R}^n_+} \partial_n u(x', x_n) \varphi - \int_{\mathbb{R}^n_-} \left(\partial_n u\right)(x', -x_n) \, \varphi(x', +x_n) \\
&\overset{def}{=} - \int_{\mathbb{R}^n_+} \partial_n \tilde{u} \varphi - \int_{\mathbb{R}^n_-} \partial_n \tilde{u} \varphi \\
&= - \int_{\mathbb{R}^n} \partial_n \tilde{u} \, \varphi
\end{aligned}$$

This shows that $\tilde{u} \in W^{1,2}(\mathbb{R}^n)$ (some details need to be ironed out, but thats the idea).

Now we compute in a similar way the PDE:

$$\partial_{x_i x_i} \tilde{u}(x) = (\partial_{x_i x_i} u)(x', |x_n|) \quad i = 1, \ldots, n-1$$

and

$$\partial_{x_n x_n} \tilde{u}(x) = \begin{cases} (\partial_{x_n x_n} u)(x', x_n) & x_n > 0 \\ - - (\partial_{x_n x_n} u)(x', -x_n) & x_n < 0 \end{cases} = (\partial_{nn} u)(x', |x_n|)$$

Thus we have

$$\Delta \tilde{u} = f(x', |x_n|) \quad \text{in } \mathbb{R}^n.$$

Now we can obtain boundary regularity by interior regularity theory.

This arguments is beautiful, but its downsides are that the reflection needs to be adapted to the PDE at hand – which can be extremely difficult (or impossible).

6.6. **Extension to more general elliptic equations.** With the above arguments one can also treat more general linear PDE (and indeed the arguments for nonlinear elliptic pde are mostly based on "using linear theory for nonlinear pde", and thus follow the general spirit of the above argument).

$$\begin{cases} -\operatorname{div}(A \nabla u) + b \cdot \nabla u + cu = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

Part 2. **PDE 2**

7. The Role of Harmonic Analysis in PDE – L^p-theory

7.1. **Short introduction to Calderón-Zygmund Theory.** Calderón-Zygmund theory is the L^p-regularity theory for elliptic equations. For example assume that u solves

$$-\Delta u = \partial_\beta f \quad \text{in } \mathbb{R}^n$$

and that $f \in L^p(\mathbb{R}^n)$, $p \in (1, \infty)$. We would like to conclude that $\nabla u \in L^p(\mathbb{R}^n)$ with the estimates

$$\|\nabla u\|_{L^p} \precsim \|f\|_{L^p}. \tag{7.1}$$

or, if

$$-\Delta u = f \quad \text{in } \mathbb{R}^n$$

then we would like to conclude

$$\|\nabla^2 u\|_{L^p} \precsim \|f\|_{L^p}. \tag{7.2}$$

Both statements are called *Calderón-Zygmund theory*. Observe that for $p = 2$ we have treated this in Section 6.

Example 7.1 (An idea for an application)**.** Assume that $u \in W^{1,2}$ solves

$$\Delta u = \partial_\alpha u$$

then we would like to conclude that u is smooth and a classical solution.

First we would get from Section 6 that since $\partial_\alpha u \in L^2$ we have $\nabla^2 u \in L^2$, i.e. $u \in W^{2,2}$ which by Sobolev embedding implies $\partial_\alpha u \in L^{2^*}$ by Sobolev embedding (Recall that $2^* = \frac{2n}{n-2}$ is the Sobolev exponent). To continue our bootstrapping we would now need (7.2) but for $p = 2^* \neq 2$.

Another way would be an approach via (7.1). By Sobolev embedding $u \in L^{2^*}$, and again we need (7.1) to conclude that $\nabla u \in L^{2^*}$ and then we could try to bootstrap our way to smoothness of u.

This theory that leads to both (7.1) and (7.2) is closely connected with harmonic analysis and *Calderón-Zygmund operators*. Denote by $I^2 = (-\Delta)^{-1}$ the *Riesz potential* (we called this *Newton potential* before) (we assume for simplicity that $n \geq 3$), we have the formula (2.4)

$$I^2 g(x) = c \int_{\mathbb{R}^n} |x-y|^{2-n}\, g(y)\, dy.$$

Then, if

$$\Delta u = f \quad \text{in } \mathbb{R}^n$$

we can (formally) write

$$\partial_\alpha u = \partial_\alpha \Delta^{-1} \Delta u = \partial_\alpha \Delta^{-1} \partial_\beta f.$$

Computing the derivative we find that

$$\partial_\alpha \Delta^{-1} \partial_\beta f(x) = \tilde{c} \int_{\mathbb{R}^n} \frac{\frac{(x-y)^\alpha}{|x-y|}\,\frac{(x-y)^\beta}{|x-y|}}{|x-y|^n}\, f(y)\, dy. \tag{7.3}$$

We will see below that (for $n \geq 2$) the operator

$$T_{\alpha,\beta} f(x) := \tilde{c} \int_{\mathbb{R}^n} \frac{\frac{(x-y)^\alpha}{|x-y|}\,\frac{(x-y)^\beta}{|x-y|}}{|x-y|^n}\, f(y)\, dy$$

is a *Calderón-Zygmund operator* which as such is a bounded linear operator from L^p to L^p, namely

$$\|T_{\alpha,\beta} f\|_{L^p(\mathbb{R}^n)} \lesssim \|f\|_{L^p(\mathbb{R}^n)} \quad \forall f \in L^p(\mathbb{R}^n), p \in (1,\infty).$$

From this we obtain immmediately that

$$\|\nabla u\|_{L^p(\mathbb{R}^n)} \lesssim \max_{\alpha,\beta \in \{1,\ldots,n\}} \|\partial_\alpha \Delta^{-1} \partial_\beta f\|_{L^p(\mathbb{R}^n)} = \max_{\alpha,\beta \in \{1,\ldots,n\}} \|T_{\alpha,\beta} f\|_{L^p(\mathbb{R}^n)} \lesssim \|f\|_{L^p(\mathbb{R}^n)}.$$

In the following we illustrate how to make the above statements precise.

7.2. **Calderón-Zygmund operators.** The typical *Calderón-Zygmund operator* is the *Riesz transform*

$$\mathcal{R}_\alpha f(x) := c \int_{\mathbb{R}^n} \frac{\frac{(x-y)^\alpha}{|x-y|}}{|x-y|^n} f(y) dy. \tag{7.4}$$

One can compute that the *Fourier symbol* of $\mathcal{R}_\alpha$ is $i\frac{\xi^\alpha}{|\xi|}$, $\alpha = 1, \ldots, n$, i.e.

$$(\mathcal{R}_\alpha f)^\wedge(\xi) = i\frac{\xi^\alpha}{|\xi|} f^\wedge(\xi).$$

In particular we have that $\partial_\alpha \partial_\beta f = c\mathcal{R}_\alpha \mathcal{R}_\beta \Delta f$, which is what we used in (7.3).

Observe that the symbol of $\mathcal{R}_\alpha$ belongs to $L^\infty(\mathbb{R}^n)$,

$$\left\| i\frac{\xi^\alpha}{|\xi|} \right\|_{L^\infty} \leq 1.$$

It is easy to show that such an operator is bounded on L^2:

Lemma 7.2. *Let $m \in L^\infty(\mathbb{R}^n)$ and define*

$$Tf := \left(m(\xi) f^\wedge(\xi)\right)^\vee.$$

Then T is a linear bounded operator on $L^2(\mathbb{R}^n)$ with

$$\|Tf\|_{L^2(\mathbb{R}^n)} \lesssim \|m\|_{L^\infty} \|f\|_{L^2(\mathbb{R}^n)}.$$

Such a T is usually called a multiplier operator, and m is the symbol.

Proof. By Plancherel identity, $\|g\|_{L^2(\mathbb{R}^n)} = \|g^\wedge\|_{L^2(\mathbb{R}^n)}$. Thus,

$$\|Tf\|_{L^2(\mathbb{R}^n)} = \|(Tf)^\wedge\|_{L^2(\mathbb{R}^n)} = \|m\, f^\wedge\|_{L^2(\mathbb{R}^n)} \leq \|m\|_{L^\infty} \|f^\wedge\|_{L^2} = \|m\|_{L^\infty} \|f\|_{L^2}.$$

□

Observe that we cannot simply replace L^2 with L^p in Lemma 7.2, since there is no Plancherel identity on L^p for $p \neq 2$.

Theorem 7.3 (Boundedness of Calderón-Zygmund-Operators)**.** *Let $T : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ be a bounded linear operator, which for $f \in C_c^\infty(\mathbb{R}^n)$ can be written as*

$$Tf(x) = \int_{\mathbb{R}^n} \frac{\Omega(x-y)}{|x-y|^n} f(y)\, dy$$

in a principal value *sense:*

$$Tf(x) = \lim_{\varepsilon \to 0^+} \int_{\varepsilon \leq |x-y| \leq \frac{1}{\varepsilon}} \frac{\Omega(x-y)}{|x-y|^n} f(y)\, dy$$

If the kernel Ω satisfies

(1) $\Omega : \mathbb{R}^n \backslash \{0\} \to \mathbb{R}$ *is bounded,* $\|\Omega\|_{L^\infty} < \infty$

(2) Ω *is homogeneous of order* 0*, i.e.* $\Omega(rz) = \Omega(z)$ *for all* $r > 0$*,* $z \in \mathbb{R}^n \backslash \{0\}$*.*

(3) $\Omega : \mathbb{R}^n \backslash \{0\}$ *is locally Lipschitz with the bound* $\sup_{z \in \mathbb{R}^n \backslash \{0\}} |z||\nabla \Omega(z)| < \infty$

then T *is*[17] *a bounded linear operator from* $L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$*.*

Exercise 7.4. *Show that each of the* Ω *defined below satisfy the conditions (1),(2),(3) of Theorem 7.3*

(1) for some $\alpha \in \{1, \ldots, n\}$*,*

$$\Omega(z) := \frac{z_\alpha}{|z|}.$$

(2) for some $\alpha, \beta \in \{1, \ldots, n\}$*,*

$$\Omega(z) := \frac{z_\alpha z_\beta}{|z|^2}.$$

(3) $\Omega(z) = 1$

Bonus question: Show that the results of Theorem 7.3 are nevertheless false for $\Omega(z) = 1$*. Why does the theorem not apply to the last case?*

We are not proving Theorem 7.3 in its full generality, but only illustrate one argument for the case we need (a relatively easy adaptation of the following does the job).

The idea is to get an extremal theorem: One at the level of BMO (which is a replacement for L^∞) or at the level of L^1 (this leads to the so-called *Calderón-Zygmund decomposition*), then we use *interpolation*. The L^2-boundedness follows from Lemma 7.2.

Proposition 7.5 (Boundedness from L^∞ to BMO)**.** *For a monomial* p *of degree* k *assume that* T *is a linear bounded operator from* $L^2(\mathbb{R}^n)$ *to* $L^2(\mathbb{R}^n)$ *which for* $f \in C_c^\infty(\mathbb{R}^n)$ *can be represented as*

$$Tf(x) := P.V. \int_{\mathbb{R}^n} \frac{p(x-y)}{|x-y|^{n+k}} f(y)\, dy$$

Then, for $f \in L^\infty(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$*,*

$$[Tf]_{BMO} \lesssim \|f\|_{L^\infty(\mathbb{R}^n)}.$$

[17] more precisely: extends to

Here[18]

$$[g]_{BMO} = \sup_{B(x,\rho)} \left(\fint_{B(x,\rho)} |g-(g)_{B(x,\rho)}|^2 \right)^{\frac{1}{2}}.$$

Lemma 7.6. *Let $f \in L^2(B(r))$ then for all $c \in \mathbb{R}$,*

$$\|f-(f)_{B(r)}\|_{L^2(B(r))} \leq \|f-c\|_{L^2(B(r))}.$$

Exercise 7.7. *Prove Lemma 7.6.*

Proof of Proposition 7.5. For $x_0 \in \mathbb{R}^n$ and $r>0$ let $f_{r,x_0}(x) := f(x_0+rx)$. Observe that by the structure of T,

$$T(f_{r,x_0})(x) = (Tf)(x_0+rx).$$

This implies that

$$\fint_{B(0,1)} T(f_{r,x_0}) = \fint_{B(x_0,r)} Tf,$$

and

$$\fint_{B(0,1)} \left|T(f_{r,x_0}) - (T(f_{r,x_0}))_{B(0,1)}\right|^2 = \fint_{B(x_0,r)} \left|T(f) - (T(f)_{B(x_0,r)}\right|^2,$$

and

$$\|f_{r,x_0}\|_{L^\infty(\mathbb{R}^n)} = \|f\|_{L^\infty(\mathbb{R}^n)}.$$

Thus, if we can show that for any $f \in L^2 \cap L^\infty(\mathbb{R}^n)$

$$\fint_{B(0,1)} |Tf-(Tf)_{B(0,1)}|^2 \precsim \|f\|^2_{L^\infty(\mathbb{R}^n)},$$

then the full claim follows via scaling and translation.

Now let

$$f := f_1 + f_2,$$

with $f_1 = \chi_{B(0,2)} f$ and $f_2 = \chi_{\mathbb{R}^n \backslash B(0,2)} f$. Then,

$$\begin{aligned}\fint_{B(0,1)} |Tf-(Tf)_{B(0,1)}|^2 \precsim & \fint_{B(0,1)} |Tf_1-(Tf_1)_{B(0,1)}|^2 + \fint_{B(0,1)} |Tf_2-(Tf_2)_{B(0,1)}|^2 \\ \precsim & 2\fint_{B(0,1)} |Tf_1|^2 + \fint_{B(0,1)} |Tf_2-(Tf_2)_{B(0,1)}|^2.\end{aligned}$$

Observe that by the L^2-boundedness, Lemma 7.2,

$$\fint_{B(0,1)} |Tf_1|^2 \precsim \|Tf_1\|^2_{L^2(\mathbb{R}^n)} \precsim \|f_1\|^2_{L^2(B(0,2))} = \|f\|^2_{L^2(B(0,2))} \precsim \|f\|^2_{L^\infty(\mathbb{R}^n)}.$$

[18]It is a major theorem, the John-Nirenberg theorem, that the power inside the integral of the BMO-seminorm does not matter: That is, if $[g]_{BMO} < \infty$ then

$$[g]_{BMO} \approx \sup_{B(x,\rho)} \left(\fint_{B(x,\rho)} |g-(g)_{B(x,\rho)}|^p \right)^{\frac{1}{p}}.$$

Now in view of Lemma 7.6,

$$\fint_{B(0,1)} |Tf_2 - (Tf_2)_{B(0,1)}|^2 \precsim \fint_{B(0,1)} |Tf_2 - Tf_2(0)|^2 \tag{7.5}$$

Now,

$$Tf_2(x) - Tf_2(0) = \int_{\mathbb{R}^n} \left(\frac{p(x-y)}{|x-y|^{n+k}} - \frac{p(-y)}{|y|^{n+k}} \right) f_2(y)\,dy = \int_{\mathbb{R}^n \setminus B(0,2)} \left(\frac{p(x-y)}{|x-y|^{n+k}} - \frac{p(-y)}{|y|^{n+k}} \right) f(y)\,dy$$

If $x \in B(0,1)$ and $y \notin B(0,2)$ then $|x-y| \approx |y| \succsim 1$. In this case we obtain from the fundamental theorem of calculus that

$$\left(\frac{p(x-y)}{|x-y|^{n+k}} - \frac{p(-y)}{|y|^{n+k}} \right) \precsim \frac{|x|}{|x-y|^{n+1}}.$$

Consequently,

$$|Tf_2(x) - Tf_2(0)| \precsim |x| \int_{\mathbb{R}^n \setminus B(0,2)} |x-y|^{-n-1} |f(y)|\,dy \precsim \|f\|_{L^\infty(\mathbb{R}^n)} \int_{|x-y| \succsim 1} |x-y|^{-n-1}\,dy.$$

Observe that

$$\sup_x \int_{|x-y| \succsim 1} |x-y|^{-n-1}\,dy < \infty.$$

So we have shown that

$$\sup_{x \in B(0,1)} |Tf_2(x) - Tf_2(0)| \precsim \|f\|_{L^\infty(\mathbb{R}^n)}.$$

which together with (7.5) implies

$$\fint_{B(0,1)} |Tf_2 - (Tf_2)_{B(0,1)}|^2 \precsim \|f\|_{L^\infty(\mathbb{R}^n)}.$$

Thus we have shown

$$\fint_{B(0,1)} |Tf - (Tf)_{B(0,1)}|^2 \precsim \|f\|_{L^\infty(\mathbb{R}^n)},$$

which by the scaling argument leads to the claim. □

Why are we happy about Proposition 7.5? Because BMO represents "almost L^∞", and we have

Theorem 7.8. *Let $1 \le p < \infty$ and T be a linear operator of "strong (p,p)-type", meaning that*

$$\|Tf\|_{L^p(\mathbb{R}^n)} \precsim \|f\|_{L^p(\mathbb{R}^n)} \quad \forall f \in L^p(\mathbb{R}^n),$$

and bounded from L^∞ to BMO, i.e

$$[Tf]_{BMO} \precsim \|f\|_{L^\infty(\mathbb{R}^n)} \quad \forall f \in L^\infty(\mathbb{R}^n),$$

Then for any $q \in [p,\infty)$, T maps $L^q(\mathbb{R}^n)$ into $L^q(\mathbb{R}^n)$ with

$$\|Tf\|_{L^q(\mathbb{R}^n)} \precsim \|f\|_{L^q(\mathbb{R}^n)} \quad \forall f \in L^q(\mathbb{R}^n).$$

Proof. This can be proven using the Marcinkiewicz interpolation theorem, but it also needs the John-Nirenberg theorem for BMO-functions – which we are not treating here. We refer to [Giaquinta and Martinazzi, 2012, Theorem 6.29]. □

Proof of Theorem 7.3. Observe that T is bounded from L^2 to L^2, Lemma 7.2, and from L^∞ to BMO, Proposition 7.5, and thus by Theorem 7.8 for any $p \in [2,\infty)$ we have

$$\|Tf\|_{L^p(\mathbb{R}^n)} \leq C_p \|f\|_{L^p(\mathbb{R}^n)}.$$

For $p < 2$ we argue *by duality*. Observe that by Riesz Representation Theorem

$$\|Tf\|_{L^p(\mathbb{R}^n)} = \sup_{g \in C_c^\infty(\mathbb{R}^n), \|g\|_{L^{p'}(\mathbb{R}^n)} \leq 1} \int_{\mathbb{R}^n} Tf\, g = \sup_{\|g\|_{L^{p'}}(\mathbb{R}^n)} \int_{\mathbb{R}^n} f\, T^*g \lesssim \|f\|_{L^p(\mathbb{R}^n)} \|T^*g\|_{L^{p'}(\mathbb{R}^n)}.$$

Now observe that T^* is of the same type of operator, so we have for any $q \in [2,\infty)$ we have

$$\|T^*g\|_{L^q(\mathbb{R}^n)} \leq C_p \|g\|_{L^q(\mathbb{R}^n)}.$$

Since for $p < 2$ we have that $q := p' > 2$ this concludes the proof. □

Remark 7.9. There is another, older, way, using the *Calderón-Zygmund decomposition* and an L^1-L^1-weak type estimate to obtain Theorem 7.3. This is what is usually done in harmonic analysis.

7.3. $W^{1,p}$-theory for the Laplace equation.

Theorem 7.10. *Let $\Omega_1 \subset\subset \Omega \subset\subset \mathbb{R}^n$ be two smoothly bounded domains, and let $p \geq 2$.*

Assume that for some $f \in L^p(\Omega)$ there is $u \in W^{1,2}(\Omega)$ that satisfies in distributional sense

$$\Delta u = \partial_\alpha f \quad \text{in } \Omega$$

Then

$$\|\nabla u\|_{L^p(\Omega_1)} \leq C(\Omega_1, \Omega, p) \left(\|f\|_{L^p(\Omega)} + \|u\|_{L^2(\Omega)}\right).$$

Remark 7.11.
- The L^2-norm for u on the right-hand side is necessary, since otherwise $f = 0$ would imply that u is constant (which is false without the assumption of appropriate boundary values).
- This statement holds for more general equations, e.g. $\partial_i(A_{ij}\partial_j u) = \partial_\alpha f$, if A is smooth enough (the sharp assumption being VMO, [Iwaniec and Sbordone, 1998])
- This is an interior statement, but it holds up to the boundary: for example if

  $$\begin{cases} \Delta u = \partial_\alpha f & \text{in } \Omega_2 \\ u = 0 & \text{on } \partial\Omega_2 \end{cases}$$

  then $\|\nabla u\|_{L^p(\Omega_2)} \leq \|f\|_{L^p(\Omega_2)}$; Cf. [Giaquinta and Martinazzi, 2012, Chapter 7].

As for the L^2-theory, the proof of Theorem 7.10 follows a sequence of cutoff arguments, such as the following

Lemma 7.12. *For $p, q \in [2, \infty)$ assume that $u \in W^{1,p}(\Omega)$ satisfies for some $f \in L^q(\Omega)$*

$$\Delta u = \partial_\alpha f \quad \text{in } \Omega$$

Let $\eta \in C_c^\infty(\Omega)$. Then for $v := \eta u$ we have

$$\Delta v = \tilde{g} + \partial_\alpha \tilde{f} \quad \text{in } \mathbb{R}^n$$

with

$$\|\tilde{f}\|_{L^q(\mathbb{R}^n)} \precsim \|f\|_{L^q(\Omega)},$$

and for $1 \le r \le \min\{p, q\}$ we have

$$\|\tilde{g}\|_{L^r(\mathbb{R}^n)} \precsim \|f\|_{L^q(\Omega)} + \|u\|_{W^{1,p}(\Omega)}$$

Moreover $\tilde{f}$ and $\tilde{g}$ have compact support, so does v and we have

$$\|v\|_{W^{1,p}(\mathbb{R}^n)} \precsim \|u\|_{W^{1,p}(\Omega)}.$$

All the constants depend on η.

Proof.

$$\Delta v = \underbrace{(\Delta\eta)u}_{L^p(\mathbb{R}^n)} + \underbrace{2\nabla\eta \cdot \nabla u}_{L^p(\mathbb{R}^n)} - \underbrace{(\partial_\alpha \eta)\, f}_{L^q(\mathbb{R}^n)} + \partial_\alpha(\underbrace{\eta f}_{L^q(\mathbb{R}^n)})$$

□

Moreover we use the following global result:

Proposition 7.13. *Let $p, q, r \in (1, \infty)$. Assume that $v \in W^{1,p}(\mathbb{R}^n)$,*

$$\Delta v = g + \partial_\alpha f \quad \text{in } \mathbb{R}^n$$

with $f \in L^q(\mathbb{R}^n)$, $g \in L^r(\mathbb{R}^n)$ all with compact support.

Then for $1 < \sigma \le q$ and if $r < n$ additionally $\sigma < \frac{nr}{n-r}$

$$\|v\|_{W^{1,\sigma}(\mathbb{R}^n)} \precsim \|f\|_{L^q(\mathbb{R}^n)} + \|g\|_{L^r(\mathbb{R}^n)}.$$

The constant depends on the compact support.

Proof. By the compact support of v we only need to estimate ∇v (the rest follows from Poincaré).

As since the equation under consideration is constant-coefficient linear equation, we may assume that v, f, $g \in C_c^\infty(\mathbb{R}^n)$[19]

[19] we used this argument already in Theorem 2.40: Denoting $v_\varepsilon := \eta_\varepsilon * v$, $f_\varepsilon = f * \eta_\varepsilon$, $g_\varepsilon = g * \eta_\varepsilon$ the usual convolution (i.e. $\eta \in C_c^\infty(B(0,1))$ nonnegative, $\int \eta = 1$, $\eta_\varepsilon := \varepsilon^{-n}\eta(\cdot/\varepsilon)$ we readily see from the compact support that $v_\varepsilon, f_\varepsilon, g_\varepsilon \in C_c^\infty(\mathbb{R}^n)$. Using η_ε as a test-function () we conclude that

$$\Delta v_\varepsilon = g_\varepsilon + \partial_\alpha f_\varepsilon \quad \textit{pointwise} \text{ in } \mathbb{R}^n.$$

Now if we were able to show

$$\|v_\varepsilon\|_{W^{1,\sigma}(\mathbb{R}^n)} \precsim \|f_\varepsilon\|_{L^q(\mathbb{R}^n)} + \|g_\varepsilon\|_{L^r(\mathbb{R}^n)}$$

By the representation theorem (since all quantities have compact support the integral converge, we assume $n > 3$ for simplicity)

$$v(x) = c\int_{\mathbb{R}^n} |x-y|^{2-n}(g+\partial_\alpha f)(y)\,dy$$

and thus (one needs to show this converges, but it does) using integration by parts

$$\begin{aligned}
\partial_\beta v(x) =&\tilde{c}\int_{\mathbb{R}^n} \frac{(x-y)_\beta}{|x-y|}|x-y|^{1-n}(g+\partial_\alpha f)(y)\,dy\\
=&\tilde{c}_1\int_{\mathbb{R}^n} \frac{(x-y)_\beta}{|x-y|}|x-y|^{1-n}g(y)\\
&+\tilde{c}_2\int_{\mathbb{R}^n} \frac{\delta_{\alpha\beta}|x-y|^2-n(x-y)_\alpha(x-y)_\beta}{|x-y|^{2+n}}f(y)\,dy\\
=:&T_1g(x)+T_2f(x).
\end{aligned}$$

Observe that $\frac{(z)_\beta}{|z|}|z|^{1-n} \in L^s_{loc}(\mathbb{R}^n)$ for any $s < \frac{n}{n-1}$. So from Young's inequality (using the compact support and the smoothness of the kernel $\frac{(z)_\beta}{|z|}|z|^{1-n}$ away from 0) for any admissible σ , choosing s so that

$$1+\frac{1}{\sigma} = \frac{1}{s}+\frac{1}{r}$$

(this is essentially Sobolev embedding), we have

$$\|T_1 g\|_{L^\sigma(\operatorname{supp} u)} \precsim \|g\|_{L^r(\mathbb{R}^n)}.$$

For T_2 we use the Calderón-Zygmund theorem, Theorem 7.3:

Observe that

$$p_{\alpha\beta}(z) := \delta_{\alpha\beta}|z|^2 - n(z)_\alpha(z)_\beta = \begin{cases}\sum_{\gamma\neq\alpha}(z_\gamma)^2+(1-n)(z_\alpha)^2 & \text{if } \alpha=\beta\\ -n\,z_\beta z_\alpha\end{cases}$$

always (recall that $n \geq 2$!) satisfies the conditions of Proposition 7.5; or, alternatively,

$$\Omega_{\alpha\beta}(z) := \frac{p_{\alpha\beta}(z)}{|z|^2} = \begin{cases}\sum_{\gamma\neq\alpha}\left(\frac{z_\gamma}{|z|}\right)^2+(1-n)(\frac{z_\alpha}{|z|})^2 & \text{if } \alpha=\beta\\ -n\frac{z_\beta}{|z|}\,\frac{z_\alpha}{|z|}\end{cases}$$

(with a constant in $\precsim$ not depending on ε!) then we have from the Young's convolution estimate (using that as usual we choose $\|\eta\|_{L^1} = 1$)

$$\sup_{\varepsilon>0}\|v_\varepsilon\|_{W^{1,\sigma}(\mathbb{R}^n)} \precsim \sup_{\varepsilon>0}\left(\|f_\varepsilon\|_{L^q(\mathbb{R}^n)}+\|g_\varepsilon\|_{L^r(\mathbb{R}^n)}\right) \leq \|f\|_{L^q(\mathbb{R}^n)}+\|g\|_{L^r(\mathbb{R}^n)}.$$

By reflexivity we conclude that there exists a sequence $\varepsilon_i \to 0$ such that v_ε weakly converges to v in $W^{1,\sigma}(\mathbb{R}^n)$. By lower semicontinuity of the norm we conclude

$$\|v\|_{W^{1,\sigma}(\mathbb{R}^n)} \leq \liminf_{\varepsilon\to 0}\|v_\varepsilon\|_{W^{1,\sigma}(\mathbb{R}^n)} \precsim \sup_{\varepsilon>0}\left(\|f_\varepsilon\|_{L^q(\mathbb{R}^n)}+\|g_\varepsilon\|_{L^r(\mathbb{R}^n)}\right) \leq \|f\|_{L^q(\mathbb{R}^n)}+\|g\|_{L^r(\mathbb{R}^n)}.$$

satisfies the assumptions of Theorem 7.3. So,

$$\|T_2 f\|_{L^q(\operatorname{supp} u)} \lesssim \|f\|_{L^q(\mathbb{R}^n)}.$$

Combining this we conclude

$$\|\nabla v\|_{L^\sigma(\mathbb{R}^n)} \lesssim_{\text{supp}} \|f\|_{L^q(\mathbb{R}^n)} + \|g\|_{L^r(\mathbb{R}^n)}$$

□

Proof of Theorem 7.10. Let $\Omega \supset\supset \Omega_1 \supset\supset \Omega_2 \supset\supset \ldots$ and take $\eta_i \in C_c^\infty(\Omega_i)$ with $\eta \equiv 1$ in Ω_{i+1}.

Following Lemma 7.12 and Proposition 7.13 we obtain that

$$\eta_1 u \in W^{1,\sigma_1}(\Omega_1)$$

where we take $\sigma_1 \leq p$ and $\sigma_1 \leq \frac{2n}{n-2}$ (if $n \geq 3$). If we can take $\sigma_1 = p$ we are done, otherwise we observe that $\sigma_1 = \frac{2n}{n-2} > 2$. We then repeat the argument for $\eta_2\eta_1 u$: from Lemma 7.12 and Proposition 7.13 we then obtain

$$\eta_2\eta_1 u \in W^{1,\sigma_2}(\mathbb{R}^n)$$

for $\sigma_2 \leq p$ and $\sigma_2 \leq \frac{\sigma_1 n}{n-\sigma_1}$ (if $\sigma_1 < n$). Again, either we can choose $\sigma_2 = p$ or $\sigma_2 = \frac{\sigma_1 n}{n-\sigma_1}$.

In this way we obtain a sequence

$$v_k := \eta_k \eta_{k-1} \ldots \eta_1 u \in W^{1,\sigma_k}(\mathbb{R}^n)$$

where

$$\sigma_k = \begin{cases} p & \text{if } \sigma_{k-1} < n \text{ or } p \leq \frac{\sigma_k n}{n-\sigma_k} \\ \frac{\sigma_{k-1} n}{n-\sigma_{k-1}} & \text{else.} \end{cases}$$

This sequence terminates after finitely many steps. Indeed, let

$$\tilde{\sigma}_k := \begin{cases} \frac{\tilde{\sigma}_{k-1} n}{n-\tilde{\sigma}_{k-1}} & \text{if } \sigma_{k-1} < n \\ \infty & \text{otherwise.} \end{cases}$$

The sequence $\tilde{\sigma}_k$ is increasing, $\tilde{\sigma}_k \geq \tilde{\sigma}_{k-1}$ and strictly increasing unless $\sigma_k = n$. The only possibility that $\tilde{\sigma}_k$ is not ∞ after finitely many steps k, is that $\sigma_k < n$ for all k – then we have a monotone, bounded sequence which has a limit $\tilde{\sigma} \leq n$, which has to satisfy

$$\tilde{\sigma} = \frac{\tilde{\sigma} n}{n - \tilde{\sigma}}$$

There is no positive, finite solution to this equation. Contradiction. So $\tilde{\sigma}_k$ is infinite for $k \geq K$ for some K, which means that $\sigma_k = p$ for $k \geq K$.

That is, we have shown that $v_K \in W^{1,p}(\mathbb{R}^n)$, and since $v_K \equiv u$ in Ω_K we have $u \in W^{1,p}(\mathbb{R}^n)$ in Ω_K.

The number K is independent of the equation, it just depends on the dimension and p, so if we choose Ω_K well, then we get the claim. □

7.4. $W^{1,p}$**-theory for a constant coefficient linear elliptic equation.** As we have seen above, we essentially, by cutoff arguments can reduce a equation to an equation in $\mathbb{R}^n$. So let us now consider

$$-\mathrm{div}\,(A\nabla u) = \partial_\sigma g \quad \text{in } \mathbb{R}^n$$

where $A \in \mathbb{R}^{n\times n}$ is a symmetric matrix with all eigenvalues strictly positive.

Here is a trick (this is a bit different from the L^2-theory, for L^2-theory we got this for free from the variational argument!): we can *rewrite* this equation in to the Laplace equation.

Indeed, linear algebra tells us that $A = P^T DP$ where $D = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$ is the diagonal matrix of eigenvalues (all positive) and $P \in SO(n)$.

Denote by

$$\sqrt{D} := \mathrm{diag}(\sqrt{\lambda_1}, \sqrt{\lambda_2}, \sqrt{\lambda_3}, \ldots, \sqrt{\lambda_n}).$$

Set now

$$v(x) := u(\sqrt{D}Px)$$

Then we have (we use Einstein's summation here!)

$$\partial_\alpha v(x) = \left(\sqrt{D}P\right)_{\alpha\gamma} (\partial_\gamma u)\,(\sqrt{D}Px)$$

and

$$\partial_\beta\partial_\alpha v(x) = \left(\sqrt{D}P\right)_{\beta\delta}\left(\sqrt{D}P\right)_{\alpha\gamma} (\partial_{\gamma\delta} u)\,(\sqrt{D}Px)$$

So

$$\begin{aligned}
\Delta v(x) =& \left(\sqrt{D}P\right)_{\alpha\delta}\left(\sqrt{D}P\right)_{\alpha\gamma} (\partial_{\gamma\delta} u)\,(\sqrt{D}Px)\\
=& \left(\left(\sqrt{D}P\right)^t\sqrt{D}P\right)_{\delta\gamma} (\partial_{\gamma\delta} u)\,(\sqrt{D}Px)\\
=& \left(P^t\sqrt{D}^t\sqrt{D}P\right)_{\delta\gamma} (\partial_{\gamma\delta} u)\,(\sqrt{D}Px)\\
=& \left(P^tDP\right)_{\delta\gamma} (\partial_{\gamma\delta} u)\,(\sqrt{D}Px)\\
=& (A)_{\delta\gamma}\ (\partial_{\gamma\delta} u)\,(\sqrt{D}Px)\\
=& \mathrm{div}\,(A\nabla u)(\sqrt{D}Px)\\
=& (\partial_\sigma g)\,(\sqrt{D}Px)\\
=& (\left(\sqrt{D}P\right)^{-1})_{\mu\sigma}\left(\partial_\mu g(\sqrt{D}Px)\right)
\end{aligned}$$

Since

$$x \mapsto \sqrt{D}Px$$

is a smooth diffeomorphism on $\mathbb{R}^n$, we see that if $g \in L^p$ then we can apply L^p-theory to Δv, and find from Theorem 7.10,

Theorem 7.14. *Let $A \in \mathbb{R}^{n\times n}$ be a symmetric matrix with all eigenvalues strictly positive, and assume that $u \in W^{1,2}_{loc}(\Omega)$ solves*

$$-\mathrm{div}\,(A\nabla u) = \partial_\sigma g \quad \textit{in } \Omega.$$

Then if $g \in L^p_{loc}(\Omega)$ we have that $\nabla u \in L^p_{loc}(\Omega)$, with the estimate for any $\Omega_1 \subset\subset \Omega_2 \subset\subset \Omega$,

$$\|\nabla u\|_{L^p(\Omega_1)} \le C(\Omega_1, \Omega, p)\,\left(\|f\|_{L^p(\Omega)} + \|u\|_{L^2(\Omega)}\right).$$

7.5. $W^{1,p}$-theory for a Hölder continuous coefficient linear elliptic equation.

7.5.1. *A priori estimates.* Let $A \in C^0(\Omega, \mathbb{R}^{n\times n})$ be a *continuous*, *symmetric* matrix valued map such that

$$\lambda := \inf_{x\in\Omega} \inf_{\xi\in\mathbb{R}^n} \langle A(x)\xi, \xi\rangle > 0.$$

Assume we have $u \in W^{1,2}(\Omega)$ solving the equation

$$-\mathrm{div}\,(A\nabla u) = \partial_\sigma g \quad \text{in } \Omega.$$

If $g \in L^p(\Omega)$ can we use the *constant coefficient* theory, Theorem 7.14, to conclude that $\nabla u \in L^p_{loc}(\Omega)$?

One approach is the so-called *freezing*-technique.

Fix $x_0 \in \Omega$, then we would like to compare the equation above to the equation for $-\mathrm{div}\,(A(x_0)\nabla u)$. Indeed, we could write

$$-\mathrm{div}\,(A(x_0)\nabla u) = -\mathrm{div}\,((A(x_0) - A)\nabla u) + \partial_\sigma g \quad \text{in } \Omega.$$

Now observe that

$$\|(A(x_0) - A)\nabla u\|_{L^p(\Omega)} \le \|A(x_0) - A\|_{L^\infty} \|\nabla u\|_{L^p(\Omega)}.$$

So if

- Ω is small enough, so that $\|A(x_0) - A\|_{L^\infty} < \varepsilon$ (continuity! such a small Ω exists!)
- and we *already know* $\nabla u \in L^p(\Omega)$

then we could envision that we obtain an estimate of the form

$$\|\nabla u\|_{L^p} \le C\,\underbrace{\|A(x_0) - A\|_{L^\infty}}_{\le\varepsilon} \|\nabla u\|_{L^p(\Omega)} + C\|g\|_{L^p}$$

So if $\varepsilon \ll 1$ we could absorb and have

$$\|\nabla u\|_{L^p} \le \tilde{C}\|g\|_{L^p}.$$

There are two problems with this approach: First, the L^p-norms on the left- and right-hand side won't be on the same set (so no absorbtion). Second, we want to prove that $\nabla u \in L^p$ – not assume it (i.e. the above argument looks like an *a priori* estimate)!

Let us assume that $A \in C^\alpha$ – this makes life easier.

Fix some $\Omega_1 \subset\subset \Omega$.

Consider $\eta \in C_c^\infty(\Omega_1)$ and set $v := \eta u$, then we have

$$\operatorname{div}(A\nabla v) = G(u,\eta) \quad \text{in } \mathbb{R}^n$$

where

$$G(u,\eta) := \eta \partial_\sigma g + \operatorname{div}(A(\nabla \eta)\, u) + \nabla \eta : A\nabla u.$$

We observe that for $\varphi \in C_c^\infty(\Omega)$ we have (using Sobolev embedding – this makes sense if $p' < n$ which we assume for illustration, otherwise its easier). Slight adaptation in the argument can be done otherwise),

$$\begin{aligned}\left|\int G(u,\eta)\varphi\right| \lesssim_{\Omega,\eta,A} &\|g\|_{L^p}\|\nabla\varphi\|_{L^{p'}(\mathbb{R}^n)} + \|u\|_{L^p}\|\nabla\varphi\|_{L^{p'}} + \|\nabla u\|_{L^{\frac{np}{p+n}}}\|\varphi\|_{L^{\frac{np'}{n-p'}}}\\ \lesssim &\left(\|g\|_{L^p} + \|\nabla u\|_{L^{\frac{np}{n+p}}}\right)\|\nabla\varphi\|_{L^{p'}}\end{aligned}$$

Now we consider $v_\delta := v * \chi_\delta$ for the usual bump function $\chi \in C_c^\infty(B(0,1),[0,1])$ and $\chi_\delta(x) := \delta^{-n}\chi(\cdot/\delta)$. Since $\operatorname{supp} v \subset\subset \mathbb{R}^n$, for small enough δ we have $v_\delta \in C_c^\infty(B(0,1))$. Moreover, setting $\Omega_{-\delta} := \{x \in \Omega, \operatorname{dist}(x,\partial\Omega) > \delta\}$ we have

$$\operatorname{div}(A\nabla v_\delta) = H(u,\eta,\delta) \quad \text{in } \Omega_{-\delta}$$

where

$$H(u,\eta,\delta) := G(u,\eta) * \chi_\delta + \operatorname{div}\left(A\left(\nabla v * \chi_\delta\right) - (A\nabla v) * \chi_\delta\right)$$

We then have for any $\varphi \in C_c^\infty(\Omega_{-\delta})$,

$$\begin{aligned}\left|\int (G(u,\eta) * \chi_\delta)\ \varphi\right| = &\left|\int G(u,\eta)\ (\varphi * \chi_\delta)\right|\\ \lesssim &\left(\|g\|_{L^p} + \|\nabla u\|_{L^{\frac{np}{n+p}}}\right)\|\nabla(\chi_\delta * \varphi)\|_{L^{p'}}\\ \lesssim &\left(\|g\|_{L^p} + \|\nabla u\|_{L^{\frac{np}{n+p}}(\Omega)}\right)\|\nabla\varphi\|_{L^{p'}(\mathbb{R}^n)}.\end{aligned}$$

Moreover,

$$\begin{aligned}|A(\nabla v * \chi_\delta) - (A\nabla v) * \chi_\delta|(x) \le &\int_{B(0,\delta)} |A(x) - A(x-z)|\,|\nabla v(x-z)|\,\chi_\delta(z)\,dz\\ \le &[A]_{C^\alpha}\int_{B(0,\delta)} |z|^\alpha|\nabla v(x-z)|\,\chi_\delta(z)\,dz\\ \le &\delta^\alpha[A]_{C^\alpha}\int_{B(0,\delta)} |\nabla v(x-z)|\,\chi_\delta(z)\,dz\\ = &\delta^\alpha[A]_{C^\alpha}|\nabla v| * \chi_\delta(x)\end{aligned}$$

Thus, with Young's inequality for convolutions

$$\begin{aligned}\|A(\nabla v * \chi_\delta) - (A\nabla v) * \chi_\delta\|_{L^p(\mathbb{R}^n)} &\precsim_A \delta^\alpha \||\nabla v| * \chi_\delta\|_{L^p(\mathbb{R}^n)} \\ &\precsim \delta^\alpha \underbrace{\|\chi_\delta\|_{L^{\frac{n}{n-\alpha}}(\mathbb{R}^n)}}_{\precsim \delta^\alpha} \|\nabla v\|_{L^{\frac{np}{n+\alpha p}}(\mathbb{R}^n)} \\ &\precsim \underbrace{\delta^\alpha \|\chi_\delta\|_{L^{\frac{n}{n-\alpha}}(\mathbb{R}^n)}}_{\precsim 1} \|u\|_{W^{1,\frac{np}{n+\alpha p}}(\Omega)}\end{aligned}$$

The important thing to note here is that while we have L^p-control on the "left-hand side", we only need $L^{\frac{np}{n+\alpha p}}$-control of the right-hand side. So this should be seen as some sort of *reverse Sobolev inequality*!

So, we find (using Hölder's inequality to estimate $\|u\|_{L^{\frac{np}{n+p}}(\Omega)} \precsim_\Omega \|u\|_{W^{1,\frac{np}{n+\alpha p}}(\Omega)}$)

$$\left|\int H(u,\eta,\delta) * \chi_\delta \varphi\right| \precsim \left(\|g\|_{L^p(\Omega)} + \|u\|_{W^{1,\frac{np}{n+\alpha p}}(\Omega)}\right) \|\nabla\varphi\|_{L^{p'}(\mathbb{R}^n)}.$$

Now we (finally!) can use the freezing argument. Fix any $x_0 \in \Omega$, then we have

$$\operatorname{div}(A(x_0)\nabla v_\delta) = H(u,\eta,\delta) + \operatorname{div}((A - A(x_0))\nabla v_\delta) \quad \text{in } \Omega_{-\delta}.$$

Since $\operatorname{supp} v \subset \Omega_1$, we can find $\delta_0 > 0$ small enough so that $\operatorname{supp} v_\delta \subset \Omega_{-\delta}$ for all $\delta < \delta_0$.

Let $R > 0$ and $B(x_0, 2R) \subset \Omega_{-\delta}$. Then by Theorem 7.14 (there: $\Omega_1 = B(x_0,R)$ and $\Omega = B(x_0,2R)$)

$$\|\nabla v_\delta\|_{L^p(B(x_0,R))} \le \left(\|g\|_{L^p(\Omega)} + \|u\|_{W^{1,\frac{np}{n+\alpha p}}(\Omega)}\right) + \|(A - A(x_0))\|_{L^\infty(B(x_0,2R))} \|\nabla v_\delta\|_{L^p(B(x_0,2R))}$$

Observe that $\|\nabla v_\delta\|_{L^p(B(x_0,2R))} < \infty$ – for this we needed all the argument above!

Fix $\varepsilon > 0$. Pick R so small that two conditions are satisfied

- $\operatorname{dist}(\operatorname{supp} v_\delta, \partial\Omega_{-\delta}) > 2R$ for all $\delta < \delta_0$ (this is doable since $\operatorname{supp} v_\delta \subset B(\operatorname{supp}\eta, \delta)$), and
- $\sup_{x_0\in\Omega} \|A - A(x_0)\|_{L^\infty(B(x_0,R))} < \varepsilon$ (this is possible from continuity of A)

Then we have the following estimate for any $x_0 \in \operatorname{supp} v_\delta$

$$\|\nabla v_\delta\|_{L^p(B(x_0,R))} \le \left(\|g\|_{L^p(\Omega)} + \|u\|_{W^{1,\frac{np}{n+\alpha p}}(\Omega)}\right) + \varepsilon \|\nabla v_\delta\|_{L^p(B(x_0,2R))}$$

Now we can cover $\operatorname{supp} v_\delta$ by finitely many balls (say L many) $(B(x_i,R))_{i=1}^N$ such that $B(x_i,R)$ overlaps with only finitely many other balls $B(x_j,R)$ (and the number of overlaps is a fixed number N depending on the dimension and $\operatorname{supp}\eta$– but not on R or δ. Then summing up we have

$$\|\nabla v_\delta\|_{L^p(\bigcup_i B(x_i,R))} \le L\left(\|g\|_{L^p(\Omega)} + \|u\|_{W^{1,\frac{np}{n+\alpha p}}(\Omega)}\right) + \varepsilon C(N) \|\nabla v_\delta\|_{L^p(\bigcup_i B(x_i,2R))}$$

Thus

$$\|\nabla v_\delta\|_{L^p(\Omega)} = \|\nabla v_\delta\|_{L^p(\operatorname{supp} v_\delta)} \leq N \left(\|g\|_{L^p(\Omega)} + \|u\|_{W^{1,\frac{np}{n+\alpha p}}(\Omega)} \right) + \varepsilon C(N) \|\nabla v_\delta\|_{L^p(\Omega)}$$

Finally, we can absorb, if we choose ε small enough (N does not depend on ε), and we have shown

$$\|\nabla v_\delta\|_{L^p(\Omega)} = \|\nabla v_\delta\|_{L^p(\operatorname{supp} v_\delta)} \leq N \left(\|g\|_{L^p(\Omega)} + \|u\|_{W^{1,\frac{np}{n+\alpha p}}(\Omega)} \right).$$

Of course this only holds if p is chosen, so that $u \in W^{1,\frac{np}{n+\alpha p}}(\Omega)$. But now we can *bootstrap*:

First we only know that $u \in W^{1,2}(\Omega)$. So choose p such that $\frac{np}{n+\alpha p} = 2$. That is $p_1 = \frac{2n}{n-\alpha} > 2$. We conclude that $\eta u \in W^{1,p_1}$, i.e. $u \in W^{1,p_1}(\Omega_1)$.

Now we run this argument on a smaller set $\Omega_2 \subset \Omega_1$. We already know $u \in W^{1,p_1}(\Omega_1)$ so we obtain $u \in W^{1,p_2}(\Omega_2)$, where $p_2 := \frac{np_1}{n-\alpha} > p_1$. We can iterate this argument *as long as $g \in L^{p_i}$*..

Observe that

$$p_{i+1} = p_i \frac{n}{n-\alpha}.$$

So $p_i \xrightarrow{i\to\infty} \infty$ – meaning that if $g \in L^r$ and $r < \infty$ then there exists $p_i > r$.

We have proven the following

Theorem 7.15. *Let $A \in C^\alpha(\mathbb{R}^n, \mathbb{R}^{n\times n})$ be a Hölder continuous symmetric matrix function with*

$$\lambda := \inf_{x\in\mathbb{R}^n} \inf_{\xi\in\mathbb{R}^n} \langle A(x)\xi, \xi\rangle > 0.$$

Assume that $u \in W^{1,2}_{loc}(\Omega)$ solves

$$-\operatorname{div}(A\nabla u) = \partial_\sigma g \quad \text{in } \Omega.$$

Then if $g \in L^p_{loc}(\Omega)$ we have that $\nabla u \in L^p_{loc}(\Omega)$, with the estimate for any $\Omega_1 \subset\subset \Omega_2 \subset\subset \Omega$,

$$\|\nabla u\|_{L^p(\Omega_1)} \leq C(\Omega_1, \Omega, p, A) \left(\|f\|_{L^p(\Omega)} + \|u\|_{L^2(\Omega)} \right).$$

Higher order estimates are then obtained via differentiating the equation, cf. Proposition 6.9.

7.6. **$W^{2,p}$-Calderón-Zygmund theory.** The same arguments as above also imply $W^{2,p}$-estimates, i.e. the L^p-version of Theorem 6.4: We illustrate this for the Laplace case only (I hope you'll appreciate after seeing the mess in the previous subsection):

Theorem 7.16 (L^2-regularity)**.** *Let $p \in [2,\infty)$, $f \in L^p(\Omega)$ and assume $u \in W^{1,2}(\Omega)$ solves*

$$\begin{cases} -\Delta u = f & \text{in } \Omega \end{cases}$$

in the distributional sense.

(1) If we assume nothing more on the boundary: Then $u \in W^{2,p}_{loc}(\Omega)$*, and we have for any* $\Omega' \subset\subset \Omega$

$$\|D^2 u\|_{L^p(\Omega')} \leq C(p, \Omega', \Omega) \left(\|u\|_{W^{1,2}(\Omega)} + \|f\|_{L^p(\Omega)} \right).$$

(2) if $u \in W^{1,2}_0(\Omega)$*, i.e.*

$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

and Ω *is a bounded set with smooth boundary* $\partial\Omega \in C^\infty$ *then*

$$\|D^2 u\|_{L^p(\Omega)} \lesssim_{\Omega,p} \|f\|_{L^p(\Omega)} + \|u\|_{W1,2(\Omega)}.$$

Let us remark that the same theorem holds also for $p \in (1,2)$ with slight modifications.

Sketch of the proof. From $W^{2,2}$-theory, we already know that $u \in W^{2,2}_{loc}$, Theorem 6.4. Using a cutoff function $\eta_1 \in C_c^\infty(\Omega)$, $\eta_1 \equiv 1$ in $\Omega_1 \subset\subset \Omega$ we have (where $2^* = \frac{2n}{n-2}$ is the Sobolev exponent)

$$-\Delta(\eta_1 u) = \eta_1 f - (\Delta \eta_1) u - 2 \underbrace{\nabla u}_{\in W^{1,2}_{loc} \subset L^{2^*}} \nabla \eta_1 \quad \text{in } \mathbb{R}^n.$$

That is

$$-\Delta(\eta_1 u) = g \in L^q$$

where $q = \min\{p, 2^*\}$.

We apply the Riesz transforms $\mathcal{R}_\alpha$ and $\mathcal{R}_\beta$ (they act in $\mathbb{R}^n$, hence we needed the cutoff) to find

$$\partial_{\alpha\beta}(\eta_1 u) = \mathcal{R}_\alpha \mathcal{R}_\beta g$$

and thus

$$\|\partial_{\alpha\beta} u\|_{L^q(\Omega_1)}\| \leq \|\partial_{\alpha\beta}(\eta_1 u)\|_{L^q(\mathbb{R}^n)} \leq \|g\|_{L^q(\mathbb{R}^n)} \lesssim \|u\|_{W^{1,2}(\Omega)} + \|f\|_{L^p}.$$

Thus $u \in W^{2,q}(\Omega_1)$, and we can repeat this argument with some cutoff function $\eta_2 \in C_c^\infty(\Omega_1)$, $\eta_2 \equiv 1$ in $\Omega_2 \subset\subset \Omega_1$. Since we already know $u \in W^{2,q}(\Omega_1)$ we find

$$-\Delta(\eta_1 u) = \eta_1 f - (\Delta \eta_1) u - 2 \underbrace{\nabla u}_{\in W^{1,q}_{loc} \subset L^{q^*}} \nabla \eta_1 \quad \text{in } \mathbb{R}^n.$$

and thus we find by the same argument as above that $u \in W^{2,q_2}(\Omega_2)$ where $q_2 = \min\{p, q^*\}$. Iterating this arguments on smaller and smaller sets, we see that we get an increasing sequence q_i that eventually is constantly p, and thus we have obtained $W^{2,p}$ in the smallest set. □

Let us remark that Theorem 7.16 is *false* for $p = 1$ and $p = \infty$.

Example 7.17. Let $n = 2$, set

$$u(x) := \log \log \frac{2}{|x|}$$

Recall that $u \in W^{1,2}(B(0,1))$, Exercise 5.7, and we have

$$-\Delta u = |\nabla u|^2 \quad \text{in } B(0,1).$$

That is for $g = |\nabla u|^2 \in L^1(B(0,1))$ we have

$$-\Delta u = g \quad \text{in } B(0,1).$$

However $u \not\in W^{2,1}_{loc}(\Omega)$. Indeed, any map $u \in W^{2,1}_{loc}(\mathbb{R}^2)$ (more generally $W^{n,1}$-functions are continuous, [Adams and Fournier, 2003, Theorem 4.12, Part I, Case A]). But u is clearly not continuous at the origin, so thats impossible.

Exercise 7.18. *Prove that if $u \in W^{n,1}(B(0,1))$ then $u \in C^0(B(0,1))$. Read the proof of* [Adams and Fournier, 2003, Theorem 4.12, Part I, Case A]*, see p.89, §4.16, and write it in your own words.*

8. Schauder theory

Consider the linear elliptic operator in *non-divergence* form

$$Lu(x) := A_{\alpha\beta}(x)\partial_{\alpha\beta}u(x) + b_\alpha \partial_\alpha u(x) + c(x)u(x)$$

Schauder theory is the theory of Hölder regularity for solutions to the above equation. Recall that the Hölder space $C^\alpha(\Omega)$ is given by

$$[f]_{C^\alpha(\Omega)} := \sup_{x \neq y \in \Omega} \frac{|f(x) - f(y)|}{|x-y|^\alpha}.$$

This is a *seminorm* (any constant f satisfies $[f]_{C^\alpha} = 0$), the norm is defined as

$$\|f\|_{C^\alpha(\Omega)} := \|f\|_{L^\infty(\mathbb{R}^d)} + [f]_{C^\alpha(\Omega)}.$$

Clearly this makes sense only if $\alpha \in (0,1]$

Exercise 8.1. *Assume $f \in C^\alpha(B(0,1))$ for some $\alpha > 1$. Show that f is necessarily constant.*

We (should) often write $C^{0,\alpha}$ to distinguish between $C^{0,1}$ (Lipschitz) and C^1 (continuously differentiable).

We say $f \in C^{k,\alpha}(\Omega)$ if $f \in C^k(\Omega)$ (i.e. k times continuously differentiable) and $D^k f \in C^\alpha(\Omega)$.

In the following we have the standing assumptions:

- Coefficient regularity: $a, b, c \in C^\alpha(\mathbb{R}^n)$, for some $\alpha \in (0,1)$, i.e. there exists $\Lambda > 0$ such that
$$\|a\|_{C^\alpha(\Omega)} + \|b\|_{C^\alpha(\Omega)} + \|c\|_{C^\alpha(\Omega)} \leq \Lambda$$
- *ellipticity*: there exists $\lambda > 0$ such that
$$a_{\alpha\beta}(x)\xi^\alpha\xi^\beta \geq \lambda|\xi|^2 \quad \forall \xi \in \mathbb{R}^n, \quad x \in \overline{\Omega}.$$

We expect from solutions $Lu = f$ that u has two more derivatives than f – in the realm of Hölder spaces, this is called *Schauder theory*.

Theorem 8.2 (Interior Schauder estimate)**.** *Let $\alpha \in (0,1)$, $\Omega \subset \mathbb{R}^n$ an open set, and assume $u \in C^{2,\alpha}(\Omega)$, $f \in C^\alpha(\overline{\Omega})$ satisfy the equation*
$$Lu(x) = f(x) \quad \text{for all } x \in \Omega.$$
Here L is a linear elliptic operator in non-divergence form as above, with $A, b, c \in L^\infty(\Omega) \cap C^\alpha(\overline{\Omega})$, $A(x) \in \mathbb{R}^{n\times n}$ uniformly elliptic, i.e. there exists $\lambda > 0$ such that
$$A_{\alpha\beta}\xi_\alpha\xi_\beta \geq \lambda|\xi|^2 \quad \forall \xi \in \mathbb{R}^n.$$
Then for any $\Omega' \subset\subset \Omega$ we have
$$[D^2u]_{C^\alpha(\Omega')} \leq C\left(\|f\|_{L^\infty(\Omega)} + [f]_{C^\alpha(\Omega)} + \|u\|_{L^\infty(\Omega)}\right).$$
where the constant C depends only on $\alpha, n, \Omega', \Omega$ and A, b, c.

8.1. Basic facts on Hölder spaces.

Exercise 8.3. *Let $\Omega \subset \mathbb{R}^n$ a set and $\alpha \in (0,1]$. Show that*

(1) $[fg]_{C^\alpha(\Omega)} \precsim \|f\|_{L^\infty(\Omega)}\,[g]_{C^\alpha(\Omega)} + \|g\|_{L^\infty(\Omega)}\,[f]_{C^\alpha(\Omega)}$
(2) $[f]_{C^\alpha(\Omega)} \precsim \|f\|_{L^\infty(\Omega)} + \|Df\|_{L^\infty(\Omega)}$.

Exercise 8.4. *Let $\Omega \subset \mathbb{R}^n$ a set and $\alpha, \beta \in (0,1]$ with $\alpha < \beta$. Show that for any $\varepsilon > 0$ there exists $C = C(\varepsilon, \beta, \alpha, \Omega)$ such that*
$$[f]_{C^\alpha(\Omega)} \leq \varepsilon[f]_{C^\beta(\Omega)} + C\,\|f\|_{L^\infty(\Omega)}.$$
Hint: Look on the internet for Ehrling lemma, and use/show that C^β compactly embedds into C^α by Arzela-Ascoli. Observe you can first show
$$\|f\|_{L^\infty(\Omega)} + [f]_{C^\alpha(\Omega)} \leq \varepsilon\left(\|f\|_{L^\infty(\Omega)} + [f]_{C^\beta(\Omega)}\right) + C\,\|f\|_{L^\infty(\Omega)}.$$
to obtain the desired result.

Exercise 8.5. *Let $\Omega \subset \mathbb{R}^n$ a set. Show that for any $\varepsilon > 0$ there exists $C = C(\varepsilon, \beta, \alpha, \Omega)$ such that*
$$\|Df\|_{L^\infty(\Omega)} \leq \varepsilon\|D^2f\|_{L^\infty(\Omega)} + C\,\|f\|_{L^\infty(\Omega)}.$$
Hint: Similar to Exercise 8.4.

Exercise 8.6. *Let $\Omega \subset \mathbb{R}^n$ a bounded open set and $\alpha \in (0,1]$. Show that for any $\varepsilon > 0$ there exists $C = C(\varepsilon, \alpha, \Omega)$ such that*

$$\|D^2 f\|_{L^\infty(\Omega)} \leq \varepsilon [D^2 f]_{C^\alpha(\Omega)} + C\, \|f\|_{L^\infty(\Omega)}.$$

Hint: Similar to Exercise 8.4.

8.2. **Fundamental Schauder estimate.** As is typical in PDE, we first consider the simplest nontrivial case, namely $L = \Delta$ and in $\mathbb{R}^n$. While the Schauder estimate can be proven via harmonic analysis methods (using potential theory and the Newton potential), we follow an elegant approach due to Leon Simon, [Simon, 1997]. It is aptly called "Schauder by scaling" (as we shall see in the proof).

Theorem 8.7 (Fundamental Schauder estimate)**.** *Let $\alpha \in (0,1)$ and assume $u \in C^{2,\alpha}(\mathbb{R}^n)$. Then there exists a constant $C = C(\alpha, n)$ such that*

$$[D^2 u]_{C^\alpha(\mathbb{R}^n)} \leq C\, [\Delta u]_{C^\alpha(\mathbb{R}^n)}$$

We will need the following

Exercise 8.8 (Liouville)**.** *Assume that $u \in C^\infty(\mathbb{R}^n, \mathbb{R})$ satisfies $\Delta u = 0$ in $\mathbb{R}^n$ and assume there exists a constant $C > 0$ and $\varepsilon > 0$ such that*

$$\sup_{B(0,r)} |u| \leq C\, r^{3-\varepsilon} \quad \text{for all } r > 0.$$

Then u is a quadratic polynomial.

Hint: Use Cauchy estimates, Lemma 2.41, see also Corollary 2.42 to show that $D^2 u(x_0) = 0$ for any x_0.

Exercise 8.9 (Taylor for Hölder functions)**.** *Assume that $f \in C^{2,\alpha}(\mathbb{R}^n)$ and $x_0 \in \mathbb{R}^n$, then for a uniform constant $C > 0$,*

$$\left| f(x) - \left(f(x_0) - (x - x_0) \cdot Dv(x_0) - \frac{1}{2}(x - x_0)^t D^2 v(x_0)(x - x_0) \right) \right| \leq C [D^2 f]_{C^\alpha} |x - x_0|^{2+\alpha}.$$

Hint: Integral formula/fundamental theorem of Taylor.

Exercise 8.10. *Assume that $f_k \in C^{2,\alpha}(\mathbb{R}^n, \mathbb{R})$ satisfies the following properties*

$$f_k(0) = 0, \quad Df_k(0) = 0, \quad D^2 f_k(0) = 0,$$

$$[D^2 f_k]_{C^{2,\alpha}(\mathbb{R}^d)} \leq 1.$$

Then, there exists a subsequence (not relabeled) and $f \in C^{2,\alpha}_{\text{loc}}(\mathbb{R}^n)$ such that f_k, Df_k and $D^2 f_k$, respectively, converge locally uniformly to f, Df, $D^2 f$, and we have

$$[D^2 f]_{C^{2,\alpha}(\mathbb{R}^d)} \leq 1$$

Proof of Theorem 8.7. Assume the claim is false, then there exists a sequence $(u_k)_{k=1}^\infty \subset C^{2,\alpha}(\mathbb{R}^n)$ such that

$$[D^2u_k]_{C^\alpha(\mathbb{R}^n)} > k\,[\Delta u_k]_{C^\alpha(\mathbb{R}^n)}$$

By assumption $u_k \in C^{2,\alpha}(\mathbb{R}^n)$, so we know (this is why we are only getting an a-priori estimate!): $[D^2u_k]_{C^\alpha(\mathbb{R}^n)} < \infty$ for each k. Then $\tilde{u}_k := u_k/[D^2u_k]_{C^\alpha(\mathbb{R}^n)} \in C^{2,\alpha}(\mathbb{R}^n)$. So we can continue to work with $\tilde{u}_k$ instead of u_k, or, equialently, w.l.o.g. assume that

$$[D^2u_k]_{C^\alpha(\mathbb{R}^n)} = 1, \quad [\Delta u_k]_{C^\alpha(\mathbb{R}^n)} < \frac{1}{k}.$$

What we'd want to do (in any blowup proof) is the following, pass somehow to the limit u, hope that $[D^2u]_{C^\alpha} = 1$ and $\Delta u = 0$, and show that this is a contradiction. But this is not so easy: we don't know that we actually can pass to the limit, under the bext expected weak convergence we cannot really hope that $[D^2u]_{C^\alpha} = 1$, but only $[D^2u]_{C^\alpha} \leq 1$, and $\Delta u = 0$ does not imply $u = 0$ or similar – that is it is not so clear how we would find that contradiction. The beauty of this method here is that those issues are relatively easily overcome:

From the condition $[D^2u_k]_{C^\alpha(\mathbb{R}^n)} = 1$ we need to find a condition that survives a weak $C^{2,\alpha}$-convergence: there must be $x_k \neq y_k \in \mathbb{R}^n$ such that

$$\frac{|D^2u_k(x_k) - D^2u_k(y_k)|}{|x_k - y_k|^\alpha} \geq \frac{1}{2}.$$

This is good, since by locally uniform convergence of D^2u_k this would survive the limit – but x_k and y_k could go to any point in $\mathbb{R}^n$, in particular to some point at ∞, so we rescale: Set $h_k := |x_k - y_k|$, and take a rotation $P_k \in O(n)$ such that $y_k - x_k = h_k P_k e_1$. If we set

$$v_k(z) := h^{-2-\alpha} u_k(x_k + h_k P_k z).$$

then (observe that $|P_k \vec{p}| = |\vec{p}|$)

$$|D^2v_k(0) - D^2v_k(e_1)| = \frac{|P_k^t\,(D^2u_k(x_k) - D^2u_k(y_k))\,P_k|}{|x_k - y_k|^\alpha} \geq \frac{1}{2}. \tag{8.1}$$

More generally, we observe the following

$$\begin{aligned}\frac{D^2v_k(z) - D^2v_k(y)}{|x-z|^\alpha} =& h_k^{-2-\alpha}\frac{h_k^2(P_k^tD^2u_kP_k)(x_k + h_kP_kz) - h_k^2(P_k^tD^2u_kP_k)(x_k + h_kP_ky)}{|x-z|^\alpha}\\ =& P_k^t\frac{(D^2u_k)(x_k + h_kP_kz) - (D^2u_k)(x_k + h_kP_ky)}{|x_k + h_kP_kz - (x_k + h_kP_ky)|^\alpha}P_k.\end{aligned}$$

so that

$$[D^2v_k]_{C^\alpha(\mathbb{R}^n)} = [D^2u_k]_{C^\alpha(\mathbb{R}^n)} = 1, \tag{8.2}$$

and by taking the trace in the above inequality

$$[\Delta v_k]_{C^\alpha(\mathbb{R}^n)} = [\Delta u_k]_{C^\alpha(\mathbb{R}^n)} < \frac{1}{k}, \tag{8.3}$$

Otherwise replacing v_k by $\tilde{v}_k(x) := v_k(x) - v(0) - \langle x, Dv(0)\rangle - \frac{1}{2}x^t D^2 v(0) x$ (which does not change any of the quantities previously considered because they are *differences* of second derivatives), we may additionally assume w.l.o.g.

$$v_k(0) = 0, \quad Dv_k(0) = 0, \quad D^2 v_k(0) = 0. \tag{8.4}$$

By Arzela-Ascoli, Exercise 8.10, up to passing to a subsequence, we find some $v \in C^{2,\alpha}_{loc}(\mathbb{R}^n)$ such that v_k, Dv_k and $D^2 v_k$ *locally* uniformly converge to v, Dv, $D^2 v$, and thus from (8.1), (8.3), (8.4) we have

$$|D^2 v(0) - D^2 v(e_1)| \geq \frac{1}{2}, \tag{8.5}$$

$$v(0) = 0, \quad Dv(0) = 0, \quad D^2 v(0) = 0. \tag{8.6}$$

$$[D^2 v]_{C^\alpha} \leq 1.$$

and (again from locally uniform convergence) we find

$$[\Delta v]_{C^\alpha} \leq \liminf_{k\to\infty} [\Delta v_k]_{C^\alpha} = 0.$$

That is we have

$$\Delta v(x) \equiv const,$$

but sine $D^2 v(0) = 0$ and $\Delta v(0) = \mathrm{tr} D^2 v(0)$ we have

$$\Delta v \equiv 0.$$

From Weyl's lemma, Theorem 2.40, we conclude $v \in C^\infty(\mathbb{R}^n)$. Since $[D^2 v]_{C^{2,\alpha}(\mathbb{R}^n)} \leq 1$ we also have a growth estimate on v using Taylor's theorem, Exercise 8.9,

$$|v(x)| = \left| v(x) - \left(\underbrace{v(0) - x \cdot Dv(0) - \frac{1}{2} x^t D^2 v(0) x}_{=0} \right) \right| \precsim \underbrace{[D^2 v]_{C^\alpha(\mathbb{R}^n)}}_{\leq 1} |x|^{-2-\alpha}.$$

Here we use that $\alpha < 1$, which implies that Lioville theorem, Exercise 8.8, is applicable. We conclude that v is a polynomial of degree 2. But then (8.6) implies that $v \equiv 0$. But this finally reaches a contradiction, since it $v \equiv 0$ contradicts (8.5)[20]. We can conclude. □

[20]Indeed, obtaining (8.5) is the crucial idea in this blow-up proof – it is a quantity that survives the limit procedure, leading to a contradiction. Compare this to the blow-up proof of Poincare inequality

8.3. Basic Schauder on a domain (interior).

Theorem 8.11 (Fundamental Schauder estimate on domain)**.** *Let $\alpha \in (0,1)$, $\Omega \subset \mathbb{R}^n$ an open set, and assume $u \in C^{2,\alpha}(\Omega)$, $f \in C^{\alpha}(\overline{\Omega})$ satisfy the equation*
$$\Delta u(x) = f(x) \quad \text{for all } x \in \Omega$$
Then for any $\Omega' \subset\subset \Omega$ we have
$$[D^2 u]_{C^\alpha(\Omega')} \leq C \left([f]_{C^\alpha(\Omega)} + \|u\|_{L^\infty(\Omega)}\right).$$
where the constant C depends only on $\alpha, n, \Omega', \Omega$.

We first establish the following building block.

Lemma 8.12. *For $\alpha \in (0,1)$, assume that $u \in C^{2,\alpha}(\overline{B(0,2)})$ and*
$$\Delta u(x) = f(x) \quad \text{for all } x \in B(0,2).$$
Then
$$[D^2 u]_{C^\alpha(B(0,1))} \leq C \left([f]_{C^\alpha(B(0,2))} + \|D^2 u\|_{L^\infty(B(0,2))} + \|u\|_{L^\infty(B(0,2))}\right).$$
where C is a constand only depending on the dimension n and α.

Proof. We argue – how could it be different, by a cutoff function. Let $\eta \in C_c^\infty(B(0,2))$, $\eta \equiv 1$ in $B(0,1)$.

Then
$$\Delta(\eta u)(x) = \eta(x) f(x) + \Delta\eta(x)\, u(x) + 2\nabla\eta(x) \cdot \nabla u(x) \quad \text{in } \mathbb{R}^n.$$
On the other hand $\eta u \in C^{2,\alpha}(\mathbb{R}^n)$, so we have from Theorem 8.7,
$$[D^2 u]_{C^\alpha(B(0,1))} \leq [D^2(\eta u)]_{C^\alpha(\mathbb{R}^n)} \lesssim_\eta [f]_{C^\alpha(R^n)} + \|D^2 u\|_{L^\infty(B(0,2))} + \|u\|_{L^\infty(B(0,2))}.$$
Here we have used several times Exercise 8.3.

□

In Lemma 8.12 we don't like the $\|D^2 u\|_{L^\infty(B(0,2))}$-term. To remove it, we are going to use the *Ehrling Lemma*, Exercise 8.4.

Observe that it is not obvious how to use Ehrling Lemma to remove the $\|D^2 u\|_{L^\infty}$ in Lemma 8.12, since the domain on the right-hand side is larger than on the left-hand side. Actually to do this we combine Ehrling Lemma and Lemma 8.12.

Lemma 8.13. *For $\alpha \in (0,1)$, assume that $u \in C^{2,\alpha}(\overline{B(0,2)})$ and*
$$\Delta u(x) = f(x) \quad \text{for all } x \in B(0,2).$$
Then
$$\|D^2 u\|_{L^\infty(B(0,1))} \leq C \left(\|u\|_{L^\infty(B(0,2))} + [f]_{C^\alpha(B(0,2))}\right).$$
where C is a constand only depending on the dimension n and α.

Proof. Assume $B(x_0, 2\rho) \subset B(0,2)$. Then

$$\tilde{u}(x) := u(x_0 + \rho x)$$

is a solution to

$$\Delta \tilde{u}(x) = \rho^2 f(x_0 + \rho x) \quad \text{in } B(0,2),$$

and thus from Lemma 8.12,

$$[D^2\tilde{u}]_{C^\alpha(B(0,1))} \leq C\,\rho^{2+\alpha}[f]_{C^\alpha(B(0,2))} + \|D^2\tilde{u}\|_{L^\infty(B(0,2))} + \|\tilde{u}\|_{L^\infty(B(0,2))}. \tag{8.7}$$

Now set (observe that $2 - |x| = \operatorname{dist}(x, \partial B(0,2))$),

$$A := \sup_{x \in B(0,2)} (2 - |x|)^2\, |D^2 u(x)|.$$

Pick any $x_0 \in B(0,2)$ and set $\rho := \frac{1}{3}(2 - |x_0|)$. Then, using the notion $\tilde{u}$ from above,

$$\begin{aligned}(2 - |x_0|)^2 |D^2 u(x_0)| \leq& 9\rho^2 \|D^2 u\|_{L^\infty(B(x_0,\rho))}\\ =& 9\|D^2\tilde{u}\|_{L^\infty(B(0,1))}.\end{aligned}$$

Using the Ehrling-Lemma, Exercise 8.6, on $B(0,1)$ we have for any $\varepsilon > 0$ (will be chosen below)

$$\leq 9\varepsilon[D^2\tilde{u}]_{C^\alpha(B(0,1))} + C(\varepsilon)\|\tilde{u}\|_{L^\infty(B(0,1))}.$$

By the estimate (8.7)

$$\begin{aligned}
&\leq 9\varepsilon\left(C\,\rho^{2+\alpha}[f]_{C^\alpha(B(0,2))} + C\|D^2\tilde{u}\|_{L^\infty(B(0,2))} + \|\tilde{u}\|_{L^\infty(B(0,2))}\right) + C(\varepsilon)\|\tilde{u}\|_{L^\infty(B(0,1))}\\
&\leq 9\varepsilon C\rho^2\|D^2 u\|_{L^\infty(B(x_0,2\rho))} + C(\varepsilon)\|u\|_{L^\infty(B(0,2))} + C\,\varepsilon[f]_{C^\alpha(B(0,2))}\\
&\leq 9\varepsilon C\rho^2 \sup_{x \in B(x_0,2\rho)} |D^2 u(x)| + C(\varepsilon)\|u\|_{L^\infty(B(0,2))} + C\,\varepsilon[f]_{C^\alpha(B(0,2))}\\
&= 9\varepsilon C\rho^2 \sup_{x \in B(x_0,2\rho)} \frac{1}{(2-|x|)^2}\, A + C(\varepsilon)\|u\|_{L^\infty(B(0,2))} + C\,\varepsilon[f]_{C^\alpha(B(0,2))}.
\end{aligned}$$

By the choice of ρ we have for any $x \in B(x_0, 2\rho)$,

$$2 - |x| \overset{\text{triangular inequality}}{\geq} 2 - |x| - |x - x_0| \geq 2 - |x_0| - 2\rho = 3\rho - 2\rho = \rho.$$

Thus, we have shown

$$(2 - |x_0|)^2 |D^2 u(x_0)| \leq 9\varepsilon C A + C(\varepsilon)\|u\|_{L^\infty(B(0,2))} + C\,\varepsilon[f]_{C^\alpha(B(0,2))}$$

This holds for any $x_0 \in B(0,2)$, so we actually have

$$A \leq 9\varepsilon C A + C(\varepsilon)\|u\|_{L^\infty(B(0,2))} + C\,\varepsilon[f]_{C^\alpha(B(0,2))}.$$

Now we take ε small enough so that $9\varepsilon C < \frac{1}{2}$, then we have shown

$$\|D^2 u\|_{L^\infty(B(0,1))} \leq A \leq 2\,C(\varepsilon)\|u\|_{L^\infty(B(0,2))} + C\,\varepsilon[f]_{C^\alpha(B(0,2))}.$$

□

Proof of Theorem 8.11. Take $\rho > 0$ such that $\operatorname{dist}(\Omega', \partial\Omega) > 10\rho$.

We then have

$$[D^2u]_{C^\alpha(\Omega')} \leq \max_{x_0\in\Omega'}[D^2u]_{C^\alpha(B(x_0,\rho))} + \max_{x_0\in\Omega'} \frac{2}{\rho^\alpha}\|D^2u\|_{L^\infty(B(x_0,\rho))} \tag{8.8}$$

So fix some $x_0 \in \Omega'$. We rescale again

$$\tilde{u}(x) := u(x_0 + \rho x), \quad \tilde{f}(x) := \rho^2 f(x_0 + \rho x).$$

Then, since $B(x_0, 2\rho) \subset \Omega$,

$$\Delta\tilde{u} = \tilde{f} \quad \text{in } B(0, 2)$$

We apply Lemma 8.12, and have

$$\begin{aligned}
\rho^{2+\alpha}[D^2u]_{C^\alpha(B(x_0,\rho))} =&[D^2\tilde{u}]_{C^\alpha(B(0,1))}\\
\precsim&\|D^2\tilde{u}\|_{L^\infty(B(0,2))} + \|\tilde{u}\|_{L^\infty(B(0,2))} + [\tilde{f}]_{C^\alpha(B(0,2))}\\
=&\rho^2\|D^2u\|_{L^\infty(B(x_0,2\rho))} + \|u\|_{L^\infty(B(0,2\rho))} + \rho^{2-\alpha}[f]_{C^\alpha(B(x_0,2\rho))}
\end{aligned}$$

Plugging this into (8.8) we have

$$[D^2u]_{C^{2,\alpha}(\Omega')} \leq C(\rho)\left([f]_{C^\alpha(\Omega)} + \max_{x_0\in\Omega'}\|D^2u\|_{L^\infty(B(x_0,2\rho))}\right) \tag{8.9}$$

To control the second term, we rescale once more

$$\tilde{u}(x) := u(x_0 + 2\rho x), \quad \tilde{f}(x) := (2\rho)^2 f(x_0 + 2\rho x).$$

Since $B(x_0, 4\rho) \subset \Omega$, we still have

$$\Delta\tilde{u} = \tilde{f} \quad \text{in } B(0, 2)$$

and thus we can apply Lemma 8.13. Then

$$\begin{aligned}
(2\rho)^{-2}\|D^2u\|_{L^\infty(B(x_0,2\rho))} =&\|D^2\tilde{u}\|_{L^\infty(B(0,1))}\\
\precsim&\|\tilde{u}\|_{L^\infty(B(0,2))} + [\tilde{f}]_{C^\alpha(B(0,2))}\\
=&\|u\|_{L^\infty(B(x_0,4\rho))} + (2\rho)^\alpha[f]_{C^\alpha(B(x_0,4\rho))}\\
\leq&C(\rho)\left(\|u\|_{L^\infty(\Omega)} + [f]_{C^\alpha(\Omega)}\right).
\end{aligned}$$

Plugging this into (8.9) we can conclude. □

8.4. **Constant coefficient Schauder estimate.** Assume next that $A \in \mathbb{R}^{n\times n}$ is a *constant* elliptic matrix, i.e. for some $\lambda > 0$,

$$\langle A\xi, \xi\rangle \geq \lambda|\xi|^2 \tag{8.10}$$

We want to prove (interior) Schauder-estimates for the equation (divergence form)

$$\operatorname{div}(A\nabla u) = f \quad \text{in } \Omega$$

or equivalently (non-divergence form)

$$A_{\alpha\beta}\partial_{\alpha\beta}u = f \quad \text{in } \Omega \tag{8.11}$$

Theorem 8.14 (Schauder estimate on domain)**.** *Let $\alpha \in (0,1)$, $\Omega \subset \mathbb{R}^n$ an open set, and assume $u \in C^{2,\alpha}(\Omega)$, $f \in C^\alpha(\overline{\Omega})$ satisfy the equation (8.11).*

Then for any $\Omega' \subset\subset \Omega$ we have

$$[D^2u]_{C^\alpha(\Omega')} \leq C\,\left([f]_{C^\alpha(\Omega)} + \|u\|_{L^\infty(\Omega)}\right).$$

where the constant C depends only on $\alpha, n, \Omega', \Omega$ and A.

Observe that we don't need to assume that A is symmetric, but we observe that we can assume that w.l.o.g.

Exercise 8.15. *Assume that $A \in \mathbb{R}^{n\times n}$ satisfy (8.10). Show that*

$$\frac{1}{2}\left(A + A^t\right)$$

still satisfies (8.10).

Exercise 8.16. *Prove Theorem 8.14. For this first assume A is symmetric (see Exercise 8.15)*

(1) Decompose $A = PDP^t$ where $P \in SO(n)$ and $D = \operatorname{diag}(\lambda_1, \ldots, \lambda_n)$ with $\lambda_i \geq \lambda$ (the ellipticity constant). Set $\sqrt{D} := \operatorname{diag}(\sqrt{\lambda_1}, \sqrt{\lambda_2}, \ldots, \sqrt{\lambda_n})$.

(2) Set $\tilde{u}(x) := u(P\sqrt{D}x)$, $\tilde{f} := f(P\sqrt{D}x)$, and

$$\tilde{\Omega} := \left(P\sqrt{D}\right)^{-1}\Omega \equiv \{P\sqrt{D}^{-1}x : \quad x \in \Omega\}.$$

Show that

$$\Delta\tilde{u} = \tilde{f} \quad \textit{in } \tilde{\Omega}$$

Hint: for any $\alpha, \gamma \in \{1, \ldots, n\}$ we have

$$\partial_{\alpha\gamma}\tilde{u}(x) = \sqrt{\lambda_\alpha}\sqrt{\lambda_\gamma}\sum_{\beta,\sigma} P_{\beta\alpha}P_{\sigma\gamma}\,(\partial_{\beta\sigma}u)(P\sqrt{D}x)$$

and thus

$$\Delta\tilde{u}(x) = \sum_{\beta,\sigma} A_{\alpha\beta}\,(\partial_{\alpha\beta}u)(P\sqrt{D}x)$$

(3) Use Theorem 8.11 on $\tilde{\Omega}$.

In case of balls, we can also make the constant independent

Corollary 8.17. *Let $\alpha \in (0,1)$. Assume that $\Omega = B(x_0, 2\rho)$ and assume $u \in C^{2,\alpha}(\Omega)$, $f \in C^\alpha(\overline{\Omega})$ satisfy the equation (8.11).*

Then

$$[D^2u]_{C^\alpha(B(x_0,\rho))} \leq C\,\left([f]_{C^\alpha(B(x_0,2\rho))} + \rho^{-2-\alpha}\|u\|_{L^\infty(\Omega)}\right).$$

where the constant C depends only on α, n, and A – but not on ρ or x_0.

Exercise 8.18. *Prove Corollary 8.17*

Hint: *Scaling.*

8.5. **Freezing the coefficient.** Assume next that $A \in C^\alpha(\Omega, \mathbb{R}^{n\times n})$, is a *non-constant* but uniformly elliptic matrix, that is for some $\lambda > 0$,

$$\langle A(x)\xi, \xi\rangle \geq \lambda |\xi|^2 \quad \forall x \in \Omega. \tag{8.12}$$

We consider the *non-divergence form* equation

$$A_{\alpha\beta}\partial_{\alpha\beta} u = f \quad \text{in } \Omega$$

The idea is simple (cf. Section 7.5.1 where we did a similar argument for $W^{1,p}$-theory): in a small ball $B(x_0, 10\rho)$ we can assume that $\|A - A(x_0)\|_{L^\infty(B(x_0,10\rho)} < \varepsilon$. In that tiny ball we rewrite the equation

$$A_{\alpha\beta}(x_0)\partial_{\alpha\beta} u = \left(A_{\alpha\beta}(x_0) - A_{\alpha\beta}\right)\partial_{\alpha\beta} u + f \quad \text{in } B(x_0, 10\rho) \tag{8.13}$$

The main observation is now

$$[\left(A_{\alpha\beta}(x_0) - A_{\alpha\beta}\right)\partial_{\alpha\beta} u]_{C^\alpha(B(x_0,\rho)} \leq \underbrace{\|A(x_0) - A\|_{L^\infty(B(x_0,10\rho))}}_{<\varepsilon} [D^2 u]_{C^\alpha} + [A]_{C^{2,\alpha}} \|D^2 u\|_{L^\infty(B(x_0,\rho)}$$

Combining this with the constant coefficient Schauder, we would hope to absorb the first term.

Lemma 8.19. *Assume that A is as above, $u \in C^{2,\alpha}$ and*

$$A_{\alpha\beta}\partial_{\alpha\beta} u = f \quad \textit{in } B(0,2)$$

Then there exists a constant C depending on A, α and the dimension such that

$$[D^2 u]_{C^\alpha(B(0,1))} \leq C\left(\|f\|_{C^\alpha(B(0,2))} + \|u\|_{L^\infty(B(0,2))} + \|D^2 u\|_{L^\infty(B(0,2))}\right).$$

Proof. Fix $x_0 \in B(0,1)$, $\rho > 0$ such that $B(x_0, 2\rho) \subset B(0,2)$.

Let $\eta \in C_c^\infty(B(x_0, 2\rho))$, $\eta \equiv 1$ in $B(x_0,\rho)$. We may assume that $\|\nabla^k \eta\|_{L^\infty} \leq C(k)\rho^{-k}$ for any $k \in \mathbb{N} \cup \{0\}$. For $v := \eta u$ we observe that from the product rule, Exercise 8.3,

$$\begin{aligned}
&[\left(A_{\alpha\beta}(x_0) - A_{\alpha\beta}\right)\partial_{\alpha\beta} v]_{C^\alpha(B(x_0,4\rho))}\\
\precsim &\|A(x_0) - A\|_{L^\infty} [D^2 v]_{C^\alpha(B(x_0,4\rho)} + [A]_{C^\alpha(B(0,2))} \|D^2 v\|_{L^\infty(B(x_0,4\rho))}\\
\precsim &\rho^\alpha [A]_{C^\alpha(B(0,2))} [D^2 v]_{C^\alpha(B(x_0,4\rho)} + [A]_{C^\alpha(B(0,2))} \|D^2 v\|_{L^\infty(B(x_0,4\rho))}
\end{aligned}$$

Now we observe that

$$A_{\alpha\beta}(x_0)\partial_{\alpha\beta} v = \left(A_{\alpha\beta}(x_0) - A_{\alpha\beta}\right)\partial_{\alpha\beta} v + A_{\alpha\beta}\partial_{\alpha\beta} v \quad \text{in } \mathbb{R}^n$$

Using the constant coefficient Schauder, Corollary 8.17, in $B(x_0, 4\rho)$ we have

$$
\begin{aligned}
&[D^2 v]_{C^\alpha(B(x_0,2\rho))}\\
\lesssim&[(A_{\alpha\beta}(x_0) - A_{\alpha\beta})\, \partial_{\alpha\beta} v]_{C^\alpha(B(x_0,4\rho))} + [A_{\alpha\beta}\partial_{\alpha\beta} v]_{C^\alpha(B(x_0,4\rho))} + \rho^{-2-\alpha}\|v\|_{L^\infty(B(x_0,2\rho))}\\
\lesssim&\rho^\alpha [A]_{C^\alpha(B(0,2))}\, [D^2 v]_{C^\alpha(B(x_0,4\rho))} + [A]_{C^\alpha(B(0,2))}\, \|D^2 v\|_{L^\infty(B(x_0,4\rho))} + [A_{\alpha\beta}\partial_{\alpha\beta} v]_{C^\alpha(B(x_0,4\rho))} + \rho^{-2-\alpha}\|v\|_{L^\infty(B(x_0,}
\end{aligned}
$$

By the support of v, we can conclude

$$
\begin{aligned}
&[D^2 v]_{C^\alpha(B(x_0,2\rho))}\\
\leq& C\rho^\alpha [A]_{C^\alpha(B(0,2))}\, [D^2 v]_{C^\alpha(B(x_0,2\rho))} + C[A]_{C^\alpha(B(0,2))}\, \|D^2 v\|_{L^\infty(B(x_0,2\rho))} + C[A_{\alpha\beta}\partial_{\alpha\beta} v]_{C^\alpha(B(x_0,2\rho))} + \rho^{-2-\alpha}C\|v\|
\end{aligned}
$$

Taking $\rho < \frac{1}{2}$ small enough so that $C\rho^\alpha [A]_{C^\alpha(B(0,2))} < \frac{1}{2}$ (this fixes ρ from now on), we can absorb the first term on the right-hand side above and find that (recall $\eta \equiv 1$ in $B(x_0, \rho)$)

$$
\begin{aligned}
&[D^2 u]_{C^\alpha(B(x_0,\rho))} \leq [D^2 v]_{C^\alpha(B(x_0,2\rho))}\\
\leq& 2C[A]_{C^\alpha(B(0,2))}\, \|D^2 v\|_{L^\infty(B(x_0,2\rho))} + 2C[A_{\alpha\beta}\partial_{\alpha\beta} v]_{C^\alpha(B(x_0,2\rho))} + \rho^{-2-\alpha}2C\|v\|_{L^\infty(B(x_0,2\rho))}
\end{aligned}
\tag{8.14}
$$

We know that $v = \eta u$, have $\|\nabla^k \eta\|_{L^\infty} \leq C(k)$, observe that by Exercise 8.5

$$
\|Du\|_{L^\infty} \lesssim \|u\|_{L^\infty} + \|D^2 u\|_{L^\infty},
$$

as well as

$$
\|f\|_{L^\infty} \leq \|A_{\alpha\beta}\partial_{\alpha\beta} u\|_{L^\infty} \lesssim \|D^2 u\|_{L^\infty}.
$$

Combining these observations with the estimate (8.14) we conclude

$$
\begin{aligned}
&[D^2 u]_{C^\alpha(B(x_0,\rho))} \leq [D^2 v]_{C^\alpha(B(x_0,2\rho))}\\
\leq& C(A,\rho)\left(\|D^2 u\|_{L^\infty(B(x_0,2\rho))} + [f]_{C^\alpha(B(x_0,2\rho))} + \|u\|_{L^\infty(B(x_0,2\rho))}\right)
\end{aligned}
\tag{8.15}
$$

Again, by a covering argument we have

$$
[D^2 u]_{C^\alpha(B(0,1))} \leq \max_{x_0 \in B(0,1)} [D^2 u]_{C^\alpha(B(x_0,\rho))} + (2\rho)^{-\alpha} 2\|D^2 u\|_{L^\infty(B(0,2))}
$$

Combining this with (8.15) we conclude (observe that ρ is dependends on A, but in no way on u). $\square$

Theorem 8.20 (Fundamental Schauder estimate on domain)**.** *Let $\alpha \in (0,1)$, $\Omega \subset \mathbb{R}^n$ an open set, and assume $u \in C^{2,\alpha}(\Omega)$, $f \in C^\alpha(\overline{\Omega})$ satisfy the equation*

$$
A_{\alpha\beta}\partial_{\alpha\beta} u(x) = f(x) \quad \text{for all } x \in \Omega.
$$

Here $A \in L^\infty(\Omega, \mathbb{R}^{n\times n}) \cap C^\alpha(\overline{\Omega}, \mathbb{R}^{n\times n})$ is uniformly elliptic, i.e. there exists $\lambda > 0$ such that

$$
A_{\alpha\beta}\xi_\alpha\xi_\beta \geq \lambda |\xi|^2 \quad \forall \xi \in \mathbb{R}^n.
$$

Then for any $\Omega' \subset\subset \Omega$ we have

$$
[D^2 u]_{C^\alpha(\Omega')} \leq C\, \left([f]_{C^\alpha(\Omega)} + \|u\|_{L^\infty(\Omega)}\right).
$$

where the constant C depends only on $\alpha, n, \Omega', \Omega$ and A.

Exercise 8.21. *Prove Theorem 8.20 using Lemma 8.19.*

Hint: You can argue similar to Theorem 8.11.

8.6. **Interior Schauder estimate: proof of Theorem 8.2.** Again, in order to prove Theorem 8.2 we first prove an estimate with C^2-term on the right hand side.

Lemma 8.22. *Assume that L is given as before by*

$$Lu(x) := A_{\alpha\beta}(x)\partial_{\alpha\beta}u(x) + b_\alpha\partial_\alpha u(x) + c(x)u(x).$$

Let $u \in C^{2,\alpha}$ and

$$Lu = f \quad \text{in } B(0,2)$$

Then there exists a constant C depending on A, α and the dimension so that

$$[D^2u]_{C^\alpha(B(0,1))} \leq C \left(\|f\|_{C^\alpha(B(0,2))} + \|u\|_{L^\infty(B(0,2))} + \|D^2u\|_{L^\infty(B(0,2))}\right).$$

Exercise 8.23. *Prove Lemma 8.22. Use that*

$$Lu = f \quad \text{in } B(0,2)$$

is equivalent to

$$A_{\alpha\beta}\partial_{\alpha\beta}u = f - b_\alpha\partial_\alpha u - cu \quad \text{in } B(0,2)$$

and estimate the right-hand side in C^α.

Exercise 8.24. *Prove Theorem 8.2 using Lemma 8.22.*

Hint: You can argue similar to Theorem 8.11.

8.7. **Schauder estimates in divergence form.** Schauder estimates also work in divergence form, simply by rewriting divergence form into non-divergence form – observe that then we need to assume more regularity on A. But this does not give optimal results

Lemma 8.25 (Schauder for divergence form – easy attempt)**.** *Let $\alpha \in (0,1)$, $\Omega \subset \mathbb{R}^n$ an open set, and assume $u \in C^{2,\alpha}(\Omega)$, $f \in C^\alpha(\overline{\Omega})$ satisfy the equation*

$$\operatorname{div}(A\nabla u)(x) = f(x) \quad \text{for all } x \in \Omega.$$

Here $A \in L^\infty(\Omega, \mathbb{R}^{n\times n}) \cap C^{1,\alpha}(\overline{\Omega}, \mathbb{R}^{n\times n})$ is uniformly elliptic, i.e. there exists $\lambda > 0$ such that

$$A_{\alpha\beta}\xi_\alpha\xi_\beta \geq \lambda|\xi|^2 \quad \forall \xi \in \mathbb{R}^n.$$

Then for any $\Omega' \subset\subset \Omega$ we have

$$[D^2u]_{C^\alpha(\Omega')} \leq C\,[f]_{C^\alpha(\Omega)} + \|u\|_{L^\infty(\Omega)}.$$

where the constant C depends only on $\alpha, n, \Omega', \Omega$ and A.

Exercise 8.26. *Prove Lemma 8.25, observe that*

$$\operatorname{div}(A\nabla u)(x) = A_{\alpha\beta}\partial_{\alpha\beta}u + \partial_\alpha A_{\alpha\beta}\partial_\beta u.$$

If we want to obtain $C^{1,\alpha}$-estimates, one again relates the PDE to the Δ-equation (we will not do this here) and one obtains, cf. [Fernández-Real and Ros-Oton, 2022, Theorem 2.28].

Theorem 8.27. *Let $\Omega \subset\subset \mathbb{R}^n$.*

Assuem $u \in C^{1,\alpha}(\Omega)$ be a weak solution to

$$\operatorname{div}(A\nabla u) = f \quad \text{in } \Omega$$

where $A \in C^\alpha$ is uniformly elliptic and bounded. Then for any $\Omega' \subset\subset \Omega$,

$$\|\nabla u\|_{C^\alpha(\Omega')} \leq C\|u\|_{L^\infty(\Omega)} + \|f\|_{L^\infty(B(0,1))}.$$

8.8. **Schauder at the boundary.** As we have discussed in the $W^{k,2}$-regularity theory, the boundary regularity can be proven by considering a boundary problem in the upper half-plane $\mathbb{R}^n_+$ and flattening the boundary. We will not go throught the details here, but we state the corresponding result (cf. [Gilbarg and Trudinger, 2001, Corollary 6.7])

Theorem 8.28 (Global Schauder a priori estimate)**.** *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary $\partial\Omega$*

Let $\alpha \in (0,1)$, $\Omega \subset \mathbb{R}^n$ an open set, and assume $u \in C^{2,\alpha}(\overline{\Omega})$, $f \in C^\alpha(\overline{\Omega})$ and $\varphi \in C^{2,\alpha}(\overline{\Omega})$ satisfy the equation

$$\begin{cases} Lu(x) = f(x) & \text{for all } x \in \Omega, \\ u = \varphi & \text{on } \partial\Omega \end{cases}$$

Here L is a linear elliptic operator in non-divergence form as above, with $A, b, c \in L^\infty(\Omega) \cap C^\alpha(\overline{\Omega})$, $A(x) \in \mathbb{R}^{n\times n}$ uniformly elliptic, i.e. there exists $\lambda > 0$ such that

$$A_{\alpha\beta}\xi_\alpha\xi_\beta \geq \lambda|\xi|^2 \quad \forall \xi \in \mathbb{R}^n.$$

Then

$$[D^2u]_{C^\alpha(\overline{\Omega})} \leq C\left(\|f\|_{L^\infty(\Omega)} + [f]_{C^\alpha(\Omega)}\right) + \|u\|_{L^\infty(\Omega)} + \|\varphi\|_{C^{2,\alpha}(\Omega)}.$$

where the constant C depends only on α, n, Ω and A, b, c.

8.9. **From a priori estimates to a posteriori estimates.** In the above theorems, we always *a priori* assumed that the solution $u \in C^{2,\alpha}$ and only obtained an estimate for the solution.

Here we illustrate how one can use those a priori estimates to obtain regularity estimates.

Definition 8.29. We say a set $\Omega \subset\subset \mathbb{R}^n$ is *admissible* for $C^{2,\alpha}$ Schauder, if for any $f \in C^\alpha$,

$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

has a solution in $C^{2,\alpha}(\overline{\Omega})$-solution u. (Observe this solution is unique, by the maximum principle!).

Any smoothly bounded set is admissible in the above sense:

Theorem 8.30. *Let Ω be a bounded set with smooth boundary. Assume that $u \in W^{1,2}_0(\Omega)$ solves*
$$\begin{cases} -\Delta u = f \qquad & in\ \Omega\\ u = 0 \qquad & on\ \partial\Omega\end{cases}$$
If $f \in C^\alpha(\overline{\Omega})$ then $u \in C^{2,\alpha}(\overline{\Omega})$ and we have
$$[u]_{C^{2,\alpha}(\overline{\Omega})} \precsim \|u\|_{L^\infty(\Omega)} + \|f\|_{C^\alpha(\Omega)} + \|g\|_{C^{2,\alpha}(\overline{\Omega})}.$$

Proof. Fix $f \in C^\alpha(\overline{\Omega})$. We can extend f to $\mathbb{R}^n$ such that
$$[f]_{C^\alpha(\mathbb{R}^n)} \leq C\left(\|f\|_{L^\infty(\Omega)} + [f]_{C^\alpha(\Omega)}\right),$$
with C a constant only depending on Ω and α, but not on f.

E.g. we could do this by observing that for some small $\delta > 0$ there must be a smooth map $\pi : B_\delta(\overline{\Omega}) \to \overline{\Omega}$ which is the identity on Ω, then set $f := \eta f(\pi(x))$ where $\eta \in C^\infty_c(B_\delta(\overline{\Omega}))$ and $\eta \equiv 1$ in $\overline{\Omega}$.

Denote by f_ε the usual smooth approximation of f, which satisfies
$$\sup_{\varepsilon\in(0,1)} \|f_\varepsilon\|_{L^\infty(\mathbb{R}^n)} + [f_\varepsilon]_{C^\alpha(\mathbb{R}^n)} \leq [f]_{C^\alpha(\mathbb{R}^n)} \leq C\left(\|f\|_{L^\infty(\Omega)} + [f]_{C^\alpha(\Omega)}\right).$$
(Observe that we do *not* have that $f_\varepsilon \to f$ in C^α, but that does not matter for us).

Now solve (e.g. by variational approach)
$$\begin{cases} \Delta u_\varepsilon = f_\varepsilon \quad \text{in } \Omega\\ u_\varepsilon = 0 \quad \text{on } \partial\Omega.\end{cases}$$
which we can do with the estimate
$$\|u_\varepsilon\|_{W^{1,2}(\Omega)} \leq \|f_\varepsilon\|_{L^2(\Omega)} \leq \|f_\varepsilon\|_{L^\infty(\Omega)} \leq \|f\|_{L^\infty(\Omega)}.$$

By $W^{k,2}$-theory, Section 6, we have $u_\varepsilon \in C^\infty(\overline{\Omega})$.

Observe that from global $W^{1,p}$-theory (Calderón-Zygmund), Theorem 7.10 but with boundary see the remark after the theorem, we conclude that for any $p \in (1,\infty)$
$$\|u_\varepsilon\|_{W^{2,p}(\Omega)} \leq \|u\|_{W^{1,2}(\Omega)} + \|f\|_{L^p(\Omega)} \precsim_p \|f\|_{L^\infty(\Omega)}.$$
In particular, choosing $p > \frac{n}{2}$ we find from Sobolev-Morrey embedding theorem
$$\|u_\varepsilon\|_{L^\infty(\Omega)} + [u_\varepsilon]_{C^{2-\frac{n}{p}}} \precsim \|u\|_{W^{2,p}(\Omega)} \precsim \|f\|_{L^\infty(\Omega)}.$$
So we can apply global (*a-priori*) Schauder, Theorem 8.28, and have
$$\begin{aligned}[D^2u_\varepsilon]_{C^\alpha(\Omega)} \precsim& \|u_\varepsilon\|_{L^\infty(\Omega)} + \|f_\varepsilon\|_{L^\infty(\Omega)} + [f_\varepsilon]_{C^\alpha(\Omega)}\\ \precsim& \|f\|_{L^\infty(\Omega)} + [f]_{C^\alpha(\Omega)}\end{aligned}$$

By Arzela-Ascoli u_ε converges in C^2 to u which (by $W^{1,2}$-convergence) satisfies

$$\begin{cases} \Delta u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

We know the solution to the above must be unique (e.g. by maximum principle or variational arguments). We have obtained the estimate

$$\begin{aligned} \|u\|_{L^\infty(\Omega)} + [D^2 u]_{C^\alpha(\Omega)} \precsim & \liminf_{\varepsilon\to 0} \|u_\varepsilon\|_{L^\infty(\Omega)} + [D^2 u_\varepsilon]_{C^\alpha(\Omega)} \\ \precsim & \|f\|_{L^\infty(\Omega)} + [f]_{C^\alpha(\Omega)}. \end{aligned}$$

□

The above argument would also work to make *a posteriori* Schauder work for $\operatorname{div}(A\nabla u)$ for quite smooth A (recall that we used $W^{k,2}$-theory to conclude that u_ε is smooth, which requires A to be relatively smooth.

Another way to do this, is the following argument.

Theorem 8.31. *Let Ω be admissible in the sense of Definition 8.29.*

Let $A \in C^\alpha(\Omega, \mathbb{R}^{n\times n})$ be uniformly elliptic and bounded, $b, c \in C^\alpha(\Omega)$, and $c \le 0$, then if $u \in C^2(\overline{\Omega})$ solves

$$\begin{cases} A_{\alpha\beta}\partial_{\alpha\beta} u + b_\alpha \partial_\alpha u + cu = f & \text{in } \Omega \\ u = g & \text{on } \partial\Omega \end{cases}$$

and we have $f \in C^\alpha(\overline{\Omega})$ and $g \in C^{2,\alpha}(\overline{\Omega})$, then $u \in C^{2,\alpha}(\overline{\Omega})$ and

$$[u]_{C^{2,\alpha}(\overline{\Omega})} \precsim \|u\|_{L^\infty(\Omega)} + \|f\|_{C^\alpha(\Omega)} + \|g\|_{C^{2,\alpha}(\overline{\Omega})}.$$

Proof. W.l.o.g. $g \equiv 0$, otherwise consider $u - g$, which leads to the above claim.

Also, $c \le 0$, so we have the maximum principle and uniqueness. That is, all we need to show that for given $f \in C^\alpha(\overline{\Omega})$ there *exists* a solution $u \in C^{2,\alpha}(\Omega)$ to

$$\begin{cases} Lu = f & \text{in } \Omega \\ u = 0 & \text{on } \Omega \end{cases}$$

which then, by the Schauder estimates Theorem 8.28, implies the claim.

We argue by a *method of continuity*, reducing the problem to Δ (which we already know, by assumptions, that we have the Schauder theory for). Consider

$$L_t u := tL + (1-t)\Delta$$

Set

$$X := \{u \in C^{2,\alpha}(\overline{\Omega}), \quad u = 0 \text{ on } \partial\Omega\}$$

equipped with the norm

$$\|u\|_X := [D^2 u]_{C^\alpha(\Omega)} + \|u\|_{L^\infty(\Omega)}.$$

Also set

$$Y := \{f \in C^\alpha(\overline{\Omega})\}.$$

equipped with the norm

$$\|f\|_Y := \|f\|_{L^\infty(\Omega)} + [f]_{C^\alpha(\Omega)}.$$

Then for each $t \in \mathbb{R}$

$$L_t : X \to Y,$$

is a bounded linear operator. An the question of finding $u_t \in C^{2,\alpha}(\overline{\Omega})$ solving

$$\begin{cases} L_t u_t = f & \text{in } \Omega \\ u = 0 & \text{on } \Omega \end{cases}$$

is actually of invertability of $L_t : X \to Y$.

Set

$$I := \{t \in [0,1] : \quad L_t : X \to Y \quad \text{is bijective}\}.$$

Observe that for each $t \in [0,1]$, L_t is uniformly elliptic. By the maximum principle (and since $c \leq 0$), L_t is always injective. So actually

$$I = \{t \in [0,1] : \quad L_t : X \to Y \quad \text{is } \textit{surjective}\}.$$

We have I is nonempty, indeed $0 \in I$, since $L_0 = \Delta$, and by our assumption on Ω.

Next we observe, I is closed: Indeed, assume that there is $t_i \in I$, $\lim_{i\to\infty} t_i = \bar{t}$. Fix any $f \in Y$, we want to show that there is $u_{\bar{t}} \in X$ with $L_{\bar{t}} u_{\bar{t}} = f$. Since $t_i \in I$ there exist $u_i \in X$ with $L_{t_i} u_i = f$. By Schauder a priori estimates

$$[D^2 u_{t_i}]_{C^\alpha(\Omega)} \precsim \|u_{t_i}\|_{L^\infty(\Omega)} + \|f\|_{L^\infty(\Omega)} + [f]_{C^\alpha(\Omega)}.$$

Here we observe that the ellipticity constant and boundedness of L_{t_i} is *uniform* for $t \in [0,1]$, and thus the constant in $\precsim$ does not depend on t_i! By maximum principle (since $c \leq 0$, Corollary 2.63) we have

$$\|u_{t_i}\|_{L^\infty(\Omega)} \precsim \|f\|_{L^\infty(\Omega)}$$

Again the constant in $\precsim$ is independent of t_i!

Using possibly the Ehrling-type Lemma, cf. Exercise 8.5 and Exercise 8.6, we find

$$\sup_i \|u_{t_i}\|_{C^{2,\alpha}(\overline{\Omega})} \precsim \|f\|_{L^\infty(\Omega)} + [f]_{C^\alpha(\Omega)}.$$

This is a case for Arzela-Ascoli, from which we conclude that u_{t_i} converges in C^2-norm to some $u_{\bar{t}} \in C^{2,\alpha}$ (in particular $u_{\bar{t}} \in X$). Then we have

$$L_{\bar{t}} u_{\bar{t}} = \lim_{i\to\infty} L_{t_i} u_{t_i} = f.$$

Thus $L_{\bar{t}}$ is still surjective, i.e. I is closed.

Lastly, I is open. Indeed let $\bar{t} \in I$. X and Y are Banach spaces, so by the open mapping theorem (or the inverse map theorem), $L_{\bar{t}} : X \to Y$ is not only invertible, the inverse $L_{\bar{t}}^{-1} : Y \to X$ is a bounded linear function. We observe that

$$L_t = L_{\bar{t}} + L_t - L_{\bar{t}} = L_{\bar{t}} \left(\mathrm{Id_X} + L_{\bar{t}}^{-1} \left(L_t - L_{\bar{t}}\right)\right) \tag{8.16}$$

Now

$$\begin{aligned}\|L_{\bar{t}}^{-1} \left(L_t - L_{\bar{t}}\right) \|_{L(X,X)} \leq & \|L_{\bar{t}}^{-1}\|_{L(Y,X)} \|L_t - L_{\bar{t}}\|_{L(X,Y)} \\ \leq & |t - \bar{t}| \, \|L_{\bar{t}}^{-1}\|_{L(Y,X)} \left(\|L\|_{L(X,Y)} + \|\Delta\|_{L(X,Y)}\right).\end{aligned}$$

In particular we can find $\delta > 0$ such that for any $t \in (\bar{t} - \delta, \bar{t} + \delta)$,

$$\|L_{\bar{t}}^{-1} \left(L_t - L_{\bar{t}}\right) \|_{L(X,X)} \leq \frac{1}{2}.$$

But then by the *von Neumann* sum argument,

$$\left(\mathrm{Id_X} + L_{\bar{t}}^{-1} \left(L_t - L_{\bar{t}}\right)\right)^{-1} = \sum_{k=0}^{\infty} \left(L_{\bar{t}}^{-1} \left(L_t - L_{\bar{t}}\right)\right)^k \in L(X,X),$$

where again we use that X is a Banach space, so the right-hand side operator converges in operator norm. But then (8.16) implies that L_t that for any $t \in (\bar{t} - \delta, \bar{t} + \delta)$ is a combination of two bijective operators, and thus bijective. And thus

$$(\bar{t} - \delta, \bar{t} + \delta) \in I.$$

That is, I is open.

We conclude that I is an nonempty, open and closed subset of $[0,1]$, and that is equivalent to saying $I = [0,1]$.

In particular $1 \in I$, and thus $L_1 = L : X \to Y$ is invertible. We can conclude. □

From the previous theorem we readily obtain Corollary 8.32.

Corollary 8.32. *Assume $u \in C^2(\Omega)$*[21] *and*

$$A_{\alpha\beta} \partial_{\alpha\beta} u = f \quad \text{in } \Omega,$$

where $A \in C^\alpha(\Omega, \mathbb{R}^{n\times n})$ is uniformly elliptic and bounded.

Then $u \in C^{2,\alpha}_{loc}(\Omega)$ and for any $\Omega' \subset\subset \Omega$ we have

$$[D^2 u]_{C^{2,\alpha}(\Omega')} \leq C(\Omega', \Omega, \alpha, n, A) \left(\|f\|_{L^\infty(\Omega)} + [f]_{C^\alpha(\Omega)} + \|u\|_{L^\infty(\Omega)}\right).$$

We also record the divergence-form version

[21] this assumption is only needed to have a reasonable notion for a solution with a maximum principle – if $u \in C^0$, the notion of *Viscosity solutions* could be used

Corollary 8.33. *Let $\Omega \subset\subset \mathbb{R}^n$.*

Assume $u \in W^{1,2}(\Omega)$ be a weak solution to

$$\operatorname{div}(A\nabla u) = f \quad \text{in } \Omega$$

where $A \in C^\alpha$ is uniformly elliptic and bounded. Then for any $\Omega' \subset\subset \Omega$,

$$\|\nabla u\|_{C^\alpha(\Omega')} \leq C(\Omega', \Omega, \alpha, n, A) \left(\|u\|_{L^\infty(\Omega)} + \|f\|_{C^\alpha(\Omega)} \right).$$

Exercise 8.34. *Prove Corollary 8.32. For this fix any $\Omega' \subset\subset \Omega$. Pick a smoothly bounded set $\Omega_0 \subset\subset \Omega$, $\partial\Omega_0 \in C^\infty$ and $\Omega_0 \supset \Omega'$.*

Take $\eta \in C_c^\infty(\Omega_0)$, $\eta \equiv 1$ in Ω. Consider the equation for ηu and use Theorem 8.31.

Lastly, we mention that the term $\|u\|_{L^\infty(\Omega)}$ in the Schauder estimate can be replaced by the L^1-norm of u, due to the following observation

Exercise 8.35. *Let $B \subset \mathbb{R}^n$ be a ball. Show that for any $\varepsilon > 0$ there exists $C = C(\varepsilon, B)$ such that*

$$\|u\|_{L^\infty(B)} \leq \varepsilon \|D^2 u\|_{L^\infty(B)} + C\|u\|_{L^1(B)}.$$

Hint: Prove first $\|f\|_{L^\infty(B)} \lesssim \|f\|_{L^1(B)} + \|Df\|_{L^\infty(B)}$. Conclude that

$$\|Df\|_{L^\infty(B)} \leq \|D^2 f\|_{L^\infty(B)} + \|Df\|_{L^1(B)}.$$

Use e.g. the Ehrling-Lemma for $W^{2,1}$ to conclude

$$\|f\|_{L^\infty(B)} \lesssim \|f\|_{L^1(B)} + \|D^2 f\|_{L^\infty(B)}$$

Then prove an Ehrling lemma, cf. Exercise 8.6.

9. Segway: Higher order Schauder estimates, nonlinear version, bootstrapping, Hilbert's 19th problem

Example 9.1 (Higher order bootstrapping, nonlinear version)**.** Assume that $A \in C^\infty(\mathbb{R}^n, \mathbb{R}^{n\times n})$ is uniformly bounded and uniformly elliptic and we have $u \in C^2$ (actually $u \in C^{1,\alpha}$ is enough, only the notion of solution needs to be made precise[22]) soving the equation

$$A_{\alpha\beta}(Du)\partial_{\alpha\beta} u = 0 \quad \text{in } \Omega.$$

If $u \in C^{1,\alpha}$ then $A_{\alpha\beta}(Du) \in C^\alpha$, by Schauder theory $u \in C^{2,\alpha}$, thus we can differentiate the equation have

$$A_{\alpha\beta}(Du)\partial_{\alpha\beta}\partial_\gamma u = 0 + \partial_\gamma \left(A_{\alpha\beta}(Du)\right) \partial_{\alpha\beta} u \in C^\alpha$$

thus again by Schauder, $\partial_\gamma u \in C^{2,\alpha}$, so $u \in C^{3,\alpha}$ and we can continue this *bootstrapping* argument to conclude $u \in C^\infty$.

But we need to initially assume $u \in C^{1,\alpha}$ to conclude $A_{\alpha\beta}(Du) \in C^\alpha$.

[22] *Viscosity solutions*, [Koike, 2004]

This actually points to a serious issue that is one of the most famous regularity results, the resulution of *Hilbert's 19th problem*.

Problem 9.2 (Hilbert 19th problem (1900)). *Consider any minimizer u of the energy functional*

$$\mathcal{E}(u) := \int_\Omega I(\nabla u)dx.$$

That is assume

$$\mathcal{E}(u) \leq \mathcal{E}(v) \quad \text{for all } v \text{ with the same boundary data, } u - v = 0 \text{ on } \partial\Omega.$$

Here $I : \mathbb{R}^n \to \mathbb{R}$ is smooth[23]*, and uniformly convex, i.e. there exist $\lambda, \Lambda \in (0,\infty)$ such that*

$$\lambda|\xi|^2 \leq \langle D^2 I(p)\xi, \xi\rangle \leq \Lambda|\xi|^2$$

and $\Omega \subset \mathbb{R}^n$ is a smoothly bounded set.

Is it true that $u \in C^\infty$?

Observe that above we did not specify the class of functions permissible. But by the direct method of calculus of variations it is easy to find a minimizer in the Sobolev space, so what we shall mean is:

Problem 9.3 (Hilbert 19th problem (1900)). *Consider any minimizer $u \in W^{1,2}(\Omega)$ of the energy functional*

$$\mathcal{E}(u) := \int_\Omega I(\nabla u)dx.$$

That is assume

$$\mathcal{E}(u) \leq \mathcal{E}(v) \quad \text{for all } v \in W^{1,2}(\Omega) \text{ with the same boundary data, } u - v \in W^{1,2}_0(\Omega).$$

Here $I : \mathbb{R}^n \to \mathbb{R}$ is smooth, and uniformly convex, i.e. there exist $\lambda, \Lambda \in (0,\infty)$ such that

$$\lambda|\xi|^2 \leq \langle D^2 I(p)\xi, \xi\rangle \leq \Lambda|\xi|^2$$

and $\Omega \subset \mathbb{R}^n$ is a smoothly bounded set.

Is it true that $u \in C^\infty$?

Remark 9.4. Observe that uniformly convex is a quite strong assumption – $f(t) = t^p$ is *not* uniformly convex for $p > 2$!

Proposition 9.5. *Fix $u_0 \in W^{1,2}(\Omega)$. There exists a unique minimizer to the above problem in*

$$X := \{u \in W^{1,2}(\Omega), \quad u - u_0 \in W^{1,2}_0(\Omega)\}.$$

$u \in X$ is a minimizer if and only if (in distributional sense)

$$\partial_\beta \Big((\partial_{p_\beta} I)(\nabla u)\Big) = 0 \quad \text{in } \Omega$$

[23]this is a simplified version, in general I could depend on x and u, to obtain a *regular variational problem*

Proof. I is in particular strictly convex, so uniqueness of a minimizer follows as in Section 6.2.

For the *Euler-Lagrange equation* we observe

$$\frac{d}{dt}\Big|_{t=0} I(\nabla u + t\nabla\varphi) = \sum_{\beta=1}^{n} (\partial_{p_\beta} I)(\nabla u)\partial_\beta \varphi.$$

For existence we argue by the direct method of calculus of variations, but we need to establish *coercivity* and *lower semicontinuity*.

Coercivity: We need to show

$$\text{(Coercivity)} \qquad \text{if } (u_k)_{k\in\mathbb{N}} \in W_0^{1,2}(\Omega) \text{ and } \sup_k \mathcal{E}(u_k) < \infty \text{ then } \sup_k \|u\|_{W^{1,2}(\Omega)} < \infty$$

Since I is uniformly convex, we have (by Tailor's theorem and since $\lambda|p-q|^2 \leq \langle D^2 I(p)p - q, p-q\rangle$)

$$I(p) \geq I(q) + DI(q)(p-q) + \frac{\lambda}{2}|p-q|^2,$$

and similarly

$$I(p) \leq I(q) + DI(q)(p-q) + \frac{\Lambda}{2}|p-q|^2,$$

and in particular $\lim_{|p|\to\infty} I(p) = +\infty$. So I must have a global minimum. Since I is in particular convex, I has a unique minimum $\bar{\gamma}$ – where we have that $DI(\bar{\gamma}) = 0$. By a simple shift in the function (shifting the energy does not change information about its infimum) we may assume $\bar{\gamma} = 0$ and $I(\bar{\gamma}) = 0$.

We conclude that we must have

$$\frac{\lambda}{2}|p|^2 \leq I(p) \leq \frac{\Lambda}{2}|p|^2$$

This readily leads to coercivity.

That is, whenever we have a minimizing sequence

$$\mathcal{E}(u_k) \xrightarrow{k\to\infty} \inf_X \mathcal{E},$$

then we can assume w.l.o.g. u_k converges weakly to some $u \in X$ in $W^{1,2}(\Omega)$ (observe that X is a convex subset of $W^{1,2}(\Omega)$, so weakly closed).

As for lower semicontinuity, we need to show

$$\mathcal{E}(u) \leq \liminf_{k\to\infty} \mathcal{E}(u_k).$$

But we observe that by Poincaré lemma, $\sqrt{\mathcal{E}(u_k)}$ is a norm on X which is equivalent to the $W^{1,2}(\Omega)$ – in particular weak convergence does not change when considering $\sqrt{\mathcal{E}(u)}$ as a the norm. By lower semicontinuity of norms w.r.t their weak convergence we have lower semicontinuity and can conclude that u is the desired minimizer. □

So minimiziation of $\mathcal{E}$ is not a problem, and we may see why one would hope Hilbert's problem has a positive answer. Everything is smooth and nice and quadratic, the simplest case is $I(\nabla u) = |\nabla u|^2$ – *What could possibly go wrong*?

As a sidenote, if I is non-convex it could be that I as two minimum values, e.g. in dimension 1, $I(1) = I(-1) = \min_{\mathbb{R}} I$. Then we can construct zig-zag minimizer without any reasonable regularity, e.g. $u(x) = |x|$, then $u'(x) \in \{-1, 1\}$ but u is not smooth at 0. So *some sort of* convexity of I is important.

We observe that at least formally (using the chain rule) the equation we have to consider is the one in Example 9.1. We can make the argument of Example 9.1 more precise to conclude from Schauder estimates

Proposition 9.6. *Assume that $u \in W^{1,2}(\Omega)$ is a weak solution of*

$$\partial_\beta \Big((\partial_{p_\beta} L)(\nabla u) \Big) = 0 \quad \text{in } \Omega$$

where L is smooth and uniformly convex. If we a priori assume $u \in C^{1,\alpha}(\Omega)$ for some $\alpha > 0$, then $u \in C^\infty(\Omega)$.

Proof. Pick $\Omega' \subset\subset \Omega$ and let $\varphi \in C_c^\infty(\Omega)$ and $|h| < \operatorname{dist}(\Omega', \partial\Omega)$. Then

$$\int_\Omega \Big(\partial_{p_\beta} L(\nabla u(x+h)) - \partial_{p_\beta} L(\nabla u(x)) \Big)\, \partial_\beta \varphi(x) = 0.$$

By the fundamental theorem of calculus,

$$\begin{aligned} &\Big(\partial_{p_\beta} L(\nabla u(x+h)) - \partial_{p_\beta} L(\nabla u(x)) \Big) \\ =& \int_0^1 \partial_{p_\beta p_\gamma} L\left(t\nabla u(x+h) + (1-t)\nabla u(x) \right) dt\ \partial_\gamma \left(u(x+h) - u(x) \right). \end{aligned}$$

Set

$$A_h(x) := \int_0^1 \partial_{\beta\gamma} L\left(t\nabla u(x+h) + (1-t)\nabla u(x) \right) dt,$$

then we see that A is uniformly elliptic, and we have (with the notation $\delta_h u(x) = u(x+h) - u(x)$),

$$\operatorname{div}\left(A \nabla \delta_h u \right) = 0$$

Our initial assumption $u \in C^{1,\alpha}$ implies $A \in C^\alpha$, so by divergence-form Schauder, Corollary 8.33, we obtain $\frac{\delta_h u}{|h|} \in C^{1,\alpha}_{loc}(\Omega)$. This holds uniform for small $|h|$, so actually $u \in C^{2,\alpha}$. We now differentiate the equation to bootstrap to C^∞. □

The above argument can be relaxed to the assumption $u \in C^1$: In that case we have $A \in C^0$, so either we can extend Schauder-type theory to this regularity [Fernández-Real and Ros-Oton, 2022, Proposition 2.32.], or extend Calderón-Zygmund theory, Theorem 7.15, to obtain to $|h|^{-1}\delta_h u \in$

$C^{1-\varepsilon}$ uniformly in h, that is $u \in C^{2-\varepsilon}$, so we can run the previous arguments to conclude $u \in C^{\infty}$.

Observe, however, that for $u \in W^{1,2}$ the above argument falls apart. Our $A(\nabla u)$ from above is merely *bounded and measurable*, and it needed two geniuses *De Giorgi* and *Nash* (independently, but at the same time) to solve this long-standing issue.

10. De Giorgi - Nash - Moser iteration and De Giorgi's theorem

A good reference for this section is [Han and Lin, 2011], also the recent [Fernández-Real and Ros-Oton, 2022] is recommended. The theory below is commonly known as *De Giorgi–Nash–Moser* theory.

For simplicity we will always assume $n \geq 2$ in the following. (If $n = 1$, Hölder continuity is always true for $W^{1,2}$-functions!).

Theorem 10.1 (De Giorgi - Nash - Moser). *Let $u \in W^{1,2}(B(0,4))$ a weak solution to the equation*

$$\operatorname{div}(A\nabla u) = 0.$$

where $A = (a_{ij}(x))_{i,j=1}^n$ is a symmetric matrix, uniformly elliptic and uniformly bounded

$$\lambda|\xi|^2 \leq \sum_{i,j=1}^n a_{ij}\xi_i\xi_j \quad \forall \xi \in \mathbb{R}^n, \text{ and } \quad |a_{ij}| \leq \Lambda$$

Assume that the dependency $x \mapsto a(x)$ is only measurable (rendering all arguments from Section 7.5 useless).

Then there exists $\alpha > 0$ such that $u \in C^{\alpha}$, and we have

$$\sup_{x \in B(0,2)} |u(x)| + \sup_{x \neq y} \frac{|u(x) - u(y)|}{|x-y|^{\alpha}} \leq C(n, \Lambda/\lambda)\|u\|_{L^2(B(0,4))}.$$

It is quite interesting, that the analogous result does not hold for too many systems, i.e. when $u : \mathbb{R}^n \to \mathbb{R}^m$, $m \geq 2$ – [Giusti, 2003, Example 6.3]. That is, the techniques we discuss here are mostly *scalar*. The reason is, that a lot of arguments we discuss below are about *superlevel sets* $\{u > \lambda\}$.

Corollary 10.2 (Resolution of Hilbert's 19th problem). *Assume that $u \in W^{1,2}(\Omega)$ is a weak solution of*

$$\partial_\beta \left((\partial_{p_\beta} L)(\nabla u) \right) = 0 \quad \text{in } \Omega$$

where L is smooth and uniformly convex. Then $u \in C^{\infty}(\Omega)$.

Proof. We argue as in Proposition 9.6. Before we used Schauder theory there (which we can't use here, because we don't know yet that $u \in C^{1,\alpha}$), we wrote the equation as

$$\operatorname{div}(A_h \nabla \delta_h u) = 0$$

for A_h was only boudned measurable. From Theorem 10.1 we obtain that actually $\delta_h u/|h| \in C^\alpha$, thus $u \in C^{1,\alpha}$, and we can conclude by the arguments in Proposition 9.6. □

It is worthwile to observe that the following arguments (one by Nash-Moser, one by De Giorgi) are *fundamentally different* from previous arguments in Calderon-Zygmund or Schauder theory. For those we always tried to "reduce" the equation at hand to the laplace equation (by blowup, freezing etc.). These are called *perturbative methods*. The new approaches don't do this. They instead try to obtain a continuous improvement in regularity that eventually leads to the desired results, working directly with the equation.

There are two steps to showing Theorem 10.1, *boundedness* and *Hölder continuity*.

We begin by stating the boundedness theorem (this can be substantially generalized, as always, the method is more important than the specific theorem)

Theorem 10.3 (Local Boundedness). *Fix $A(x) = (a_{ij}(x))_{i,j=1}^n$ bounded and measurable, uniformly elliptic and bounded with ellipticity constants $\lambda, \Lambda > 0$, i.e.*

$$\lambda|\xi|^2 \le a_{ij}\xi_i\xi_j \quad \forall \xi \in \mathbb{R}^n, \quad |a_{ij}| \le \Lambda$$

Assume $u \in W^{1,2}(B(0,1))$ is a solution to

$$-\mathrm{div}\,(A\nabla u) = 0 \quad \text{in } B(0,1),$$

in distributional sense, i.e.

$$\int_{B(0,1)} a_{ij}\partial_i u\,\partial_j\varphi = 0$$

for all $\varphi \in C_c^\infty(B(0,1))$ in $B(0,1)$.

Then

$$\sup_{B(0,1/2)} |u| \le C\,\|u\|_{L^2(B(0,1))}$$

The constant C depends on λ, Λ, p, n but not on u, f or otherwise on A.

One important generalization we want to mention here, is that it the above result applies to

$$-\mathrm{div}\,(A\nabla u) = f$$

where $f \in L^p$ for some $p > \frac{n}{2}$, in which case we find that

$$\sup_{B(0,1/2)} |u| \le C\left(\|u\|_{L^2(B(0,1))} + \|f\|_{L^q(B(0,1))}\right)$$

The assumption $f \in L^p$ for $p > \frac{n}{2}$ is justified by Laplace theory. If $\Delta u \in L^p$ then we can hope that $u \in W^{2,p}_{loc}$. By Sobolev embedding $W^{2,p}_{loc} \hookrightarrow L^\infty$ if $p > \frac{n}{2}$ (but not if $p < \frac{n}{2}$.)

10.1. **Boundedness by Moser-Itaration: First proof of Theorem 10.3.** We begin by a few observations, that we sucessively sharpen to actually obtain a theorem that implies Theorem 10.3.

The first observation is that a PDE such as the one under consideration implies an somewhat unnatural phenomenon: a specific Hölder inequality for u is true "in the wrong direction". This is called a *reverse Hölder inequality*. A function satisfying a reverse Hölder inequality must have some special properties (one result that we mention in passing is *Gehring's Lemma*).

Example 10.4 (A first reverse inequality)**.** Let $n \geq 3$.

Assume $u \in W^{1,2}(B(0,1))$ is a solution to

$$-\mathrm{div}\,(A\nabla u) = 0 \quad \text{in } B(0,1),$$

in distributional sense, i.e.

$$\int_{B(0,1)} a_{ij}\partial_i u\,\partial_j \varphi = 0$$

for all $\varphi \in C_c^\infty(B(0,1))$ in $B(0,1)$.

Then for any $0 < r < R \leq 1$

$$\|u\|_{L^{\frac{2n}{n-2}}(B(0,r))} \leq C(\Lambda,\lambda)\,\frac{1}{R-r}\|u\|_{L^2(B(0,R))}.$$

Proof. Let $\eta \in C_c^\infty(B(0,R))$ be as from Exercise 10.5, $\eta \equiv 1$ in $B(0,r)$ and $|D\eta| \leq \frac{C}{R-r}$.

Consider[24]

$$\eta^2 u \in W_0^{1,2}(B(0,1)).$$

By density we then have

$$\int_{B(0,1)} a_{ij}\partial_i u\,\partial_j(\eta^2 u) = 0$$

Testing with $\eta^2 u$ (or the resulting inequality) is often called *Cacciopoli inequality*; also sometimes it is referred to simply as the *energy inequality*. Using the product rule

$$\int_{B(0,1)} \eta^2 a_{ij}\partial_i u\,\partial_j u = -2\int_{B(0,1)} \eta a_{ij}\partial_i u\,\partial_j \eta\, u.$$

By ellipticity on the left and Hölder's inequality on the right we have

$$\lambda \int_{B(0,1)} \eta^2|\nabla u|^2 \leq C(\Lambda)\|\nabla \eta\|_{L^\infty}\left(\int_{B(0,1)} \eta^2|\nabla u|^2\right)^{\frac{1}{2}}\|u\|_{L^2(B(0,R))}$$

[24] η^2 instead for η is mainly for stylistic reasons to not need to deal with roots, η^2 has essentially the same qualities as η

By Young's inequality, $2ab \leq \delta a^2 + \frac{1}{\delta} b^2$, we then have (recall that constants C change from line to line), we obtain for any $\varepsilon > 0$,

$$\lambda \int_{B(0,1)} \eta^2 |\nabla u|^2 \leq \varepsilon \int_{B(0,1)} \eta^2 |\nabla u|^2 + \frac{C(\Lambda) \|\nabla \eta\|_{L^\infty}^2}{\varepsilon} \int_{B(0,R)} |u|^2$$

For $\varepsilon = \frac{\lambda}{2}$ we then find

$$\int_{B(0,1)} \eta^2 |\nabla u|^2 \leq C(\Lambda, \lambda) \left(\frac{1}{(R-r)^2} \int_{B(0,R)} |u|^2 \right)$$

We now apply Sobolev inequality for the left-hand side (here we use, mainly for simplicity, $n \geq 3$)

$$\|\eta u\|_{L^{\frac{2n}{n-2}}(\mathbb{R}^n)} \leq C(n) \|\nabla(\eta u)\|_{L^2(\mathbb{R}^n)} \leq C(n) \|\eta |\nabla u|\|_{L^2(B(0,1))} + \|\nabla \eta\|_{L^\infty} \|u\|_{L^2(B(0,R))}.$$

Then we have shown

$$\|\eta u\|_{L^{\frac{2n}{n-2}}}^2 \leq C(\Lambda, \lambda) \left(\frac{1}{(R-r)^2} \int_{B(0,R)} |u|^2 \right)$$

and thus

$$\|u\|_{L^{\frac{2n}{n-2}}(B(0,r))} \leq C(\Lambda, \lambda) \frac{1}{R-r} \|u\|_{L^2(B(0,R))}.$$

□

Exercise 10.5. *Show that for any $0 < r < R < \infty$ there exists nonnegative $\eta \in C_c^\infty(B(0,R))$, $\eta \equiv 1$ in $B(0,r)$ and*

$$|D^k \eta(x)| \leq \frac{C(k)}{(R-r)^k} \quad \forall k = 1, 2, \ldots$$

where $C(k)$ is a constant which depends on k and possibly the dimension n, but not on r and R.

Example 10.4 is a reverse Hoelder inequality, but it is not clear what it really tells us with respect to boundedness. However we may hope that if we get a control of $\|u\|_{L^p}$ for some very large p (or maybe as $p \to \infty$), then we almost control the L^∞-norm of u, thanks to the following

Exercise 10.6. *Let $\Omega \subset \mathbb{R}^n$ be any open bounded set and $f : \Omega \to \mathbb{R}$ be measurable. Then*

$$\|f\|_{L^\infty(\Omega)} = \lim_{p \to \infty} \left(\int_\Omega |f|^p \right)^{\frac{1}{p}}$$

In particular show that

$$\|f\|_{L^\infty(\Omega)} \leq \liminf_{p \to \infty} \left(\int_\Omega |f|^p \right)^{\frac{1}{p}}.$$

(I.e. when the right-hand side is finite, then the left-hand side is finite and the inequality holds).

The main point of what we call the *Moser iteration* is to use the same method as in Example 10.4 but testing with "powers of u", i.e. with $\eta^2 u^{\beta+1}$ instead of $\eta^2 u$. Here we use the notation $u^{\beta+1} := u|u|^\beta$ for some $\beta \geq 0$. This then could lead to an inequality chain on successively smaller balls

$$\|u\|_{L^{p_K}(B(0,r_K))} \precsim \ldots\ldots \precsim \|u\|_{L^{p_2}(B(0,r_2))} \precsim \|u\|_{L^{\frac{2n}{n-2}}(B(0,r_1))} \precsim \|u\|_{L^2(B(0,1))}$$

This is the *Moser iteration*

The basic building block of this idea is the following observation: It is important to observe that in the following we need to assume u to be *a priori bounded* (which is not a good assumption if that is what we want to show). A further refinement then will lead to the real argument.

Example 10.7 (The (a priori) reverse inequality)**.** Let $n \geq 3$.

Assume $u \in W^{1,2}(B(0,1))$ is a solution to

$$-\mathrm{div}\,(A\nabla u) = 0 \quad \text{in } B(0,1),$$

in distributional sense, i.e.

$$\int_{B(0,1)} a_{ij}\partial_i u\, \partial_j \varphi = 0$$

for all $\varphi \in C_c^\infty(B(0,1))$ in $B(0,1)$.

Then for any $\gamma \geq 2$, if we additionally assume $u \in L^\infty(B(0,1))$, we have the estimate for any $0 < r < R \leq 1$.

$$\|u\|_{L^{\frac{\gamma n}{n-2}}(B(0,r))} \leq (C(\lambda,\Lambda)\,\gamma)^{\frac{1}{\gamma}}\ \frac{1}{(R-r)^2}\|u\|_{L^\gamma(B(0,R))}.$$

Proof. Let $\eta \in C_c^\infty(B(0,R))$ be as above, i.e. from Exercise 10.5, $\eta \equiv 1$ in $B(0,r)$ and $|D\eta| \leq \frac{C}{R-r}$.

As discussed we consider as testfunction φ

$$\eta^2 u|u|^\beta.$$

Observe that

$$\partial_j(u|u|^\beta) = (\beta+1)|u|^\beta \partial_j u$$

Now the important observation is: since we assume already that $u \in L^\infty$ we know that $\eta^2|u|^\beta u \in W_0^{1,2}(B(0,1))$.

By density we then have

$$\int_{B(0,1)} a_{ij}\partial_i u\, \partial_j(\eta^2 u|u|^\beta) = 0.$$

Using the product rule

$$(\beta+1)\int_{B(0,1)} \eta^2 a_{ij}|u|^\beta \partial_i u\, \partial_j u = -2\int_{B(0,1)} \eta a_{ij}\partial_i u\, \partial_j\eta\, u|u|^\beta.$$

By ellipticity on the left and Hölder's inequality[25] on the right we have

$$\lambda(\beta+1)\int_{B(0,1)} \eta^2 |\nabla u|^2 |u|^\beta \leq C(\Lambda) \|\nabla \eta\|_{L^\infty} \left(\int_{B(0,1)} \eta^2 |u|^\beta |\nabla u|^2\right)^{\frac{1}{2}} \||u|^{\frac{\beta}{2}} u\|_{L^2(B(0,R))}$$

By Young's inequality, we again absorb, and find (the constant is independent of β since $\beta \geq 0$))

$$(\beta+1)\int_{B(0,1)} \eta^2 |\nabla u|^2 |u|^\beta \leq C(\lambda,\Lambda)\ \frac{1}{(R-r)^2} \||u|^{\frac{\beta}{2}} u\|^2_{L^2(B(0,R))}$$

We simplify this

$$(\beta+1)\int_{B(0,1)} \eta^2 |\nabla u|^2 |u|^\beta \leq C(\lambda,\Lambda) \frac{1}{(R-r)^2} \|u\|^{2+\beta}_{L^{2+\beta}(B(0,R))}$$

Next, as above we plan to apply Sobolev inequality. Observe that

$$|\nabla |u|^{\frac{\beta+2}{2}}|^2 = \left(\frac{\beta+2}{2}\right)^2 |u|^\beta |\nabla u|^2$$

So actually we have (observe $\beta+1 \approx \beta+2$ since $\beta \geq 0$)

$$\int_{B(0,1)} \eta^2 |\nabla |u|^{\frac{\beta+2}{2}}|^2 \leq (\beta+2) C(\lambda,\Lambda) \frac{1}{(R-r)^2} \|u\|^{2+\beta}_{L^{2+\beta}(B(0,R))}$$

By Sobolev inequality (observe we use the L^2-Sobolev inequality, so the additional constants are independent of β!)

$$\|\eta |u|^{\frac{\beta+2}{2}}\|^2_{L^{\frac{2n}{n-2}}(\mathbb{R}^n)} \leq C(n) \|\eta \nabla |u|^{\frac{\beta+2}{2}}|\|^2_{L^2(B(0,1))} + \|\nabla \eta\|^2_{L^\infty}\ \||u|^{\frac{\beta+2}{2}}\|^2_{L^2(B(0,R))}. \tag{10.1}$$

Together, (again, observe that $\beta+1 \approx \beta+2$ for $\beta \geq 0$) we have shown

$$\|\eta |u|^{\frac{\beta+2}{2}}\|^2_{L^{\frac{2n}{n-2}}(\mathbb{R}^n)} \leq (\beta+2)\, C(\lambda,\Lambda)\, \frac{1}{(R-r)^2} \|u\|^{2+\beta}_{L^{2+\beta}(B(0,R))}.$$

That is

$$\|u\|^{2+\beta}_{L^{\frac{(\beta+2)n}{n-2}}(B(0,r))} \leq (\beta+2)\, C(\lambda,\Lambda)\, \frac{1}{(R-r)^2} \|u\|^{2+\beta}_{L^{2+\beta}(B(0,R))}.$$

That is

$$\|u\|_{L^{\frac{(\beta+2)n}{n-2}}(B(0,r))} \leq \left(\frac{(\beta+2)\, C(\lambda,\Lambda)}{(R-r)^2}\right)^{\frac{1}{2+\beta}} \|u\|_{L^{2+\beta}(B(0,R))}.$$

Setting $\gamma := \beta + 2$,

$$\|u\|_{L^{\frac{\gamma n}{n-2}}(B(0,r))} \leq \left(\frac{(\beta+2)\, C(\lambda,\Lambda)}{(R-r)^2}\right)^{\frac{1}{2+\beta}} \|u\|_{L^{\gamma}(B(0,R))}.$$

□

[25] $\int_{B(0,1)} \eta a_{ij} \partial_i u\, \partial_j \eta\, u |u|^\beta = \int_{B(0,1)} \eta a_{ij} |u|^{\frac{\beta}{2}} \partial_i u\, \partial_j \eta\, u |u|^{\frac{\beta}{2}}$

Example 10.7 looks very promising, but it has the very serious a priori assumption $u \in L^\infty$. But, as we have seen now a few times, a priori estimates (which are easier being obtained, and should be seen as the best way to gather some intuition about the PDE at hand) can sometimes be transformed into a posteriori estimates. This is our next goal, it will follow all the above arguments, but in order to make the test functions bounded it will cut off the testfunction at a certain height. We observe (this idea will also appear in the De Giorgi method):

Exercise 10.8. *Let Ω be an open bounded set. Let $u \in W^{1,p}(\Omega)$ and $k, \ell \in \mathbb{R}$. Show that*

$$\min\{\max\{u, -k\}, \ell\} \in W^{1,2} \cap L^\infty(\Omega)$$

You can (and should) use one of the following: If $f \in W^{1,p}$ and g is Lipschitz, then $g \circ f$ belongs to $W^{1,p}$. Or, if $f \in W^{1,p}$ then $|f| \in W^{1,p}$.

Moroever, the above min-max cutoff does not change the derivative

Exercise 10.9. *Let $f, g \in W^{1,p}(\Omega)$, $p \geq 1$, where Ω is any open set. Assume there is a measurable set $A \subset \Omega$ such that $f = g$. Show that*

$$Df = Dg \quad a.e.\ in\ A.$$

Hint: Use Lemma 5.18.

Exercise 10.10. *Let $f(x) = x$ and $g(x) = -x$. Show*

- *$f, g \in W^{1,p}((-1,1))$.*
- *$f'(0) \neq g'(0)$*
- *Why doesn't that contradict the claim in Exercise 10.9?*

Exercise 10.11. *Let Ω be an open bounded set. Let $u \in W^{1,p}(\Omega)$ and $k, \ell \in \mathbb{R}$. By Exercise 10.8,*

$$\min\{\max\{u, k\}, \ell\} \in W^{1,2} \cap L^\infty(\Omega)$$

Show that for the distributional derivative a.e. in Ω we have

$$\partial_j \min\{\max\{u, -k\}, \ell\} = \begin{cases} \partial_j u & in\ \{x \in \Omega : \ \ell < u(x) < k\} \\ 0 & otherwise \end{cases}$$

Hint: Use Exercise 10.9

We are ready to prove the Moser iteration approach to boundedness. To illustrate the power of this method, we will consider a *differential inequality*. So we will get that for a subsolution we gain control on u_+ – this has some similarities to the structure of a maximum principle).

Proposition 10.12 (Moser iteration step)**.** *Let $n \geq 2$. Fix $A(x) = (a_{ij}(x))_{i,j=1}^n$ bounded and measurable, uniformly elliptic and bounded with ellipticity constants $\lambda > 0$, i.e.*

$$\lambda |\xi|^2 \leq a_{ij} \xi_i \xi_j \quad \forall \xi \in \mathbb{R}^n, \quad |a_{ij}| \leq \Lambda$$

Assume $u \in W^{1,2}(B(0,1))$ *is a solution to*

$$-\mathrm{div}\,(A\nabla u) \leq 0 \quad \text{in } B(0,1),$$

in distributional sense, i.e.

$$\int_{B(0,1)} a_{ij}\partial_i u\,\partial_j \varphi \leq 0$$

for all nonnegative $\varphi \in C_c^\infty(B(0,1))$ *in* $B(0,1)$.

Pick any $p > 1$ *if* $n = 2$ *or any* $p \in (1, \frac{n}{n-2}]$ *if* $n \geq 3$, *then we have for any* $\gamma \geq 2$,

$$\|u_+\|_{L^{\gamma p}(B(0,r))} \leq \left(\frac{C(\lambda,\Lambda,p)\,\gamma}{(R-r)^2}\right)^{\frac{1}{\gamma}} \|u_+\|_{L^\gamma(B(0,R))}.$$

In particular whenever the right-hand side is finite, the left-hand side is finite.

As a corollary,

Corollary 10.13 (Moser iteration step)**.** *Let* $n \geq 2$. *Fix* $A(x) = (a_{ij}(x))_{i,j=1}^n$ *bounded and measurable, uniformly elliptic and bounded with ellipticity constants* $\lambda > 0$, *i.e.*

$$\lambda|\xi|^2 \leq a_{ij}\xi_i\xi_j \quad \forall \xi \in \mathbb{R}^n, \quad |a_{ij}| \leq \Lambda$$

Assume $u \in W^{1,2}(B(0,1))$ *is a solution to*

$$-\mathrm{div}\,(A\nabla u) = 0 \quad \text{in } B(0,1),$$

in distributional sense, i.e.

$$\int_{B(0,1)} a_{ij}\partial_i u\,\partial_j \varphi = 0$$

for all $\varphi \in C_c^\infty(B(0,1))$ *in* $B(0,1)$.

Pick any $p > 1$ *if* $n = 2$ *or any* $p \in (1, \frac{n}{n-2}]$ *if* $n \geq 3$, *then we have for any* $\gamma \geq 2$ *and any* $0 < r < R \leq 1$

$$\|u\|_{L^{\gamma p}(B(0,r))} \leq \left(\frac{C(\lambda,\Lambda,p)\,\gamma}{(R-r)^2}\right)^{\frac{1}{\gamma}} \|u\|_{L^\gamma(B(0,R))}.$$

In particular whenever the right-hand side is finite, the left-hand side is finite.

Exercise 10.14. *Prove that Corollary 10.13 is a consequence of Proposition 10.12.*

Proof of Proposition 10.12. For $m > 0$ we set

$$\bar{u}_m := \min\{u_+, m\}$$

By Exercise 10.8 $\bar{u}_m \in L^\infty \cap W^{1,2}(B(0,1))$. Thus, taking η as in Example 10.4 and 10.7, we test the equation with

$$\eta^2 \bar{u}_m^\beta u_+ \in W_0^{1,2}(B(0,1)).$$

Indeed observe[26] that since $\beta \geq 0$,

$$\nabla\left(\bar{u}_m^\beta u_+\right) = \beta \underbrace{\nabla \bar{u}_m}_{=0 \text{ if } u>m} \underbrace{u_+ \bar{u}_m^{\beta-1}}_{\in[0,u_+)} + \underbrace{\bar{u}_m^\beta}_{\in L^\infty} \underbrace{\nabla u_+}_{\in L^2} \in L^2(B(0,1)).$$

By density, from the PDE we have

$$\int_{B(0,1)} a_{ij}\partial_i u\, \partial_j\left(\eta^2 u_m^\beta u_+\right) \leq 0$$

That is

$$\int_{B(0,1)} \eta^2\, \bar{u}_m^\beta\, a_{ij}\partial_i u\, \partial_j u_+ + \beta \int_{B(0,1)} \eta^2\, \bar{u}_m^{\beta-1} u_+\, a_{ij}\partial_i u\, \partial_j u_m$$
$$\leq -\int_{B(0,1)} a_{ij}\partial_i u\, 2\,\eta\, \partial_j \eta\, u_m^\beta u_+$$

Now we would like to use ellipticity on the left-hand side, but we are worried because there are two different vectors multiplied to a_{ij}. But, they are actually not different: By Exercise 10.11,

$$D\bar{u}_m = \begin{cases} Du & \text{a.e. in } \{x \in B(0,1) : 0 < u(x) < m\} \\ 0 & \text{a.e. anywhere else.} \end{cases}$$

and similarly for u_+, so we have

$$\int_{B(0,1)} \eta^2\, \bar{u}_m^\beta\, a_{ij}\partial_i u_+\, \partial_j u_+ + \beta \int_{B(0,1)} \eta^2\, \bar{u}_m^{\beta-1} u_+\, a_{ij}\partial_i u_m\, \partial_j u_m$$
$$\leq -\int_{B(0,1)} a_{ij}\partial_i u\, 2\,\eta\, \partial_j \eta\, u_m^\beta u_+$$

Now we use ellipticity,

$$\lambda\left(\int_{B(0,1)} \eta^2\, \bar{u}_m^\beta\, |Du_+|^2 + \beta \int_{B(0,1)} \eta^2\, \bar{u}_m^{\beta-1} u_+\, |Du_m|^2\right)$$
$$\leq -\int_{B(0,1)} a_{ij}\partial_i u\, 2\,\eta\, \partial_j \eta\, u_m^\beta u_+$$

Now for the right-hand side we observe that

$$\partial_i u\, u_+ = \partial_i u_+\, u_+$$

So we arrive at

$$\lambda\left(\int_{B(0,1)} \eta^2\, \bar{u}_m^\beta\, |Du_+|^2 + \beta \int_{B(0,1)} \eta^2\, \bar{u}_m^{\beta-1} u_+\, |Du_m|^2\right)$$
$$\leq C(\Lambda)\|D\eta\|_{L^\infty} \int_{B(0,1)} \eta\, |D\bar{u}_+|\, \bar{u}_m^\beta\, u_+.$$

[26] if you are uncomfortable about the set $\{u = 0\}$, then use

$$(\min\{u^+ + k, k+m\})^\beta (u_+ + k) - k^{\beta+1}$$

for a tiny $k > 0$.

Now, as before, by Young's inequality we arrive at

$$\begin{aligned}&\int_{B(0,1)} \eta^2\, \bar{u}_m^\beta\, |Du_+|^2 + \beta \int_{B(0,1)} \eta^2\, \bar{u}_m^{\beta-1} u_+\, |Du_m|^2\\ \leq& C(\lambda,\Lambda)\|D\eta\|_{L^\infty}^2 \int_{B(0,R)} \bar{u}_m^\beta\, |u_+|^2.\end{aligned}$$

Set

$$w := \bar{u}_m^{\frac{\beta}{2}} u_+$$

Then

$$|Dw|^2 \leq (1+\beta)\left(\beta \bar{u}_m^\beta |D\bar{u}_m|^2 + \bar{u}_m^\beta |D\bar{u}|^2\right)$$

Thus, we find

$$\begin{aligned}&\int_{B(0,1)} \eta^2\, |Dw|^2\\ \leq& C(\lambda,\Lambda)(1+\beta)\|D\eta\|_{L^\infty}^2 \int_{B(0,R)} |w|^2.\end{aligned}$$

This is the same inequality as in Example 10.4, and we have with the same argument as there for any $q \in (1, \frac{2n}{n-2}]$ (or $q < \infty$ if $n = 2$)

$$\|w\|_{L^q(B(0,r))}^2 \leq C(\lambda,\Lambda)(1+\beta)\frac{1}{(R-r)^2}\|w\|_{L^2(B(0,R))}^2.$$

Since $w := \bar{u}_m^{\frac{\beta}{2}} u_+$ we find

$$\|w\|_{L^q(B(0,r))}^2 \leq C(\lambda,\Lambda)(1+\beta)\frac{1}{(R-r)^2}\|u_+\|_{L^{2+\beta}(B(0,R))}^{2+\beta}.$$

Letting $m \to \infty$ (using monotone convergence theorem) we obtain

$$\|u_+\|_{L^{\frac{2+\beta}{2}q}(B(0,r))}^{2+\beta} \leq C(\lambda,\Lambda)(1+\beta)\frac{1}{(R-r)^2}\|u_+\|_{L^{2+\beta}(B(0,R))}^{2+\beta}.$$

That is,

$$\|u_+\|_{L^{\frac{(\beta+2)}{2}q}(B(0,r))} \leq \left(\frac{C(\lambda,\Lambda,q)\,(\beta+2)}{(R-r)^2}\right)^{\frac{1}{2+\beta}} \|u_+\|_{L^{2+\beta}(B(0,R))}.$$

Again, we set $\gamma := \beta + 2 \geq 2$, and then have for $p := \frac{q}{2} > 1$ (more specifically we can pick any $p > 1$ if $n = 2$ and any $p \in (1, \frac{n}{n-2})$ if $n \geq 3$)

$$\|u_+\|_{L^{\gamma p}(B(0,r))} \leq \left(\frac{C(\lambda,\Lambda,q)\,\gamma}{(R-r)^2}\right)^{\frac{1}{\gamma}} \|u_+\|_{L^\gamma(B(0,R))}.$$

We can conclude. □

By an iteration we now obtain the following local boundedness result, which readily implies Theorem 10.3.

Theorem 10.15 (Local Boundedness (inequality))**.** *Fix $A(x) = (a_{ij}(x))_{i,j=1}^n$ bounded and measurable, uniformly elliptic and bounded with ellipticity constants $\lambda, \Lambda > 0$, i.e.*

$$\lambda|\xi|^2 \leq a_{ij}\xi_i\xi_j \quad \forall \xi \in \mathbb{R}^n, \quad |a_{ij}| \leq \Lambda$$

Assume $u \in W^{1,2}(B(0,1))$ is a solution to

$$-\mathrm{div}\,(A\nabla u) \leq 0 \quad \text{in } B(0,1),$$

in distributional sense, i.e.

$$\int_{B(0,1)} a_{ij}\partial_i u\,\partial_j \varphi \leq 0$$

for all nonnegative $\varphi \in C_c^\infty(B(0,1))$ in $B(0,1)$.

Then

$$\sup_{B(0,1/2)} u_+ \leq C\|u\|_{L^2(B(0,1))}$$

Where C depends on λ, Λ, n but not on u, or otherwise on A.

Exercise 10.16. *Show that Theorem 10.15 implies Theorem 10.3*

Proof of Theorem 10.15. As discussed before, we will use the reverse Hölder inequality from Proposition 10.12 on sucessively smaller balls and larger exponents.

Fix $p > 1$ (either $p = \frac{n}{n-2}$ if $n \geq 3$ or any $p > 2$).

We set

$$r_i := \frac{1}{2} + 2^{-i-1}.$$

and

$$\gamma_i := 2p^i$$

From Proposition 10.12 we then have

$$\|u_+\|_{L^{\gamma_i}(B(0,r_i))} \leq \left(\frac{C(\lambda,\Lambda,p)\,\gamma_{i-1}}{(r_{i-1}-r_i)^2}\right)^{\frac{1}{\gamma_{i-1}}} \|u_+\|_{L^{\gamma_{i-1}}(B(0,r_{i-1}))}.$$

This translates into

$$\|u_+\|_{L^{2^ip}(B(0,r_i))} \leq e^{\frac{\log(8\,C(\lambda,\Lambda,p))+i\log 2+(i-1)\log p}{2p^{i-1}}} \|u_+\|_{L^{\gamma_{i-1}}(B(0,r_{i-1}))}.$$

Iterating this estimates implies (observe that $r_0 = 1$ and $\gamma_0 = 2$)

$$\|u_+\|_{L^{2^ip}(B(0,r_i))} \leq e^{\sum_{j=1}^i \frac{\log(8\,C(\lambda,\Lambda,p))+j\log 2+(j-1)\log p}{2p^{j-1}}} \|u_+\|_{L^2(B(0,1))}.$$

Now we observe that since $p > 1$,

$$\Gamma(\lambda,\Lambda,p) := \sum_{j=1}^\infty \frac{\log(8\,C(\lambda,\Lambda,p)) + j\log 2 + (j-1)\log p}{2p^{j-1}} < \infty$$

and then we have shown

$$\|u_+\|_{L^{2^ip}(B(0,\frac{1}{2}))} \leq e^\Gamma \|u_+\|_{L^2(B(0,1))}$$

This holds for all $i \in \mathbb{N}$, so in particular, Exercise 10.6,

$$\|u_+\|_{L^\infty(B(0,\frac{1}{2}))} \leq \limsup_{i\to\infty} \|u_+\|_{L^{2^i p}(B(0,\frac{1}{2}))} \leq e^\Gamma \|u_+\|_{L^2(B(0,1))}$$

We can conclude. □

An adapatation of the above argument can also be used for inhomogeneous right-hand side, even with right-hand side depending on u. For details see [Han and Lin, 2011, Chapter 4].

Theorem 10.17 (Local Boundedness (inequality)). *Fix $A(x) = (a_{ij}(x))_{i,j=1}^n$ bounded and measurable, uniformly elliptic and bounded with ellipticity constants $\lambda, \Lambda > 0$, i.e.*

$$\lambda|\xi|^2 \leq a_{ij}\xi_i\xi_j \quad \forall \xi \in \mathbb{R}^n, \quad |a_{ij}| \leq \Lambda$$

Let $f \in L^q(B(0,1))$ for some $q > \frac{n}{2}$.

Assume $u \in W^{1,2}(B(0,1))$ is a solution to

$$-\mathrm{div}\,(A\nabla u) \leq f \quad \text{in } B(0,1),$$

in distributional sense, i.e.

$$\int_{B(0,1)} a_{ij}\partial_i u\, \partial_j \varphi \leq \int f\varphi$$

for all nonnegative $\varphi \in C_c^\infty(B(0,1))$ in $B(0,1)$.

Then

$$\sup_{B(0,1/2)} u_+ \leq C(\|u\|_{L^2(B(0,1))} + \|f\|_{L^q(B(0,1))})$$

Where C depends on λ, Λ, p, n but not on u, f or otherwise on A.

The assumption $q > \frac{n}{2}$ is natural: Even for $\Delta u = f \in L^q$ we get $u \in W^{2,q}$. If $q > \frac{n}{2}$ then by Sobolev-Morrey embedding, Theorem 5.27, $W^{2,q} \hookrightarrow C^{0,2-\frac{n}{q}}$, i.e. we can hope for u to be bounded.

10.2. **Boundedness by De Giorgi's method: Second proof of Theorem 10.3.** The other method to obtain Theorem 10.3 is by De Giorgi (both methods were developed around the same time). It is important to know *both* approaches, because in more complicated situations it may be that one performs better than the other.

Similar to Moser, the underlying effect that we use is the Cacciopoli estimate. In contrast to Moser, De Giorgi's method does not use powers, but different superlevel sets of u.

Lemma 10.18 (Cacciopoli inequality). *Fix $A(x) = (a_{ij}(x))_{i,j=1}^n$ bounded and measurable, uniformly elliptic and bounded with ellipticity constants $\lambda, \Lambda > 0$, i.e.*

$$\lambda|\xi|^2 \leq a_{ij}\xi_i\xi_j \quad \forall \xi \in \mathbb{R}^n, \quad |a_{ij}| \leq \Lambda$$

There exists $\delta > 0$ depending only on n, λ, and Λ such that the following holds.

Assume $u \in W^{1,2}(B(0,1))$ is a solution to

$$-\mathrm{div}\,(A\nabla u) \leq 0 \quad \textit{in } B(0,1),$$

in distributional sense, i.e.

$$\int_{B(0,1)} a_{ij}\partial_i u\, \partial_j \varphi \leq 0$$

for all nonnegative $\varphi \in C_c^\infty(B(0,1))$ in $B(0,1)$.

Let $k > 0$ and $\eta \in C_c^\infty(B(0,1))$. Then we have for

$$v := (u-k)_+$$

$$\int_{B(0,1)} |\nabla(\eta v)|^2 \leq C\, \|\nabla \eta\|_{L^\infty}^2 \int_{\mathrm{supp}\,\eta} |v|^2$$

Proof. From Exercise 10.11 we again have that

$$\partial_j v = \begin{cases} \partial_j u & \text{in } \{u < k\} \\ 0 & \text{otherwise} \end{cases}$$

That for a.e. x with $v(x) \neq 0$ we have $\partial_j u = \partial_j v$.

In particular we will use below

$$(\partial_j u) v = (\partial_j v)\, v \quad \text{a.e.}$$

As before, $\eta^2 v$ is a permissible test-function and we have

$$\begin{aligned}
\int_{B(0,1)} \eta^2 a_{ij}\partial_i v \partial_j v =& \int_{B(0,1)} \eta^2 a_{ij}\partial_i u \partial_j v \\
=& \int_{B(0,1)} a_{ij}\partial_i u \partial_j (\eta^2 v) - 2\int_{B(0,1)} \eta a_{ij}\partial_i u \partial_j \eta\ v \\
\leq& - 2\int_{B(0,1)} \eta a_{ij}\partial_i u \partial_j \eta\ v \\
=& - 2\int_{B(0,1)} \eta a_{ij}\partial_i v \partial_j \eta\ v.
\end{aligned}$$

By the same absorption argument as in the Moser-iteration argument we then obtain

$$\int_{B(0,1)} \eta^2 |\nabla v|^2 \leq C\, \|\nabla \eta\|_{L^\infty} \int_{\mathrm{supp}\,\eta} |v|^2$$

with a constant C depending on Λ, λ, but not on v or η.

Using the product rule we find (the constant C changes from line to line observe that $\nabla \eta \equiv 0$ in $\mathbb{R}^n \setminus \mathrm{supp}\,\eta$)

$$\int_{B(0,1)} |\nabla(\eta v)|^2 \leq C\, \|\nabla \eta\|_{L^\infty}^2 \int_{\mathrm{supp}\,\eta} |v|^2$$

□

Proposition 10.19 (De Giorgi's method)**.** *Fix $A(x) = (a_{ij}(x))_{i,j=1}^n$ bounded and measurable, uniformly elliptic and bounded with ellipticity constants $\lambda, \Lambda > 0$, i.e.*

$$\lambda |\xi|^2 \leq a_{ij}\xi_i\xi_j \quad \forall \xi \in \mathbb{R}^n, \quad |a_{ij}| \leq \Lambda$$

There exists $\delta > 0$ depending only on n, λ, and Λ such that the following holds.

Assume $u \in W^{1,2}(B(0,1))$ is a solution to

$$-\mathrm{div}\,(A\nabla u) \leq 0 \quad \text{in } B(0,1),$$

in distributional sense, i.e.

$$\int_{B(0,1)} a_{ij}\partial_i u\, \partial_j \varphi \leq 0$$

for all nonnegative $\varphi \in C_c^\infty(B(0,1))$ in $B(0,1)$.

If

$$\|u_+\|_{L^2(B(0,1))} < \delta \tag{10.2}$$

then

$$u \leq 1 \quad \text{a.e. in } B(0, \frac{1}{2})$$

Exercise 10.20. *Show that Proposition 10.19 implies Theorem 10.15 (and thus of Theorem 10.3). Hint: Apply Proposition 10.19 to $\tilde{u} := \delta \frac{u}{2\|u\|_{L^2(B(0,1))}}$.*

Proof of Proposition 10.19. We again consider the balls between $B(0,1/2)$ and $B(0,1)$:

Set

$$r_i := \frac{1}{2} + 2^{-i-1},$$

and

$$\mu_i := 1 - 2^{-i}$$

and take $\eta_i \in C_c^\infty(B(0,r_{i-1}))$, $\eta_i \equiv 1$ in $B(0,r_i)$ nonnegative bump functions with $|\nabla \eta_i| \lesssim_n 2^i$.

We apply Cacciopoli inequality, Lemma 10.18, and have

$$\int_{B(0,1)} |\nabla(\eta_i(u-\mu_i)_+)|^2 \leq C\, 2^{2i} \int_{\operatorname{supp}\eta_i} |(u-\mu_i)_+|^2 \tag{10.3}$$

Fix some $p \in (2, \frac{2n}{n-2}]$ (if $n > 3$, any $p > 2$ if $n = 2$). From Sobolev embedding (with a constant depending on p, but not on i; recall constants change from line to line!), and using that $\eta_{i-1} \equiv 1$ in $\operatorname{supp} \eta_i$,

$$\begin{aligned}
\left(\int_{B(0,1)} |\eta_i(u-\mu_i)_+|^p\right)^{\frac{2}{p}} &\overset{(10.3),\text{ Sob.}}{\leq} C\, 2^{2i} \int_{\operatorname{supp}\eta_i} |(u-\mu_i)_+|^2 \\
&= C\, 2^{2i} \int_{\operatorname{supp}\eta_i} \underbrace{\eta_{i-1}{}^2}_{\equiv 1} |(u-\mu_i)_+|^2 \qquad (10.4)\\
&\leq C 2^{2i} \int_{B(0,1)} (\eta_{i-1})^2\, |(u-\mu_i)_+|^2.
\end{aligned}$$

Now observe that $\mu_i > \mu_{i-1}$, so $u(x)-\mu_i < u(x)-\mu_{i-1}$ – and thus $|(u-\mu_i)_+|^2 \leq |(u-\mu_{i-1})_+|^2$

Then we have obtained

$$\left(\int_{B(0,1)} |\eta_i(u-\mu_i)_+|^p\right)^{\frac{1}{p}} \leq C 2^i \left(\int_{B(0,1)} |\eta_{i-1}(u-\mu_{i-1})_+|^2\right)^{\frac{1}{2}}. \tag{10.5}$$

Observe, on the left-hand side $p > 2$! So again this is an inverse Hölder inequality! We have from (normal) Hölder inequality.

(10.6)

$$\begin{aligned}
\left(\int_{B(0,1)} |\eta_i(u-\mu_i)_+|^2\right)^{\frac{1}{2}} &= \left(\int_{\{\eta_i(u-\mu_i)_+>0\}} |\eta_i(u-\mu_i)_+|^2\right)^{\frac{1}{2}} \\
&\leq |\{\eta_i(u-\mu_i)_+ > 0\}|^{\frac{1}{2}-\frac{1}{p}} \left(\int_{B(0,1)} |\eta_i(u-\mu_i)_+|^p\right)^{\frac{1}{p}} \\
&\overset{(10.5)}{\leq} C 2^i\, |\{\eta_i(u-\mu_i)_+ > 0\}|^{\frac{1}{2}-\frac{1}{p}} \left(\int_{B(0,1)} |\eta_{i-1}(u-\mu_{i-1})_+|^2\right)^{\frac{1}{2}}.
\end{aligned}$$

Now if $x \in B(0,1)$ such that $\eta_i(x)(u-\mu_i)_+(x) > 0$ then $\eta_i(x) > 0$, so in particular $\eta_{i-1}(x) = 1$. Moroever, we must have $u(x) > \mu_i$, so

$$u(x) - \mu_{i-1} > \mu_i - \mu_{i-1} = 2^{-i}.$$

That is we have

$$\{\eta_i(u-\mu_i)_+ > 0\} \subset \{\eta_{i-1}(u-\mu_{i-1})_+ > 2^{-i}\}$$

and thus (by Chebychev's inequality)

$$|\{\eta_i(u-\mu_i)_+ > 0\}| \leq 2^{2i} \int_{B(0,1)} (\eta_{i-1}(u-\mu_{i-1})_+)^2. \tag{10.7}$$

Combining this with (10.6),

$$\left(\int_{B(0,1)} |\eta_i(u-\mu_i)_+|^2\right)^{\frac{1}{2}} \leq C 2^i \left(2^{2i}\right)^{\frac{1}{2}-\frac{1}{p}} \left(\int_{B(0,1)} |\eta_{i-1}(u-\mu_{i-1})_+|^2\right)^{\frac{1}{2}+\frac{1}{2}-\frac{1}{p}}.$$

That is

$$\left(\int_{B(0,1)} |\eta_i(u-\mu_i)_+|^2\right)^{\frac{1}{2}} \leq C2^i \left(2^{2i}\right)^{\frac{1}{2}-\frac{1}{p}} \left(\int_{B(0,1)} |\eta_{i-1}(u-\mu_{i-1})_+|^2\right)^{\frac{1}{2}\left(2-\frac{2}{p}\right)}.$$

That is, for some $\Gamma > 0$ (depending only on irrevant data) and setting $\gamma := 1 - \frac{2}{p}$ we have shown that for any $i \in \mathbb{N}$.

$$\|\eta_i(u-\mu_i)_+\|_{L^2(B(0,1))} \leq \Gamma^i \|\eta_{i-1}(u-\mu_{i-1})_+\|_{L^2(B(0,1))}^{1+\gamma}$$

By Exercise 10.21 if δ is small enough, our assumption (10.2) implies that

$$\lim_{i\to\infty} \|\eta_i(u-\mu_i)_+\|_{L^2(B(0,1))} = 0.$$

We are ready to conclude: If there was some $\sigma > 1$ and a measurable set $A \subset B(0, 1/2)$ such that $u \geq \sigma$ in A, then we have

$$\underbrace{|\sigma - 1|}_{>0} \sqrt{\mathcal{L}^n(A)} \leq \|(u-\mu_i)_+\|_{L^2(A)} \leq \|\eta_i(u-\mu_i)_+\|_{L^2(B(0,1))} \xrightarrow{i\to\infty} 0.$$

so $\mathcal{L}^n(A) = 0$. Since

$$\{u > 1\} \cap B(0,1/2) = \bigcup_{\ell=1}^{\infty} \{u > 1 + \frac{1}{\ell}\} \cap B(0,1/2),$$

and the countable union of zero-sets is still a zero set, we conclude that $\mathcal{L}^n(\{u > 1\} \cap B(0,1/2)) = 0$ and thus $u \leq 1$ a.e. in $B(0,1/2)$.

$\square$

Exercise 10.21. *Show that for any $\Gamma > 0$, $\gamma > 0$ there exists $\delta > 0$ such that the following holds.*

Assume $(a_i)_{i=0}^{\infty}$ is a nonnegative sequence in $[0,\infty)$ with the following growth law

$$a_i \leq \Gamma^i \left(a_{i-1}\right)^{1+\gamma} \quad \forall i = 1, \ldots.$$

Assume moreover that $a_0 < \delta$.

Then $\lim_{i\to\infty} a_i = 0$.

10.3. **Hölder continuity (Moser's method).** Moser's technique to obtain Hölder continuity essentially is to test with negative powers $u^{-\beta}$, [Han and Lin, 2011, Chapter 4, Theorem 4.17], to obtain *Harnack's inequality*.

Lemma 10.22 (Composition and sub/supersolutions)**.** *Let $\Omega \subset \mathbb{R}^n$ be an open set, $A = (a_{ij}) : \Omega \to \mathbb{R}^{n\times n}$ be a bounded measurable (possible degenerate) elliptic matrix function, with constants $0 \leq \lambda < \Lambda < \infty$ and*

$$a_{ij}\xi_i\xi_j \geq \lambda|\xi|^2, \quad \forall \xi \in \mathbb{R}^n,\ x \in \Omega$$

and

$$|a_{ij}| \leq \Lambda \quad \forall x \in \Omega.$$

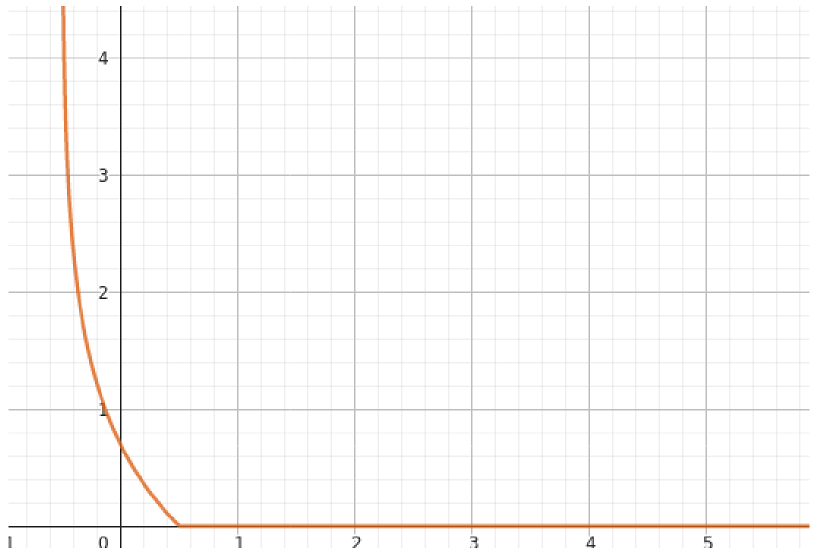


FIGURE 10.1. Φ from Corollary 10.23

Assume that $u \in W^{1,2}(\Omega)$ *and*

$$-\mathrm{div}\,(A\nabla u) \geq 0 \quad \text{in } \Omega.$$

Let $\Phi \in C^\infty(\mathbb{R})$ *be a convex function, which is moreover nonincreasing* ($\Phi' \leq 0$)*, with globally bounded first and second derivatives,* $\sup_{\mathbb{R}} |\Phi'| + |\Phi''| < \infty$*. Then* $\Phi(u) \in W^{1,2}(\Omega)$ *and we have*

$$-\mathrm{div}\,(A\nabla\, \Phi(u)) \leq 0 \quad \text{in } \Omega$$

Proof. Let $\varphi \in C_c^\infty(\Omega)$, $\varphi \geq 0$, then we have

$$\begin{aligned}\int a_{ij}\partial_i \Phi(u)\, \partial_j\varphi =& \int a_{ij}\Phi'(u)\partial_i u\, \partial_j \varphi\\ =& \int a_{ij}\partial_i u\, \partial_j\left(\Phi'(u)\varphi\right) - \int a_{ij}\partial_i u\, \Phi''(u)\partial_j u\varphi\\ =& -\int a_{ij}\partial_i u\, \partial_j\left(-\Phi'(u)\varphi\right) - \int a_{ij}\partial_i u\, \Phi''(u)\partial_j u\varphi\end{aligned}$$

Since $-\Phi'(u)\varphi \geq 0$ (and $\Phi'(u)\varphi \in W^{1,2}$ because Φ'' is bounded) we have from the PDE

$$-\int a_{ij}\partial_i u\, \partial_j\left(-\Phi'(u)\varphi\right) \leq 0.$$

Also, by ellipticity ($\Phi'' \geq 0$ by convexity)

$$-\int a_{ij}\partial_i u\, \Phi''(u)\partial_j u\varphi \leq -\lambda \int |\nabla u|^2 \Phi''(u)\varphi \leq 0$$

So we have

$$\int a_{ij}\partial_i \Phi(u)\, \partial_j \varphi \leq 0$$

$\square$

By approximation, we also obtain

Corollary 10.23. *Let A be as in Lemma 10.22. Consider for any fixed $\delta > 0$ $\Phi(t) = -\min\{0, \log(t+\delta)\}$. Cf. Figure 10.1. Assume that $u \in W^{1,2}(\Omega)$, $u \geq 0$ and*

$$-\mathrm{div}\,(A\nabla u) \geq 0 \quad \text{in } \Omega.$$

Then $\Phi(u) \in W^{1,2}(\Omega)$ and

$$-\mathrm{div}\,(A\nabla \Phi(u)) \leq 0.$$

Proof. Set

$$v := (\log(u+\delta))_-$$

Observe since $u \geq 0$ we know that $0 \leq v \leq \log 1/\delta$, and thus

$$v = (\log(u+\delta))_- \equiv -\min\{\log u, 0\} \in W^{1,2}(B(0,1)).$$

Also, as we see from the picture, Figure 10.1, we can approximate Φ by smooth nondecreasing Φ_ε, with bounded first and second derivatives (just mollify the kink). More precisely, we may assume

- $\Phi_\varepsilon \to \Phi$ uniformly in $(-\frac{\delta}{2}, \infty)$.
- $\sup_\varepsilon \sup_{[0,\infty)} |\Phi_\varepsilon'| < \infty$
- $\sup_{[0,\infty)} |\Phi_\varepsilon''| \leq C(\varepsilon)$

We then see that

$$\Phi_\varepsilon(u) \xrightarrow{\varepsilon \to 0} \Phi(u) \quad \text{a.e. in } \Omega$$

On the other hand we have

$$\sup_\varepsilon \|\Phi_\varepsilon(u)\|_{W^{1,2}(\Omega)} \lesssim \|u\|_{W^{1,2}(\Omega)} < \infty,$$

so (up to subsequence) we can assume $\Phi_\varepsilon(u)$ weakly converges to $\Phi(u)$ w.r.t $W^{1,2}(\Omega)$-topology.

But then for any nonnegative $\varphi \in C_c^\infty(\Omega)$ from Lemma 10.22 we obtain that

$$\int_\Omega a_{ij}\partial_i \Phi(u)\ \partial_j\varphi = \lim_{\varepsilon \to 0} \int_\Omega a_{ij}\partial_i \Phi_\varepsilon(u)\ \partial_j\varphi \leq 0.$$

We can conclude. □

Both, De Giorgi's method and Moser's method rely on the following *density theorem*. It should be interpreted as a control on oscillation.

Lemma 10.24 (Density theorem)**.** *Let $u \in W^{1,2}(B(0,2))$ with*

$$\inf_{B(0,2)} u \geq 0$$

and

$$-\mathrm{div}\,(A\nabla u) \geq 0 \quad \text{in } B(0,2)$$

such that

$$\mathcal{L}^n(B(0,1) \cap \{u \geq 1\}) \geq \mu\, \mathcal{L}^n(B(0,1))$$

for some $\mu > 0$. *Then there exists a constant* $\gamma = \gamma(\mu, n, \Lambda, \lambda) \in (0,1)$ *such that*

$$\inf_{B(0,1/2)} u \geq \gamma.$$

Proof. Fix $\delta \in (0,1)$ and set

$$v := (\log(u+\delta))_- \equiv -\min\{\log u, 0\} \in W^{1,2}(B(0,1)).$$

Observe since $u \geq 0$ we know that $0 \leq v \leq \log 1/\delta$.

By Corollary 10.23, v is a subsolution, i.e.

$$-\mathrm{div}\,(A\nabla v) \leq 0 \quad \text{in } B(0,2).$$

By the boundedness result, Theorem 10.15, we have (recall $v \geq 0$)

$$\sup_{B(0,1/2)} v \lesssim \|v\|_{L^2(B(0,1))}.$$

Now observe that

$$\{v = 0\} = \{u + \delta \geq 1\} \subset \{u \geq 1\},$$

and thus we have from the assumptions

$$\mathcal{L}^n\left(\{v = 0\} \cap B(0,1)\right) \geq \mu \mathcal{L}^n(B(0,1)).$$

Thus we work in a Poincaré-Lemma type cone, i.e. we have Poincaré lemma, see Exercise 10.25, i.e.

$$\|v\|_{L^2(B(0,1))} \leq C(\mu, n)\, \|\nabla v\|_{L^2(B(0,1))}.$$

So we have

$$\sup_{B(0,1/2)} v \leq C(\mu, n) \|\nabla v\|_{L^2(B(0,1))}. \tag{10.8}$$

We need to show the right-hand side is bounded. Fix $\eta \in C_c^\infty(B(0,2))$, $\eta \geq 0$ and $\eta \equiv 1$ in $B(0,1)$ the typical bump function. Then $\frac{\eta}{u+\delta} \in W_0^{1,2}(B(0,2))$, and thus[27]

$$\begin{aligned}
0 &\leq \int a_{ij} \partial_i u\, \partial_j \left(\frac{\eta^2}{u+\delta}\right) \\
&= -\int a_{ij}\, \partial_i u\, \frac{\eta^2}{(u+\delta)^2}\, \partial_j u + 2\int a_{ij} \frac{\partial_i u}{u+\delta}\, \eta \partial_j \eta \\
&= -\int a_{ij}\, \partial_i\, (u+\delta)\ \frac{\eta^2}{(u+\delta)^2}\, \partial_j\, (u+\delta) + 2\int a_{ij} \frac{\partial_i\, (u+\delta)}{u+\delta}\, \eta \partial_j \eta \\
&= -\int a_{ij}\, \partial_i v\, \eta^2\, \partial_j v + 2\int a_{ij}\, \partial_i v\, \eta \partial_j \eta
\end{aligned}$$

[27]Observe, similar to the boundedness in Moser's method follows from testing with u^β for $\beta > 0$, now we *de facto*test with u^{-1}

That is, by ellipticity, and Young's inequality for any $\varepsilon > 0$

$$\lambda \int \eta^2 |\nabla v|^2 \leq C(\Lambda) \int \eta |\nabla v| \, |D\eta| \leq \varepsilon \int \eta^2 |\nabla v|^2 + C(\Lambda, \varepsilon) \int |D\eta|^2.$$

Taking $\varepsilon < \lambda$ we can absorb and obtain

$$\int \eta^2 |\nabla v|^2 \leq C(\Lambda, \lambda) |D\eta|^2.$$

Plugging this into (10.8), using that $\eta \equiv 1$ in $B(0,1)$ (and that η is fixed), we have found

$$\sup_{B(0,1/2)} v \leq C(\mu, n\lambda, \Lambda)$$

Observe that nothing depends on $\delta > 0$ here!

By definition of v, we conclude that in $B(0,1/2)$

$$-\min\{\log(u+\delta), 0\} \leq C(\mu, n, \lambda, \Lambda)$$

thus

$$\log(u+\delta) \geq -C(\mu, n, \lambda, \Lambda)$$

and thus

$$u + \delta \geq e^{-C(\mu,n,\lambda,\Lambda)} =: \gamma.$$

This holds for any $\delta > 0$, and thus letting $\delta \to 0^+$ we conclude. □

Exercise 10.25. *Prove the following version of Poincaré inequality:*

For any $\mu > 0$, $p \in (1,\infty)$ and $\Omega \subset \mathbb{R}^n$ open and bounded with smooth boundary there exists a constant $C = C(\mu, p, \Omega)$ such that whenever $f \in W^{1,p}(\Omega)$ and

$$\mathcal{L}^n\left(\{x \in \Omega : f(x) = 0\}\right) \geq \mu \mathcal{L}^n(\Omega).$$

then

$$\|f\|_{L^p(\Omega)} \leq C \, \|\nabla f\|_{L^p(\Omega)}$$

Hint: Use Theorem 5.20.

Replacing u by $1-u$ we see that an equivalent formulation of Lemma 10.24 is

Lemma 10.26 (Density Theorem (Revisited))**.** *Let A be as above and let $\mu > 0$. There exists $\gamma > 0$ depending only on λ, Λ, n and μ such that the following holds.*

Assume $u \in W^{1,2}(B(0,2))$ satisfies

$$\sup_{B(0,2)} u \leq 1, \quad -\operatorname{div}(A\nabla u) \leq 0 \quad \text{in } B(0,2).$$

If

$$|\{u \leq 0\} \cap B(0,1)| \geq \mu$$

then

$$\sup_{B(0,1/2)} u \leq 1 - \gamma.$$

As we have discussed before, Example 2.26, while boundedness is about the size of u, Hölder continuity is about the *oszillation* of u. And if we had a decay estimate before for the boundedness, we will prove now a decay estimate for the oscillation. Recall that

$$\operatorname{osc}_A u := \sup_A u - \inf_A u.$$

(Both notions are *essential* sups and infs) The oscillation is always finite in our situation, by boundedness, Theorem 10.3.

Observe, that for (one-sided) boundedness we only needed an PDE *inequality*. For Hölder continuity we need a real solution.

The following is all we need to conclude the proof of Theorem 10.1, and it is a consequence of Lemma 10.26.

Proposition 10.27. *Fix $A(x) = (a_{ij}(x))_{i,j=1}^n$ bounded and measurable, uniformly elliptic and bounded with ellipticity constants $\lambda, \Lambda > 0$, i.e.*

$$\lambda|\xi|^2 \leq a_{ij}\xi_i\xi_j \quad \forall \xi \in \mathbb{R}^n, \quad |a_{ij}| \leq \Lambda$$

There exists a small $\theta \in (0,1)$ depending only on n, λ, Λ such that the following holds.

Assume $u \in W^{1,2}(B(0,2))$ is a solution to

$$-\operatorname{div}(A\nabla u)=0 \quad \text{in } B(0,2),$$

in distributional sense, i.e.

$$\int_{B(0,1)} a_{ij}\partial_i u\, \partial_j \varphi = 0$$

for all $\varphi \in C_c^\infty(B(0,2))$ in $B(0,2)$.

Then

$$\operatorname{osc}_{B(0,1/2)} u \leq (1-\theta) \operatorname{osc}_{B(0,2)} v$$

Exercise 10.28. *Show that Theorem 10.3 combined with Proposition 10.27 implies Theorem 10.1.*

Hint: Cf. Example 2.26.

Proof of Proposition 10.27. Set

$$w(x) := \frac{2}{\operatorname{osc}_{B(0,2)} u}\left(u(x) - \frac{\sup_{B(0,2)} u + \inf_{B(0,2)} u}{2}\right).$$

Then

$$-1 \leq w \leq 1 \quad \text{in } B(0,2),$$

and

$$\operatorname{div}(A\nabla w) = 0.$$

One of the following must be true

$$|\{w \leq 0\} \cap B(0,1)| \geq \frac{1}{2}|B(0,1)|, \quad \text{or} \quad |\{-w \leq 0\} \cap B(0,1)| \geq \frac{1}{2}|B(0,1)|.$$

By Lemma 10.26 we then obtain one of the following

$$\sup_{B(0,1/2)} w \leq 1-\gamma \quad \text{or} \sup_{B(0,1/2)} -w \leq 1-\gamma.$$

that is

$$\sup_{B(0,1/2)} w \leq 1-\gamma \quad \text{or} - \inf_{B(0,1/2)} w \leq 1-\gamma.$$

In either case we then have (recall that we still have $-1 \leq w \leq 1$)

$$\operatorname*{osc}_{B(0,1/2)} w = \sup_{B(0,1/2)} w - \inf_{B(0,1/2)} w \leq 2-\gamma.$$

By the definition of w this implies

$$\operatorname*{osc}_{B(0,1/2)} \left(u \frac{2}{\operatorname{osc}_{B(0,2)} u} \right) \leq 2-\gamma.$$

and thus

$$\operatorname*{osc}_{B(0,1/2)} u \leq \frac{2-\gamma}{2} \operatorname*{osc}_{B(0,2)} u.$$

This is the claim and we can conclude. □

10.4. **Hölder continuity (De Giorgi's method).** For DeGiorgi's version of the proof of Lemma 10.26, we begin by what is sometimes referred to as *De Giorgi's isoperimetric inequality*. We already know that $W^{1,2}$-functions cannot jump, this lemma quantifies this.

Lemma 10.29 (De Giorgi's insoperimetric inequality). *Let $f \in W^{1,2}(B(0,1))$. Then*

$$\left(\mathcal{L}^n(\{f \leq 0\})\right) \left(\mathcal{L}^n(\{f \geq 1\})\right) \leq C(n) \left(\mathcal{L}^n(\{0 < f < 1\})\right)^{\frac{1}{2}} \|\nabla f\|_{L^2(B(0,1))}.$$

Proof. Set

$$g := \min\{\max\{f,0\},1\} \in W^{1,2}(B(0,1)).$$

Then

$$\begin{aligned}
\mathcal{L}^n(\{f \leq 0\}) \left(\mathcal{L}^n(\{f \geq 1\})\right) &= \int_{\{f\leq 0\}} \int_{\{f \geq 1\}} |g(x)-g(y)| \, dx \, dy \\
&\leq \int_{B(0,1)} \int_{B(0,1)} |g(x)-g(y)| \, dx \, dy
\end{aligned}$$

Denote by $(g)_{B(0,1)} := |B(0,1)|^{-1} \int_{B(0,1)} g$. Then from triangular inequality and Poincaré inequality,

$$\begin{aligned}
\mathcal{L}^n(\{f \leq 0\}) \left(\mathcal{L}^n(\{f \geq 1\})\right) \leq& 2 \int_{B(0,1)} |g(z) - (g)_{B(0,1)}| \, dz \\
\leq& C(n) \int_{B(0,1)} |\nabla g|
\end{aligned}$$

Now observe, Exercise 10.11, $\nabla g = \nabla f$ a.e. in $\{0 < f < 1\}$ and $\nabla g = 0$ a.e. anywhere else. So we have

$$\int_{B(0,1)} |\nabla g| = \int_{\{0<f<1\}} |\nabla f| \leq \mathcal{L}^n \left(\{0 < f < 1\}\right)^{\frac{1}{2}} \|\nabla f\|_{L^2(B(0,1))}.$$

We can conclude. □

Proof of the density theorem, Lemma 10.26. For $k \in \mathbb{N}$ set

$$w_k := 2^k(u - (1 - 2^{-k}))_+.$$

We first collect some properties of w_k

- we still have $w_k \leq 1$ in $B(0,2)$ (since $u \leq 1$ in $B(0,2)$)
- we also have

  $$\{w_k \leq 0\} \cap B(0,1) = \{u \leq (1 - 2^{-k})\} \cap B(0,1) \supset \{u \leq 0\} \cap B(0,1)$$

  so that

$$|\{w_k \leq 0\} \cap B(0,1)| \geq \mu. \tag{10.9}$$

- We can write $w_k = (2^k u - 2^k + 1)_+$, and conclude that

  $$\begin{aligned} &w_k \geq \frac{1}{2}\\ \Leftrightarrow &2w_k - 1 \geq 0\\ \Leftrightarrow &w_{k+1} \geq 0, \end{aligned}$$

  i.e.

$$\{w_k \geq \frac{1}{2}\} = \{w_{k+1} \geq 0\} \tag{10.10}$$

- We have $w_{k+1} = (2w_k - 1)_+$. Hence,

  $$w_k(x) < \frac{1}{2} \quad \Rightarrow w_{k+1}(x) = 0.$$

  This implies that

$$\{0 < w_k(x) < \frac{1}{2}\} \cap \{0 < w_j(x) < \frac{1}{2}\} = \emptyset \quad k \neq j \tag{10.11}$$

- For any $\mu \geq 0$, taking a cutoff-function $\eta \in C_c^\infty(B(0,2))$, $\eta \equiv 1$, and applying Lemma 10.18 to $\eta(u - \mu)_+$ we obtain

  $$\int_{B(0,1)} |\nabla (u - \mu)_+|^2 \leq C(n) \int_{B(0,2)} |(u - \mu)_+|^2.$$

  In particular,

$$\int_{B(0,1)} |\nabla w_k|^2 \leq C(n) \int_{B(0,2)} |w_k|^2 \leq C(n)|B(0,2)|. \tag{10.12}$$

- From De Giorgi's isoperimetric inequality, Lemma 10.29, we obtain

$$(\mathcal{L}^n(\{w_k \leq 0\}))^2 \left(\mathcal{L}^n(\{w_k \geq \frac{1}{2}\})\right)^2 \leq C(n)\, \mathcal{L}^n(\{0 < w_k < \frac{1}{2}\})\, \|\nabla w_k\|^2_{L^2(B(0,1))}.$$

In view of (10.12), (10.10), (10.9) we conclude

$$\begin{aligned} C(n)\, \mathcal{L}^n(\{0 < w_k < \frac{1}{2}\} \cap B(0,1)) \geq& (\mathcal{L}^n(\{w_k \leq 0\}))^2 \left(\mathcal{L}^n(\{w_k \geq \frac{1}{2}\}) \cap B(0,1)\right)^2 \\ \geq& \mu \left(\mathcal{L}^n(\{w_{k+1} \geq 0\}) \cap B(0,1)\right)^2 \end{aligned}$$

Since $0 \leq w_{k+1} \leq 1$ we have

$$\mathcal{L}^n(\{w_{k+1} \geq 0\} \cap B(0,1)) \geq \int_{B(0,1)} (w_{k+1})^2.$$

so that we have

$$\int_{B(0,1)} (w_{k+1})^2 \leq \frac{C(n)}{\mu}\, \mathcal{L}^n(\{0 < w_k < \frac{1}{2}\} \cap B(0,1)). \tag{10.13}$$

Fix now some $\delta > 0$ (to be specified later). We claim that there exists a number k_0 (depending on δ, n, Λ, λ, but otherwise independent) so that for some $\bar{k} \in \{1, \ldots, k_0\}$ we have

$$\int_{B(0,1)} (w_{\bar{k}})^2 < \delta^2.$$

Indeed if we have

$$\int_{B(0,1)} (w_k)^2 \geq \delta^2 \quad \text{for all } k = 1, \ldots, k_0$$

we conclude from (10.13) that

$$\frac{\mu}{C(n)}\, \delta^2 \leq \mathcal{L}^n(\{0 < w_k < \frac{1}{2}\} \cap B(0,1)) \quad \text{for all } k = 1, \ldots, k_0$$

But by (10.11) we have disjointness, so

$$k_0 \frac{\mu}{C(n)}\, \delta^2 \leq \sum_{k=1}^{k_0} \mathcal{L}^n(\{0 < w_k < \frac{1}{2}\} \cap B(0,1)) \overset{(10.11)}{\leq} \mathcal{L}^n(B(0,1)).$$

This leads to a contradiction if

$$k_0 := \mathcal{L}^n(B(0,1)) \frac{C(n)}{\mu\, \delta^2}$$

So there must be some $\bar{k} \in \{1, \ldots, k_0\}$ such that

$$\int_{B(0,1)} (w_{\bar{k}})^2 < \delta^2.$$

Since $w_{\bar{k}} = (2^{\bar{k}} u - 2^{\bar{k}} + 1)_+$ and $(2^{\bar{k}} u - 2^{\bar{k}} + 1)$ is a subsolution, we have boundedness, Theorem 10.15. That is we have

$$\|w_{\bar{k}}\|_{L^\infty(B(0,1/2))} \leq C(n, \Lambda, \lambda)\, \|w_{\bar{k}}\|_{L^2(B(0,1/2))} < C(n, \Lambda, \lambda)\delta.$$

So if we choose δ small enough, we can ensure that there exists some $\bar{k} \in \{1, \ldots, k_0\}$ (where k_0 is a constant depending only on permissible data), such that

$$\|w_{\bar{k}}\|_{L^\infty(B(0,1/2))} \leq \frac{1}{2}$$

But then for $x \in B(0, 1/2)$

$$\left(u(x) - 1 + 2^{-\bar{k}}\right)_+ = 2^{-\bar{k}} w_{\bar{k}} < 2^{-1-\bar{k}}.$$

and thus

$$u_+(x) \leq 2^{-1-\bar{k}} + 1 - 2^{-\bar{k}} = 1 - \left(2^{-\bar{k}} - 2^{-1-\bar{k}}\right).$$

Setting

$$\gamma := \min_{k=1,\ldots,k_0} \left(2^{-k} - 2^{-1-k}\right) > 0$$

(and observe once more that k_0 only depends on the data) we conclude. $\square$

As a remark in passing (see [Fernández-Real and Ros-Oton, 2022]) a version *Harnack's inequality*, called Moser's Harnack inequality (see [Moser, 1961], [Han and Lin, 2011, Chapter 4.4]) can be proven in a similar fashion, cf. [Li and Zhang, 2017].

11. A semilinear equation, mountain pass theorem, and nonexistence

Let $\Omega \subset \mathbb{R}^n$ be a nicely bounded set and $p \in (1, \infty)$. We consider the *semilinear* equation[28]

$$\begin{cases} -\Delta u = |u|^{p-1} u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{11.1}$$

Clearly $u \equiv 0$ is a solution to (11.1), but we might want to find a *nontrivial solution*, i.e. $u \not\equiv 0$.

First we observe that the sign on the right-hand side of the equation is extremely important to make the above question meaningful.

Exercise 11.1. *Let $u \in W_0^{1,2}(\Omega) \cap L^{p+1}(\Omega)$, $p \in (1, \infty)$. Show that if u solves*

$$\begin{cases} -\Delta u = -|u|^{p-1} u & \textit{in } \Omega \\ u = 0 & \textit{on } \partial\Omega, \end{cases} \tag{11.2}$$

then $u \equiv 0$.

Hint: Multiply (11.2) by u and integrate by parts.

[28] This equation, also called *Lane-Emden equation*, is related to the asymptotic behavior of the *porous medium equation*. This is still an actively investigated PDE with many open questions, see e.g. [Brasco et al., 2022]

We will discuss the following results[29]

- If $1 < p < \frac{n+2}{n-2}$ there exists a nontrivial solution u to (11.1)
- (Under geometric assumptions on Ω), if $p > \frac{n+2}{n-2}$ there is *no nontrivial* solution to (11.1).

What changes for $p < \frac{n+2}{n-2}$ to $p > \frac{n+2}{n-2}$?

For formal considerations, we observe, that by testing (11.1) with u, we have

$$\int_\Omega |\nabla u|^2 = \int_\Omega |u|^{p+1}.$$

By Sobolev embedding,

$$W_0^{1,2}(\Omega) \hookrightarrow L^{\frac{2n}{n-2}}(\Omega) = L^{\frac{n+2}{n-2}+1}(\Omega)$$

So if $p < \frac{n+2}{n-2}$ then $W_0^{1,2}(\Omega) \hookrightarrow L^{p+1}$ is *compact*, and if $p > \frac{n+2}{n-2}$ then $W_0^{1,2}(\Omega) \not\hookrightarrow L^{p+1}$.

We call the case $p < \frac{n+2}{n-2}$ *subcritical* for (11.1) (the left-hand side of (11.1) dominates the right-hand side). If $p > \frac{n+2}{n-2}$ we call *supercritical* for (11.1) (the right-hand side dominates), and the case $p = \frac{n+2}{n-2}$ we call *critical* (neither sides dominates).

We begin by the observation that solutions to (11.1) are critical points of an energy E (in the variational sense).

Exercise 11.2. *Set*

$$\mathcal{E}(u) := \frac{1}{2}\int_\Omega |\nabla u|^2 - \frac{1}{p+1}\int_\Omega |u|^{p+1}.$$

Set $X := \{v \in W_0^{1,2}(\Omega) \cap L^{p+1}(\Omega)\}$. Show that if u is a minimizer of $\mathcal{E}$ in X, i.e. $u \in X$ and $\mathcal{E}(u) \leq \mathcal{E}(v)$ for all $v \in X$, then u solves (11.1) (in the distributional sense).

So we could just try to run the direct method of the Calculus of Variations to solve (11.1), find minimizers (hope they are nonzero) and then we are done?

Here is the problem: *there exists no minimizer of $\mathcal{E}$*:

Lemma 11.3. *Let $p > 1$ and set*

$$\mathcal{E}(u) := \frac{1}{2}\int_\Omega |\nabla u|^2 - \frac{1}{p+1}\int_\Omega |u|^{p+1}.$$

For any $u \in W_0^{1,2}(\Omega) \cap L^{p+1}(\Omega)$ with $u \not\equiv 0$, if we set

$$v_\lambda := \lambda u$$

we have

$$\lim_{\lambda \to \infty} \mathcal{E}(v_\lambda) = -\infty.$$

[29]We are vague here with the assumptions of what regularity u is – on purpose!

In particular

$$\inf_{u \in W_0^{1,2}(\Omega)} \mathcal{E}(u) = -\infty.$$

Exercise 11.4. *Prove Lemma 11.3.*

Instead the idea is to look not for minimizers, but for *critical point*s of $\mathcal{E}$. Just like for a function $f : \mathbb{R} \to \mathbb{R}$ we say that x_0 is a critical point of f if $f'(x_0) = 0$ (then x_0 *might* be a min or a max, but it could also be a saddle point). The method we shall use is called the *mountain pass theorem* (which – as usual is a very interesting technique on its own, applicable in many other situations – see in particular [Struwe, 2008]).

11.1. **Mountain Pass theorem: Finite dimensional case.** To illustrate the basic principle of the Mountain Pass theorem (or min-max method) we first restrict to finite dimensional cases. The following is taken from [Struwe, 2008], who credits [Courant, 1950]. It is a good idea to think of $E(u) \in \mathbb{R}$ as the elevation of a landscape at the point $u \in \mathbb{R}^d$.

Theorem 11.5 (Mountain Pass Theorem). *Let $E \in C^1(\mathbb{R}^d, \mathbb{R})$ be coercive: that is for any $\Lambda > 0$ assume that*

$$\{u \in \mathbb{R}^d : \quad E(u) < \Lambda\} \quad \text{is a bounded set in } \mathbb{R}^d.$$

Assume that there are $u_1 \neq u_2 \in \mathbb{R}^d$ at which E has strict local minima, i.e.

$$E(u_i) < E(v) \quad \forall v \approx u_i,\ v \neq u_i.$$

Then there exists $u_3 \in \mathbb{R}^3 \setminus \{u_1, u_2\}$ a critical point of E, i.e. $DE(u_3) = 0$, but u_3 is not a local minimum.

Indeed we have

$$E(u_3) = \inf_{\gamma \in \mathcal{P}} \max_{v \in \gamma} E(v) \tag{11.3}$$

where we call $\mathcal{P}$ the class of paths connecting u_1 and u_2. More precisely,

$$\mathcal{P} = \{\gamma \subset \mathbb{R}^d : \quad u_1, u_2 \in \gamma, \quad \gamma \text{ is compact and connected}\}.$$

Cf. Figure 11.1.

Exercise 11.6. *Draw a picture and prove (without using the argument below) Theorem 11.5 in one-dimension.*

That is, let $E \in C^1(\mathbb{R})$ and assume $u_1, u_2 \in \mathbb{R}$ are strict minima. Show that there exist either maxima (in higher dimensions it could be a saddle-point) $u_3 \in (u_1, u_2)$, i.e.

$$DE(u_3) = 0$$

and draw a typical graph of E. What is the meaning of (11.3) in this situation?

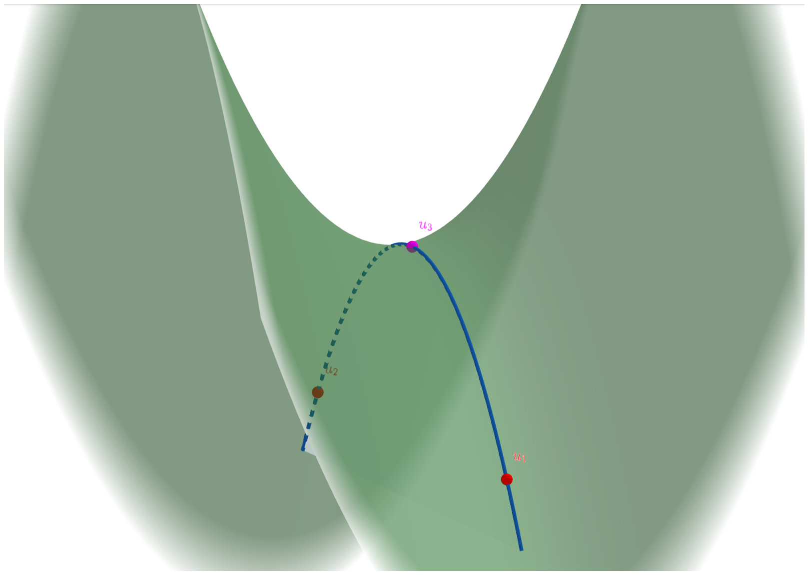


FIGURE 11.1. A rough illustration of the situation in Theorem 11.5

Proof of Theorem 11.5. Clearly $\mathcal{P}$ is nonempty. So we can find $(\gamma_k)_{k=1}^{\infty} \subset \mathcal{P}$ be a sequence of paths such that

$$\beta := \inf_{\gamma \in \mathcal{P}} \max_{v \in \gamma} E(v) = \lim_{k \to \infty} \max_{v \in \gamma_k} E(v).$$

We may assume that

$$\beta \leq \max_{v \in \gamma_k} E(v) \leq \beta + 1 \quad \forall k \in \mathbb{N}.$$

In particular,

$$\sup_{v \in \bigcup_k \gamma_k} E(v) < \infty,$$

and thus by coercivity we have that

$$\bigcup_{k=1}^{\infty} \gamma_k \quad \text{is bounded.}$$

Set now

$$\bar{\gamma} := \bigcap_{m \in \mathbb{N}} \overline{\bigcup_{\ell \geq m} p_\ell}. \tag{11.4}$$

It is clear that $u_1, u_2 \in \bar{\gamma}$, and $\bar{\gamma}$ is still compact. It is an exercise, Exercise 11.8, to show that $\bar{\gamma}$ is also connected. That is, $\bar{\gamma} \in \mathcal{P}$.

Thus,

$$\beta = \inf_{\gamma \in \mathcal{P}} \max_{v \in \gamma} E(v) \leq \max_{v \in \bar{\gamma}} E(v).$$

On the other hand, by continuity and the construction of $\bar{\gamma}$

$$\max_{v \in \bar{\gamma}} E(v) \leq \limsup_{m \to \infty} \max_{v \in \gamma_m} E(v) = \beta.$$

That is, we have shown

$$\max_{v\in\bar{\gamma}} E(v) = \beta \equiv \inf_{\gamma\in\mathcal{P}} \max_{v\in\gamma} E(v).$$

Let $u_3 \in \bar{\gamma}$ (exists by compactness and continuity) such that

$$E(u_3) = \max_{v\in\bar{\gamma}} E(v) = \inf_{\gamma\in\mathcal{P}} \max_{v\in\gamma} E(v).$$

Since $E(u_1), E(u_2)$ are strict local minima and $\bar{\gamma}$ is connected, we conclude that

$$\beta = \max_{v\in\bar{\gamma}} E(v) > \max\{E(u_1), E(u_2)\},$$

and thus the set

$$K := \{u \in \bar{\gamma} : \quad E(u) = \beta\}$$

does not contain u_1 and u_2. Since $\bar{\gamma}$ is compact and E is continuous the set K is compact.

We claim that there exists $u_3 \in K$ which is a critical point, i.e. $DE(u_3) = 0$.

Assume this is not the case, then (since $Du(\cdot)$ is continuous and K is compact)

$$\delta := \frac{1}{2} \inf_K |DE| > 0.$$

By continuity of E, there is some small $\varepsilon > 0$ for which

$$B_\varepsilon(K) := \{u \in \mathbb{R}^d : \quad \operatorname{dist}(u, K) < \varepsilon\}$$

is an open set, *not containing u_1 and u_2*, and taking ε even smaller we can ensure that

$$|DE| > \delta \quad \text{in } B_\varepsilon(K).$$

The idea is now that we shift the path $\bar{\gamma}$ (including its surrounding $B_\varepsilon(K)$) to attain a strictly smaller value (which is a contradiction to the construction of $\bar{\gamma}$).

Let $\eta \in C_c^\infty(B_\varepsilon(K))$, $\eta \equiv 1$ in a small neighborhood of K, be the usual nonnegative bump function.

We define the *deformation*

$$\Phi(v, t) := v - t\eta(v)DE(v) \tag{11.5}$$

If we apply Φ to points in $\mathbb{R}^d$ then it moves only the points in $\operatorname{supp}\eta$, and those move into the direction of the negative gradient (by Taylor that should decrease the energy).

Indeed, we have

$$\begin{aligned}\frac{d}{dt}E(\Phi(v,t)) =& -\eta(v)\langle DE(v - t\eta(v)DE(v)), DE(v)\rangle \\ =& -\eta(v)|DE(v)|^2 + \eta(v)\langle DE(v) - DE(v - t\eta(v)DE(v)), DE(v)\rangle\end{aligned}$$

Now observe that if $\eta(v) \neq 0$ then $v \in U_\varepsilon$ and thus $|DE(v)| \geq \delta$. Thus,

$$\begin{aligned}\frac{d}{dt}E(\Phi(v,t)) =& -\eta(v)\langle DE(v-t\eta(v)DE(v)), DE(v)\rangle\\ \leq& -\eta(v)|DE(v)|^2 + \eta(v)\,|DE(v) - DE(v-t\eta(v)DE(v))|\,|DE(v)|\\ \leq& -\eta(v)|DE(v)|^2 + \eta(v)\frac{|DE(v) - DE(v-t\eta(v)DE(v))|}{\delta}|DE(v)|^2\\ =& -\eta(v)|DE(v)|^2\left(1 - \frac{|DE(v) - DE(v-t\eta(v)DE(v))|}{\delta}\right)\end{aligned}$$

Since DE is continuous (and the support of η is compact) we conclude that there exists some $\tau > 0$ such that

$$\frac{d}{dt}E(\Phi(v,t)) \leq -\frac{1}{2}\eta(v)|DE(v)|^2, \quad \forall t \in [-\tau,\tau].$$

In particular we have from the fundamental theorem,

$$\begin{aligned}E(\Phi(v,\tau)) \leq& E(v) + \int_0^\tau \left(-\frac{1}{2}\eta(v)|DE(v)|^2\right) dt\\ =& E(v) - \frac{\tau}{2}\eta(v)|DE(v)|^2\end{aligned}$$

We conclude that

$$\max_{v\in\bar{\gamma}} E(\Phi(v,\tau)) \leq \max_{v\in\bar{\gamma}} E(v) = \beta,$$

thus

$$\max_{v\in\bar{\gamma}} E(\Phi(v,\tau)) \leq \beta,$$

And actually, equality is impossible. Because if $v \in K$ (where we know that $\eta(v) = 1$)

$$E(\Phi(v,\tau)) \leq \underbrace{E(v)}_{=\beta} - \frac{\tau}{2}\delta^2 < \beta$$

and if $v \in \bar{\gamma} \setminus K$ then

$$E(\Phi(v,\tau)) \leq E(v) \overset{v\in\bar{\gamma}\setminus K}{<} \beta.$$

Thus,

$$\max_{v\in\bar{\gamma}} E(\Phi(v,\tau)) < \beta. \tag{11.6}$$

But observe that since $u_1, u_2 \notin B_\varepsilon(K)$ we have that $\Phi(u_1,\tau) = u_1$ and $\Phi(u_2,\tau) = u_2$, moreover $\Phi(\cdot,\tau)$ is continuous so $\bar{\gamma}_2$ is a permissible path,

$$\bar{\gamma}_2 := \Phi(\bar{\gamma},\tau) \in \mathcal{P}.$$

Moreover,

$$\max_{v\in\bar{\gamma}_2} E(v) \overset{(11.6)}{<} \beta \overset{\text{def}}{=} \inf_{\gamma\in\mathcal{P}} \max_{v\in\gamma} E(v)$$

This is a contradiction, so K must have at least one critical point u with $DE(u) = 0$ (which cannot be u_1, u_2 because they don't belong to K).

Denote the set of critical points in K by $\tilde{K}$,

$$\tilde{K} := \{u \in K : \quad DE(u) = 0\} \neq \emptyset.$$

To conclude, we need to find a point in $\tilde{K}$ which is not a local minimum (observe no strictness is assumed, so this is not obvious by the $\max_{\bar{\gamma}}$-definition!).

Yet again we assume to the contrary that all u in $\tilde{K}$ (which is nonempty) only consists of local minima.

If that was the case, for any $u \in \tilde{K}$ by the minimum property there exists a small ball $B(u, r)$ such that

$$\beta = E(u) \leq E(w) \quad \forall w \in B(u, r).$$

But then for any $v \in B(u, r) \cap \bar{\gamma}$ we have

$$\beta = E(u) \leq E(v) \overset{def}{\leq} \beta,$$

and thus $E(v) = E(u)$, which implies that v is also a local min:

$$E(v) = E(u) \leq E(w) \quad \forall w \in B(u, r).$$

Thus $DE(v) = 0$, which implies $v \in \tilde{K}$ – and thus $B(u, r) \cap \bar{\gamma} \subset \tilde{K}$. In particular, $\tilde{K}$ would be relatively open in $\bar{\gamma}$. On the other hand, let $u_k \in \tilde{K}$ converge to $u \in p$. Then we have

$$\beta = E(u_k) \xrightarrow{k \to \infty} E(u)$$

$$0 = DE(u_k) \xrightarrow{k \to \infty} DE(u),$$

that is $\tilde{K}$ also relatively closed in $\bar{\gamma}$. That is $\tilde{K} \subset \bar{\gamma}$ is a relatively closed and open set. Since $\bar{\gamma}$ is connected, this implies that $\tilde{K} = \bar{\gamma}$. But observe that $u_1, u_2 \in \bar{\gamma} \setminus K \subset \bar{\gamma} \setminus \tilde{K}$ so this is impossible.

Hence there must be at least on point $u_3 \in \tilde{K}$ which is not a local minimum.

We can conclude. $\square$

Remark 11.7. Struwe gives the following nice geometric interpretation (taken verbatim from [Struwe, 2008, p.76])

*It is useful to think of $E(u)$ as measuring the elevation at a point u in a landscape. Our two minima u_1, u_2 then correspond to two villages at the deepest points of two valleys, separated from each other by a mountain ridge. If now we walk along a path p from u_1 to u_2 with the property that the maximal elevation $E(u)$ at points u on p is minimal among all such paths we will cross the ridge at a mountain pass u_3 which is a saddle point of E. Because of this geometric interpretation Theorem 11.5 is sometimes called the finite dimensional "*mountain pass theorem*".*

Exercise 11.8. *Show that $\bar{\gamma}$ from (11.4) is indeed compact and connected.*

Our finite dimensional example above uses strongly that closed, bounded sets are compact – and thus obtains compactness from coercivity (and continuity). As we know this is an issue in infinite dimensions, and making assumptions of compactness would be to rigid. The assumption to mitigate this issue (introduced below after the definitions) will be the *Palais-Smale condition*.

11.2. **Mountain Pass theorem: Infinite dimensional case.** Recall that we would like to find a nontrivial solution of

$$\begin{cases} -\Delta u = |u|^{p-1}u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{11.1}$$

In Exercise 11.2 we have observed that solutions to (11.1) are indeed critical points of an energy E (in the variational sense), given by

$$\mathcal{E}(u) := \frac{1}{2}\int_\Omega |\nabla u|^2 - \frac{1}{p+1}\int_\Omega |u|^{p+1}.$$

We want to use the mountain pass theorem for this energy – but we first need to extend the relevant notions and ideas to infinite dimensions.

That is, instead of working in $\mathbb{R}^d$ as in the previous section, we will now work in $W^{1,2}(\Omega)$, where $\Omega \subset \mathbb{R}^n$.

Throughout this section we will assume that the dimension $n \geq 2$ (this is not a serious restriction, it is easy to adapt what we do here to $n = 1$ but thats not the most interesting case).

For simplicity we restrict to a Hilbert space since we care about $W_0^{1,2}(\Omega)$ – we refer to [Struwe, 2008] for the Banach space version, which is not much more difficult (but the notation becomes less pleasant).

Definition 11.9. Let H be a Hilbert space, and assume $\mathcal{E} : H \to \mathbb{R}$ is a (nonlinear) functional on H.

We say that $\mathcal{E}$ is *differentiable* at a point $u \in H$ if there exists a vector $D\mathcal{E}(u) \in H$ such that for all $w \in H$, we have a first order Taylor approximation,

$$\mathcal{E}(w) = \mathcal{E}(u) + \langle D\mathcal{E}(u), w - u\rangle + o(\|w - u\|).$$

This is the same as *Fréchet differentiable*.

Observe that if we wanted to define differentiability of a functional $\mathcal{E}$ on a *Banach space* X then $D\mathcal{E}(u)$ would belong to the dual space X^*, which complicates notation, so we don't want to do that. Since H is a Hilbert space, H^* can be identified with H via Riesz representation theorem and the scalar product, which is what we do above.

Exercise 11.10. *Show that if such a map $D\mathcal{E}(u)$ exists, then it is unique.*

Exercise 11.11. *Show that for any differentiable $\mathcal{E}$ as in Definition 11.9, if u is a local minimizer and if $\mathcal{E}$ is differentiable at u then $D\mathcal{E}(u) = 0$.*

Exercise 11.12. *Assume $\mathcal{E} : \mathbb{R}^d \to \mathbb{R}$ is a C^1 function. Show that $D\mathcal{E}(u) = \nabla\mathcal{E}(u)$ (where ∇ denotes the usual gradient).*

Example 11.13. Let $p \in (1, \frac{n+2}{n-2}]$ ($p < \infty$ if $n = 2$). Then

$$\mathcal{E}(u) := \frac{1}{2}\int_\Omega |\nabla u|^2 - \frac{1}{p+1}\int_\Omega |u|^{p+1}$$

is differentiable in the sense of (11.9) for $H = W_0^{1,2}(\Omega)$, where

$$\langle D\mathcal{E}(u), v\rangle = \left(-\Delta u - |u|^{p-1}u\right)[v] = \int_\Omega \nabla u \cdot \nabla v - |u|^{p-1}uv \quad \forall v \in H \tag{11.7}$$

Proof. Since $p \le \frac{n+2}{n-2}$ we observe that $W_0^{1,2}(\Omega) \subset L^{p+1}(\Omega)$ by Sobolev embedding.

We first observe that there is exactly one $D\mathcal{E}(u) \in W_0^{1,2}(\Omega)$ such that (11.7) holds. Clearly,

$$W_0^{1,2}(\Omega) \ni v \mapsto \int_\Omega \nabla u \cdot \nabla v - |u|^{p-1}uv$$

is a bounded linear functional on $W_0^{1,2}(\Omega)$. Since $W_0^{1,2}(\Omega)$ is a Hilbert space, by Riesz representation theorem for Hilbert spaces there exists a unique $f \in W_0^{1,2}(\Omega)$ such that

$$\langle f, v\rangle_{W_0^{1,2}(\Omega)} = \int_\Omega \nabla u \cdot \nabla v - |u|^p uv \quad \forall v \in W_0^{1,2}(\Omega).$$

We simply set $D\mathcal{E}(u) := f$.

So, all we need to show is

$$\mathcal{E}(w) - \mathcal{E}(u) - \int_\Omega \nabla u \cdot \nabla(w-u) - |u|^{p-1}u(w-u) = o(\|w-u\|) \tag{11.8}$$

We have

$$\begin{aligned}
&\mathcal{E}(w) - \mathcal{E}(u) - \int_\Omega \nabla u \cdot \nabla(w-u) - |u|^{p-1}u(w-u)\\
=&\frac{1}{2}\int_\Omega |\nabla w|^2 - \frac{1}{2}\int_\Omega |\nabla u|^2 - \int_\Omega \nabla u \cdot \nabla(w-u)\\
&-\frac{1}{p+1}\int_\Omega |w|^{p+1} + \frac{1}{p+1}\int_\Omega |u|^{p+1} + \int_\Omega |u|^{p-1}u(w-u)\\
=&\frac{1}{2}\int_\Omega |\nabla(w-u)|^2\\
&-\int_\Omega \left(\frac{1}{p+1}|w|^{p+1} - \frac{1}{p+1}|u|^{p+1} - |u|^{p-1}u(w-u)\right)
\end{aligned}$$

For the last term consider

$$\varphi(t) := \frac{1}{p+1}|t|^{p+1}$$

Since $p > 1$, $\varphi \in C^2(\mathbb{R})$, and thus

$$|\varphi(w) - \varphi(u) - \varphi'(u)(w-u)| \leq \max\{|w|, |u|\}^{p-1}|w-u|^2$$

So, we have shown (using Hölder's inequality)

$$\begin{aligned}
&\left|\mathcal{E}(w) - \mathcal{E}(u) - \int_\Omega \nabla u \cdot \nabla(w-u) - |u|^{p-1}u(w-u)\right| \\
\precsim &\|\nabla(w-u)\|_{L^2(\Omega)}^2 + \left(\|w\|_{L^{p+1}(\Omega)}^{p-1} + \|u\|_{L^{p+1}(\Omega)}^{p-1}\right) \|w-u\|_{L^{p+1}(\Omega)}^2 \\
\precsim &\|\nabla(w-u)\|_{L^2(\Omega)}^2 + \left(\|\nabla w\|_{L^2(\Omega)}^{p-1} + \|\nabla u\|_{L^2(\Omega)}^{p-1}\right) \|\nabla(w-u)\|_{L^2(\Omega)}^2 \\
\precsim &\left(1 + \|\nabla w\|_{L^2(\Omega)}^{p-1} + \|\nabla u\|_{L^2(\Omega)}^{p-1}\right) \|\nabla(w-u)\|_{L^2(\Omega)}^2
\end{aligned}$$

In the second to last line we used again Sobolev embedding and that $p \leq \frac{n+2}{n-2}$. This establishes (11.8) and we can conclude. $\square$

Definition 11.14. We say that $\mathcal{E} \in C^1(H)$ (i.e. $\mathcal{E}$ is *continuously differentiable*) if for each $u \in H$, $D\mathcal{E}(u) \in H$ exists (in the sense of Definition 11.9), and moreover the map

$$H \ni u \mapsto D\mathcal{E}(u) \in H$$

is continuous.

We say that $D\mathcal{E}$ is *Lipschitz continuous on bounded subsets of H* if for any $\Lambda > 0$ there exists $C(\Lambda) > 0$ such that

$$\|D\mathcal{E}(u_1) - D\mathcal{E}(u_2)\|_H \leq C(\Lambda)\|u_1 - u_2\|_H \quad \forall u_1, u_2 \in H \text{ s.t. } \|u_1\|_H, \|u_2\|_H \leq \Lambda.$$

Continuous differentiability (i.e. $\mathcal{E} \in C^1(H)$) is enough for the mountain pass theory to work, but we will focus on $D\mathcal{E}$ is Lipschitz on bounded sets, because it fits our application and makes life simpler.

Example 11.15. Take the energy from Example 11.13, i.e. for $p \in (1, \frac{n+2}{n-2}]$ ($p < \infty$ if $n = 2$) set

$$\mathcal{E}(u) := \frac{1}{2}\int_\Omega |\nabla u|^2 - \frac{1}{p+1}\int_\Omega |u|^{p+1}$$

For $H = W_0^{1,2}(\Omega)$ we have $\mathcal{E} \in C^1(H)$ and $D\mathcal{E}$ is Lipschitz continuous on bounded subsets.

Proof. We only discuss the case $n \geq 3$. The case $n = 2$ is a minor adaptation, Exercise 11.16.

We have by duality

$$\|D\mathcal{E}(u_1) - D\mathcal{E}(u_2)\|_H = \sup_{\|v\|_H \leq 1} \langle D\mathcal{E}(u_1) - D\mathcal{E}(u_2), v\rangle$$

In view of (11.7) we have (we only treat the case $n \geq 3$, the case $n = 2$ is Exercise 11.16)

$$\begin{aligned}
\langle D\mathcal{E}(u_1) - D\mathcal{E}(u_2), v\rangle =& \int_\Omega \nabla(u_1 - u_2) \cdot \nabla v - \left(|u_1|^{p-1}u_1 - |u_2|^{p-1}u_2\right) v \\
\leq& \|u_1 - u_2\|_{W^{1,2}(\Omega)} \|v\|_{W^{1,2}(\Omega)} + \| \left(|u_1|^{p-1}u_1 - |u_2|^{p-1}u_2\right) \|_{L^{\frac{2n}{n+2}}(\Omega)} \|v\|_{L^{\frac{2n}{n-2}}(\Omega)} \\
\overset{\text{Sobolev}}{\lesssim}& \left(\|u_1 - u_2\|_{W^{1,2}(\Omega)} + \| \left(|u_1|^{p-1}u_1 - |u_2|^{p-1}u_2\right) \|_{L^{\frac{2n}{n+2}}(\Omega)}\right) \|v\|_{W^{1,2}(\Omega)}.
\end{aligned}$$

Now we use Exercise 11.17 and conclude that

$$\begin{aligned}
& \| \left(|u_1|^{p-1}u_1 - |u_2|^{p-1}u_2\right) \|_{L^{\frac{2n}{n+2}}(\Omega)} \\
\lesssim& \| \left(|u_1|^{p-1} + |u_2|^{p-1}\right) |u_1 - u_2| \|_{L^{\frac{2n}{n+2}}(\Omega)} \\
\lesssim& \left(\|u_1\|^{p-1}_{L^{p\frac{2n}{n+2}}(\Omega)} + \|u_2\|^{p-1}_{L^{p\frac{2n}{n+2}}(\Omega)}\right) \|u_1 - u_2\|_{L^{p\frac{2n}{n+2}}(\Omega)} \\
\overset{p<\frac{n+2}{n-2}}{\lesssim_\Omega}& \left(\|u_1\|^{p-1}_{L^{\frac{2n}{n-2}}(\Omega)} + \|u_2\|^{p-1}_{L^{\frac{2n}{n-2}}(\Omega)}\right) \|u_1 - u_2\|_{L^{\frac{2n}{n-2}}(\Omega)} \\
\overset{\text{Sobolev}}{\lesssim_\Omega}& \left(\|u_1\|^{p-1}_{W^{1,2}(\Omega)} + \|u_2\|^{p-1}_{W^{1,2}(\Omega)}\right) \|u_1 - u_2\|_{W^{1,2}(\Omega)}.
\end{aligned}$$

In conclusion, we have show that

$$\|D\mathcal{E}(u_1) - D\mathcal{E}(u_2)\|_H \lesssim \left(1 + \|u_1\|^{p-1}_{W^{1,2}(\Omega)} + \|u_2\|^{p-1}_{W^{1,2}(\Omega)}\right) \|u_1 - u_2\|_{W^{1,2}(\Omega)}$$

This clearly implies continuity (and Local Lipschitz continuity of $D\mathcal{E}$) w.r.t. $H = W_0^{1,2}(\Omega)$. □

Exercise 11.16. *Prove Example 11.15 for $n = 2$.*

Exercise 11.17. *Show that for any $p \geq 1$ there exists a constant $C(p)$ such that*

$$\left||a|^{p-1}a - |b|^{p-1}b\right| \leq C(p)\, |a - b| \max\{|a|^{p-1}, |b|^{p-1}\} \quad \forall a, b \in \mathbb{R}.$$

As mentioned in the finite dimensional case, we need a suitable "coercivity/compactness" replacement. This is the so-called

Definition 11.18 (Palais-Smale condition)**.** A functional $\mathcal{E} \in C^1(H, \mathbb{R})$ satisfies the *Palais-Smale condition* or *Palais-Smale compactness condition* if

any sequence $(u_k)_{k\in\mathbb{N}} \subset H$ (which we will call *Palais Smale sequence*) with the two conditions

(1) $\sup_k |\mathcal{E}(u_k)| < \infty$
(2) $\lim_{k\to\infty} D\mathcal{E}(u_k) = 0$ with respect to convergence in H.

is pre-compact in H.

Observe that we request quite a lot: pre-compactness *in H*, and not *weakly in H*.

Example 11.19. Let Ω be an open, bounded set with smooth boundary.

Let $p \in (1, \frac{n+2}{n-2})$ ($p < \infty$ if $n = 2$). Then

$$\mathcal{E}(u) := \frac{1}{2}\int_\Omega |\nabla u|^2 - \frac{1}{p+1}\int_\Omega |u|^{p+1}$$

Then $\mathcal{E}$ satisfies the Palais-Smale condition in $W^{1,2}_0(\Omega)$

Proof. So let $(u_k)_{k\in\mathbb{N}}$ be a sequence in $W^{1,2}_0(\Omega)$ with

$$\sup_{k\in\mathbb{N}} |\mathcal{E}(u_k)| < \infty,$$

that is

$$\sup_{k\in\mathbb{N}} \left|\frac{1}{2}\int_\Omega |\nabla u_k|^2 - \frac{1}{p+1}\int_\Omega |u_k|^{p+1}\right| < \infty, \tag{11.9}$$

and

$$D\mathcal{E}(u_k) \xrightarrow{k\to\infty} 0 \quad \text{in } W^{1,2}_0(\Omega). \tag{11.10}$$

If infinitely many u_k satisfy $\|u_k\|_{W^{1,2}(\Omega)} = 0$ then we have a subsequence converging to 0 in $W^{1,2}_0(\Omega)$ and there is nothing to show.

So from now on we will assume

$$\inf_k \|u_k\|_{W^{1,2}(\Omega)} > 0.$$

We first claim

$$\sup_k \|u_k\|_{W^{1,2}(\Omega)} < \infty \tag{11.11}$$

In order to see (11.11), we observe that (11.10) implies by duality

$$\|D\mathcal{E}(u_k)\|_{W^{1,2}(\Omega)} = \sup_{\|v\|_{W^{1,2}_0(\Omega)}\leq 1} \left|\int_\Omega \nabla u_k \cdot \nabla v - |u_k|^{p-1}u_k v\right| \xrightarrow{k\to\infty} 0.$$

In particular we can choose $v = \frac{u_k}{\|u_k\|_{W^{1,2}(\Omega)}}$ and conclude

$$\frac{1}{\|u_k\|_{W^{1,2}(\Omega)}}\left|\int_\Omega |\nabla u_k|^2 - \int_\Omega |u_k|^{p+1}\right| \xrightarrow{k\to\infty} 0.$$

or, more suggestively

$$\frac{1}{p+1}\frac{1}{\|u_k\|_{W^{1,2}(\Omega)}}\left|\int_\Omega |\nabla u_k|^2 - \int_\Omega |u_k|^{p+1}\right| \xrightarrow{k\to\infty} 0. \tag{11.12}$$

Now since $\inf_k \|u_k\|_{W^{1,2}} > 0$ we have that (11.9) implies

$$\sup_{k\in\mathbb{N}} \frac{1}{\|u_k\|_{W^{1,2}(\Omega)}} \left|\frac{1}{2}\int_\Omega |\nabla u_k|^2 - \frac{1}{p+1}\int_\Omega |u_k|^{p+1}\right| < \infty \tag{11.13}$$

Combining (11.12) and (11.13) we obtain

$$\begin{aligned}
&\sup_{k\in\mathbb{N}} \|u_k\|_{W^{1,2}(\Omega)}\\
\lesssim &\sup_{k\in\mathbb{N}} \frac{1}{\|u_k\|_{W^{1,2}(\Omega)}} \left(\frac{1}{2} - \frac{1}{p+1}\right)\int_\Omega |\nabla u_k|^2\\
= &\sup_{k\in\mathbb{N}} \frac{1}{\|u_k\|_{W^{1,2}(\Omega)}} \left(\frac{1}{2}\int_\Omega |\nabla u_k|^2 - \frac{1}{p+1}\int_\Omega |\nabla u_k|^2\right)\\
= &\sup_{k\in\mathbb{N}} \frac{1}{\|u_k\|_{W^{1,2}(\Omega)}} \left(\frac{1}{2}\int_\Omega |\nabla u_k|^2 - \frac{1}{p+1}\int_\Omega |u_k|^{p+1}\right) - \frac{1}{p+1}\frac{1}{\|u_k\|_{W^{1,2}(\Omega)}}\left(\int_\Omega |\nabla u_k|^2 - \int_\Omega |u_k|^{p+1}\right)\\
\overset{(11.12),(11.13)}{<} &\infty
\end{aligned}$$

That is, we have established (11.11) is established, and thus u_k is uniformly bounded in $W^{1,2}_0(\Omega)$ and up to taking a subsequence (not relabeled) we have that u_k *weakly* converges to some $u \in W^{1,2}_0(\Omega)$.

Since $p < \frac{n+2}{n-2}$ we have $p+1 < \frac{2n}{n-2} = 2^*$, thus by the compactness of Sobolev embedding (or simply: Rellich's theorem) we have (again up to a non-relabeled subsequence)

$$u_k \xrightarrow{k\to\infty} u \quad \text{in } L^{p+1}(\Omega).$$

Even more, we have

$$|u_k|^{p-1}u_k \xrightarrow{k\to\infty} |u|^{p-1}u \quad \text{in } L^{\frac{2n}{n+2}}(\Omega). \tag{11.14}$$

Indeed, as above we can use use Exercise 11.17 and conclude that

$$\begin{aligned}
&\| \left(|u_k|^{p-1}u_k - |u|^{p-1}u\right) \|_{L^{\frac{2n}{n+2}}(\Omega)}\\
\lesssim &\| \left(|u_k|^{p-1} + |u|^{p-1}\right) |u_k - u| \|_{L^{\frac{2n}{n+2}}(\Omega)}\\
\lesssim &\left(\|u_k\|^{p-1}_{L^{p\frac{2n}{n+2}}(\Omega)} + \|u\|^{q-1}_{L^{p\frac{2n}{n+2}}(\Omega)}\right) \|u_k - u\|_{L^{p\frac{2n}{n+2}}(\Omega)}\\
\overset{p<\frac{n+2}{n-2}}{\lesssim_\Omega} &\left(\|u_k\|^{p-1}_{L^{\frac{2n}{n-2}}(\Omega)} + \|u\|^{q-1}_{L^{\frac{2n}{n-2}}(\Omega)}\right) \|u_k - u\|_{L^{p\frac{2n}{n+2}}(\Omega)}\\
\overset{\text{Sobolev}}{\lesssim_\Omega} &\left(\|u_k\|^{p-1}_{W^{1,2}(\Omega)} + \|u\|^{p-1}_{W^{1,2}(\Omega)}\right) \|u_k - u\|_{L^{p\frac{2n}{n+2}}(\Omega)}.
\end{aligned}$$

Since $p < \frac{n+2}{n-2}$ we have $p\frac{2n}{n+2} < \frac{2n}{n-2} = 2^*$, so from the compactness of Sobolev embedding we have $u_k \xrightarrow{k\to\infty} u$ in $L^{p\frac{2n}{n+2}}(\Omega)$. This shows (11.14).

Solve now $w_k \in W_0^{1,2}(\Omega)$

$$\begin{cases} -\Delta w_k := |u_k|^{p-1}u_k & \text{in } \Omega \\ w_k = 0 & \text{on } \partial\Omega \end{cases}$$

This is possible by the the usual variational method, Theorem 6.1, or Lax-Milgram, since (the argument here is for $n \geq 3$, but easy to extend to $n = 2$)

$$\begin{aligned} \||u_k|^{p-1}u_k\|_{(W_0^{1,2}(\Omega))^*} \leq & \sup_{\|\varphi\|_{W_0^{1,2}(\Omega)} \leq 1} \int |u_k|^{p-1}u_k\varphi \\ \leq & \||u_k|^p\|_{L^{\frac{2n}{n+2}}(\Omega)} \sup_{\|\varphi\|_{W_0^{1,2}(\Omega)} \leq 1} \|\varphi\|_{L^{\frac{2n}{n-2}}(\Omega)} \\ \overset{\text{Sobolev}}{\lesssim} & \|u_k\|^p_{L^{p\frac{2n}{n+2}}(\Omega)} \underbrace{\sup_{\|\varphi\|_{W_0^{1,2}(\Omega)} \leq 1} \|\varphi\|_{W^{1,2}(\Omega)}}_{\leq 1} \\ \overset{p<\frac{n+2}{n-2}}{\lesssim} & \|u_k\|^p_{L^{p\frac{2n}{n+2}}(\Omega)} \overset{(11.14)}{<} \infty \end{aligned}$$

Similarly we can solve $w \in W_0^{1,2}(\Omega)$ with

$$\begin{cases} -\Delta w := |u|^{p-1}u & \text{in } \Omega \\ w = 0 & \text{on } \partial\Omega \end{cases}$$

Then we have

$$\begin{cases} -\Delta(w_k - w) = |u_k|^{p-1}u_k - |u|^{p-1}u & \text{in } \Omega \\ (w_k - w) = 0 & \text{on } \partial\Omega \end{cases}$$

and by existence and uniqueness theory, Theorem 6.1 we have as above

$$\|w_k - w\|_{W^{1,2}(\Omega)} \leq \||u_k|^{p-1}u_k - |u|^{p-1}u\|_{L^{\frac{2n}{n+2}}(\Omega)} \xrightarrow{k\to\infty} 0 \quad \text{by (11.14).}$$

We apply again (11.10)

$$\begin{aligned} & \|u_k - w\|_{W^{1,2}(\Omega)} \\ \leq & \|u_k - w_k\|_{W^{1,2}(\Omega)} + \|w_k - w\|_{W^{1,2}(\Omega)} \\ & \sup_{\|v\|_{W_0^{1,2}(\Omega)} \leq 1} \left| \int_\Omega \nabla(u_k - w_k) \cdot \nabla v \right| + \|w_k - w\|_{W^{1,2}(\Omega)} \\ = & \sup_{\|v\|_{W_0^{1,2}(\Omega)} \leq 1} \left| \int_\Omega \nabla u_k \cdot \nabla v - |u_k|^{p-1}u_k v \right| + \|w_k - w\|_{W^{1,2}(\Omega)} \\ = & \|D\mathcal{E}(u_k)\|_{W^{1,2}(\Omega)} + \|w_k - w\|_{W^{1,2}(\Omega)} \xrightarrow{k\to\infty} 0. \end{aligned}$$

That is u_k converges to w (*strongly*) in $W^{1,2}(\Omega)$, and we can conclude. □

We now want to extend the idea of the deformation Φ in (11.5) (the "gradient flow") that we used in the finite dimensional mountain pass theorem to our case. While this was almost a triviality in the finite dimensional case, this is essentially the heart of the matter for our version of the mountain pass theorem. There we said that if at a certain energy level $\mathcal{E} = \beta$ there are no critical points, we can deform the energy so that the energy values decrease in a controlled way. This is still true:

Theorem 11.20 (Deformation Theorem)**.** *Let H be a Hilbert space, $\mathcal{E} \in C^1(H,\mathbb{R})$ with $D\mathcal{E}$ is Lipschitz continuous (as defined in Definition 11.14). Assume moreover that $\mathcal{E}$ satisfies the Palais-Smale condition in H.*

Fix $\beta \in \mathbb{R}$ and set
$$\tilde{K}_\beta := \{u \in H : \quad \mathcal{E}(u) = \beta, \quad D\mathcal{E}(u) = 0\}$$
the collection of $\mathcal{E}$-critical points u at the energy level β.

If
$$\tilde{K}_\beta = \emptyset$$
then for any small $\varepsilon > 0$ there exists some $\delta \in (0,\varepsilon)$ and a map (the deformation, sometimes also referred to as pseudo-gradient flow)
$$\Phi \in C^0([0,1] \times H, H)$$
such that

(1) $\Phi(0,u) = u$ for all $u \in H$
(2) $\Phi(1,u) = u$ whenever $\mathcal{E}(u) < \beta - \varepsilon$ or $\mathcal{E}(u) > \beta + \varepsilon$
(3) $\mathcal{E}(\Phi(t,u)) \leq \mathcal{E}(u)$ for all $u \in H$, $t \in [0,1]$
(4) If $u \in H$ and $\mathcal{E}(u) \leq \beta + \delta$ then $\mathcal{E}(\Phi(u,1)) \leq \beta - \delta$.

We begin by proving intermediate statements (that should be reminiscent of the finite dimensional case).

Lemma 11.21. *Under the same assumptions as in Theorem 11.20, there exists $\sigma, \theta \in (0,1)$ such that*
$$\|D\mathcal{E}(u)\| \geq \sigma \quad \text{for all } u \text{ with } \mathcal{E}(u) \in [\beta - \theta, \beta + \theta]$$

Proof. We argue by contradiction. Assume this is not the case, then we there must be a sequence $u_k \in H$ such that
$$\mathcal{E}(u_k) \in (\beta - \frac{1}{k}, \beta + \frac{1}{k}]$$
and
$$\|D\mathcal{E}(u_k)\| \leq \frac{1}{k}.$$
Then $(u_k)_{k\in\mathbb{N}}$ is a Palais-Smale sequence (as in Definition 11.18), and since we assume the Palais-Smale condition we have that (up to subsequence, not relabeled) u_k converges to some u in H.

Since $\mathcal{E} \in C^1$ we conclude that $\mathcal{E}(u) = \beta$ and $D\mathcal{E}(u) = 0$. That is $u \in \tilde{K}_\beta$ – but by assumption $\tilde{K}_\beta = \emptyset$. $\square$

Lemma 11.22. *Under the same assumptions as in Theorem 11.20,*

Let $B \subset H$ be a bounded set, let $\lambda < \mu$ then

$$\inf\{\|u-v\|_H : \quad u,v \in B, \quad \mathcal{E}(u) \leq \lambda, \quad \mathcal{E}(v) \geq \mu\} > 0.$$

Proof. Without loss of generality we can assume that B is also convex (take a huge ball), then we have from the fundamental theorem and the assumption of Lipschitz continuity on bounded set, Definition 11.14,

$$0 < \mu - \lambda \leq \mathcal{E}(v) - \mathcal{E}(u) \leq \max_{t \in [0,1]} \|D\mathcal{E}(tv + (1-t)u)\|_H \|u-v\|_H \leq C(B)\|u-v\|_H,$$

where for any fixed $u_0 \in B$ and for B^{conv} the convex hull of B (which is still bounded)

$$\|D\mathcal{E}(tv + (1-t)u)\|_H \leq \underbrace{\|D\mathcal{E}(tv + (1-t)u) - D\mathcal{E}(u_0)\|_H}_{\precsim_{B^{conv}} \|tv+(1-t)u-u_0\|_H} + \|D\mathcal{E}(u_0)\|_H =: C(B).$$

This readily implies

$$\inf\{\|u-v\|_H : \quad u,v \in B, \quad \mathcal{E}(u) \leq \lambda, \quad \mathcal{E}(u) \geq \mu\} \geq \frac{\mu - \lambda}{C(B)}.$$

$\square$

Proof of Theorem 11.20. Take σ and θ in $(0,1)$ from Lemma 11.21 and assume $\varepsilon < \theta$.

Fix

$$0 < \delta < \min\{\varepsilon, \frac{\sigma^2}{2}\} \tag{11.15}$$

and define two sets:

$$A := \{u \in H : \quad \mathcal{E}(u) \leq \beta - \varepsilon \text{ or } \mathcal{E}(u) \geq \beta + \varepsilon\}.$$

and

$$B := \{u \in H : \quad \beta - \delta \leq \mathcal{E}(u) \leq \beta + \delta\}.$$

Take any $R > 0$ and let $u \in B(0,R) \subset H$. Then

$$\text{dist}\,(u,A) > R \quad \text{or} \quad \text{dist}\,(u,B) > R$$

or

$$\text{dist}\,(u,A) + \text{dist}\,(u,B) = \inf_{v_A \in A \cap B(0,2R)} \inf_{v_B \in B \cap B(0,2R)} \|u - v_A\| + \|u - v_B\| \geq \inf_{v_A \in A \cap B(0,2R)} \inf_{v_B \in B \cap B(0,2R)} \|v_A - v_B\|.$$

In view of Lemma 11.22 we see that the latter condition implies

$$\inf_{u \in B(0,R)} \text{dist}\,(u,A) + \text{dist}\,(u,B) \geq c_R.$$

So we can construct the cutoff function

$$g(u) := \frac{\mathrm{dist}\,(u,A)}{\mathrm{dist}\,(u,A) + \mathrm{dist}\,(u,B)},$$

which satisfies

$$0 \le g \le 1, \quad g \equiv 0 \text{ on } A, \quad \text{and } g \equiv 1 \text{ on } B. \tag{11.16}$$

Set

$$h(t) := \begin{cases} 1 & 0 \le t \le 1 \\ \frac{1}{t} & t \ge 1. \end{cases}$$

We now define the ("gradient field")

$$V(u) := -g(u)h(\|D\mathcal{E}(u)\|)\, D\mathcal{E}(u) \in H$$

This is well-defined for any $u \in H$, and we observe that

$$\sup_{u \in H} \|V(u)\|_H < \infty$$

Instead of constructing $\Phi(t,u)$ explicitely (as in the finite dimensional case), we now define it via a *flow*.

Consider the ODE

$$\begin{cases} \frac{d}{dt}\Phi(u,t) := V(\Phi(u,t)) & t > 0 \\ \Phi(0,u) = u & t = 0 \end{cases} \tag{11.17}$$

Since V is bounded and Lipschitz continuous (since the distance is a Lipschitz continuous map), we can solve this ODE for each fixed u – simply by repeating the proof of the Picard-Lindeloeff/Cauchy–Lipschitz theorem.

As a fun fact, by uniqueness we also have the semigroup property

$$\Phi(t+s,u) = \Phi(t, \Phi(s,u)).$$

Then we have a map $\Phi : [0,1] \times H \to H$. The map Φ is differentiable in t and thus continuous in time, continuity in H follows from continuous dependence of data.

From the conditions on Φ

(1) $\Phi(0,u) = u$ for all $u \in H$ – this is obvious by definition.
(2) Whenever $\mathcal{E}(u) < \beta - \varepsilon$ we have $u \in A$ and thus $V(u) = 0$ – and thus $\Phi(t,u) = u$. Similarly if $\mathcal{E}(u) > \beta + \varepsilon$
(3) Fix now some $u \in H$ and compute

$$\begin{aligned} \frac{d}{dt}\mathcal{E}(\Phi(t,u)) =& \langle D\mathcal{E}(\Phi(t,u)), \frac{d}{dt}\Phi(t,u)\rangle_H \\ =& \langle D\mathcal{E}(\Phi(t,u)), V(\Phi(u,t))\rangle_H \\ =& - g(\Phi(t,u))h(\|D\mathcal{E}(\Phi(t,u))\|)\,\|D\mathcal{E}(\Phi(t,u))\|_H^2 \\ \le& \, 0 \end{aligned}$$

Thus we have *monotonicity*, namely $t \mapsto \mathcal{E}(\Phi(t,u))$ is decreasing, in particular: $\mathcal{E}(\Phi(t,u)) \leq \mathcal{E}(u)$ for all $u \in H$, $t \in [0,1]$

(4) Now fix any $u \in H$ such that $\mathcal{E}(u) \leq \beta+\delta$. We need to show that $\mathcal{E}(\Phi(u,1)) \leq \beta-\delta$.

Assume first there is any $t \in [0,1]$ such that $\Phi(u,t) \not\in B$. By monotonicity we know already that $\mathcal{E}(\Phi(u,t)) \leq \mathcal{E}(\Phi(u,0)) \leq \beta+\delta$, so the only way that $\Phi(u,t) \not\in B$ is that $\mathcal{E}(\Phi(u,t)) < \beta-\delta$. Again by monotonicity we conclude that then

$$\mathcal{E}(\Phi(u,1)) \leq \mathcal{E}(\Phi(u,t)) < \beta - \delta.$$

So we are only interested in the situation that $\Phi(u,t) \in B$ for all $t \in [0,1]$. Then $g(u) = 1$, so as before we compute

$$\frac{d}{dt}\mathcal{E}(\Phi(t,u)) = -\underbrace{g(\Phi(t,u))}_{=1} h(\|D\mathcal{E}(\Phi(u,t))\|)\, \|D\mathcal{E}(\Phi(t,u))\|_H^2$$

If $\|D\mathcal{E}(\Phi(u,t))\| \geq 1$ then $h(\|D\mathcal{E}(\Phi(u,t))\|) = \|\|D\mathcal{E}(\Phi(u,t))\|^{-1}$, so that in this case

$$\frac{d}{dt}\mathcal{E}(\Phi(t,u)) = -\|D\mathcal{E}(\Phi(t,u))\|_H.$$

Now from the definition of σ from Lemma 11.21 (observe that $\varepsilon < \theta$ so this is applicable) we have

$$\|D\mathcal{E}(\Phi(t,u))\|_H \geq \sigma \quad \text{if } \Phi(t,u) \in B,$$

so that we conclude: for each $t \in [0,1]$ for which $\|D\mathcal{E}(\Phi(u,t))\| \geq 1$ we have

$$\frac{d}{dt}\mathcal{E}(\Phi(t,u)) \leq -\sigma \leq -\sigma^2$$

If $t \in [0,1]$ and $\|D\mathcal{E}(\Phi(u,t))\| \leq 1$ we have $h(\|D\mathcal{E}(\Phi(u,t))\|) = 1$, so that

$$\frac{d}{dt}\mathcal{E}(\Phi(t,u)) = -\|D\mathcal{E}(\Phi(t,u))\|_H^2$$

and again by the definition of σ we conclude again

$$\frac{d}{dt}\mathcal{E}(\Phi(t,u)) = -\sigma^2.$$

Integrating this up we have

$$\mathcal{E}(\Phi(1,u)) - \mathcal{E}(u) \leq -\sigma^2$$

which implies

$$\mathcal{E}(\Phi(1,u)) \leq \mathcal{E}(u) - \sigma^2 \leq \beta + \delta - \sigma^2 < \beta - \delta + 2\delta - \sigma^2 < \beta - \delta + \underbrace{\frac{\sigma^2}{2} - \sigma^2}_{\overset{(11.15)}{\leq} 0}.$$

Thus, also in this last case we conclude

$$\mathcal{E}(\Phi(1,u)) < \beta - \delta.$$

$\square$

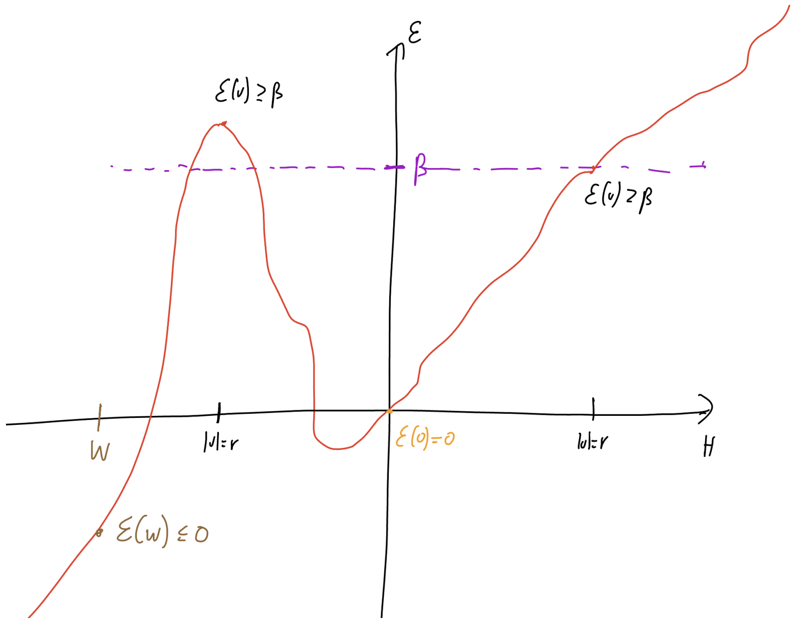


FIGURE 11.2. The setup in Theorem 11.23 (Picture by: Armin, age 39)

The deformation theorem, Theorem 11.20, is the crucial ingredient for the infinite dimensional mountain pass theorem. Since the deformation Φ is obtained as a solution to an ODE (11.17) where V is essentially (a truncated version) of $D\mathcal{E}$, some people call Φ a *pseudo-gradient flow*.

Theorem 11.23 (Mountain Pass theorem)**.** *Let H be a Hilbert space, $\mathcal{E} \in C^1(H,\mathbb{R})$ with $D\mathcal{E}$ is Lipschitz continuous (as defined in Definition 11.14). Assume moreover that $\mathcal{E}$ satisfies the Palais-Smale condition in H.*

Assume additionally

(1) $\mathcal{E}(0) = 0$
(2) there exist constants $r > 0$ and $\alpha > 0$ such that

$$\mathcal{E}(u) \geq \alpha \quad \textit{if } \|u\|_H = r$$

(3) there exists an element $w \in H$ with

$$\|w\| > r \quad \textit{and } \mathcal{E}(w) \leq 0$$

Cf. Figure 11.2.

Define the set of paths (this time real, continuous paths)

$$\mathcal{P} := \{\gamma \in C([0,1], H) \quad \gamma(0) = 0, \quad \gamma(1) = w\}$$

Then

$$\beta := \inf_{\gamma \in \mathcal{P}} \max_{t \in [0,1]} \mathcal{E}(\gamma(t)) > 0$$

is a critical value of $\mathcal{E}$, that is there exists $u \in H$ such that

$$\mathcal{E}(u) = \beta$$

and

$$D\mathcal{E}(u) = 0.$$

The above is taken from [Evans, 2010, §8.5 b, Theorem 2]. One can generalize this statement substantially, see [Struwe, 2008, Theorem 4.2].

Proof. Clearly $\beta \geq \alpha$, since any path $\gamma \in \mathcal{P}$ must traverse the sphere $\{u : \|u\|_H = r\}$. On the other hand, we see that

$$\gamma(t) := tw \in \mathcal{P}$$

and by continuity of $\mathcal{E}$ and compactness of $[0,1]$ we have

$$\beta \leq \max_{t \in [0,1]} \mathcal{E}(tw) < \infty.$$

That is $\beta \in [\alpha, \infty)$.

If we assume that β is not a critical value of $\mathcal{E}$ then the set

$$\tilde{K}_\beta := \{u \in H : \quad \mathcal{E}(u) = \beta, D\mathcal{E}(u)\} = \emptyset.$$

Pick any number $\varepsilon < \frac{\alpha}{2}$, and apply the deformation theorem Theorem 11.20.

Then for some $\delta \in (0, \varepsilon)$ we find a the *deformation*/pseudo-gradient flow with

$$\Phi \in C^0([0,1] \times H, H)$$

with

(1) $\Phi(0, u) = u$ for all $u \in H$
(2) $\Phi(1, u) = u$ whenever $\mathcal{E}(u) < \beta - \varepsilon$ or $\mathcal{E}(u) > \beta + \varepsilon$
(3) $\mathcal{E}(\Phi(s, u)) \leq \mathcal{E}(u)$ for all $u \in H$, $s \in [0,1]$
(4) If $u \in H$ and $\mathcal{E}(u) \leq \beta + \delta$ then $\mathcal{E}(\Phi(u, 1)) \leq \beta - \delta$.

Let now $\gamma \in \mathcal{P}$ such that

$$\max_{t \in [0,1]} \mathcal{E}(\gamma(t)) \leq \beta + \delta$$

This exists by the definition of β as an infimum over all paths in $\mathcal{P}$.

Set

$$\tilde{\gamma}(t) := \Phi(1, \gamma(t)).$$

Since $\gamma(0) = 0$, $\mathcal{E}(\gamma(0)) = 0 < \alpha - \varepsilon \leq \beta - \varepsilon$, so $\tilde{\gamma}(0) = \gamma(0) = 0$. Similarly, since $\mathcal{E}(\gamma(1)) \leq 0 \leq \beta - \varepsilon$ we have also $\tilde{\gamma}(1) = \gamma(1) = w$.

That is, $\tilde{\gamma} \in \mathcal{P}$ and thus

$$\max_{t\in[0,1]} \mathcal{E}(\tilde{\gamma}(t)) \geq \beta. \tag{11.18}$$

Moreover, since for any $t \in [0,1]$ we have that $\mathcal{E}(\gamma(t)) \leq \beta + \delta$, we have $\mathcal{E}(\tilde{\gamma}(t)) \leq \beta - \delta$. But then

$$\beta \overset{(11.18)}{\leq} \max_{t\in[0,1]} \mathcal{E}(\tilde{\gamma}(t)) \leq \beta - \delta,$$

a contradiction. So

$$\tilde{K}_\beta := \{u \in H : \quad \mathcal{E}(u) = \beta, D\mathcal{E}(u)\} \neq \emptyset,$$

which is all we wanted to show. □

Now we are ready to reap the fruits of our argument to find solutions to the equation $-\Delta u = |u|^{p-1}u$

Theorem 11.24 (Existence for the semilinear PDE)**.** *Let $p \in (1, \frac{n+2}{n-2})$ and $\Omega \subset\subset \mathbb{R}^n$ be a smoothly bounded set. Then there exists $u \in W^{1,2}_0(\Omega)$, $u \not\equiv 0$, which solves*

$$\begin{cases} -\Delta u = |u|^{p-1}u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{11.1}$$

Remark 11.25. While we are happy with *one* nontrivial solution, let us point out that by slight extensions of the argument here, the mountain pass theory is extended to show there are *infinitely many* solutions. See [Struwe, 2008, Theorem 5.8]. Most of these solutions are sign-changing, so one might be interested in finding nontrivial solutions $u \geq 0$.

Proof of Theorem 11.24. We set

$$\mathcal{E}(u) := \frac{1}{2}\int_\Omega |\nabla u|^2 - \frac{1}{p}\int_\Omega |u|^{p+1}.$$

defined on the Hilbert space $H := W^{1,2}_0(\Omega)$ which we equip with the norm

$$\|u\|_H := \|\nabla u\|_{L^2(\Omega)}.$$

Observe that this is indeed a norm since all the constants are 0 by the boundary data assumption. Moreover, thanks to Poincaré inequality, this norm is equivalent to the usual Sobolev norm

$$\|u\|_H \approx \|u\|_{L^2(\Omega)} + \|\nabla u\|_{L^2(\Omega)} \quad \text{for all } u \in H.$$

Let's discuss properties of $\mathcal{E}$. We already have shown:

- $\mathcal{E}$ is differentiable in H, Example 11.13
- Actually $\mathcal{E} \in C^1(H)$ and $D\mathcal{E}$ is Lipschitz continuous on bounded subsets of H, Example 11.15.
- $\mathcal{E}$ satisfies the Palais-Smale condition, Example 11.19.

We plan to use the mountain pass theorem, Theorem 11.23, so we need to establish the assumptions:

(1) It is obvious that $\mathcal{E}(0) = 0$.

(2) We need to find two constants $r > 0$ and $\alpha > 0$ such that

$$\mathcal{E}(u) \geq \alpha \quad \text{if } \|u\|_H = r \tag{11.19}$$

Observe that for $\|u\|_H = r$ we have

$$\mathcal{E}(u) = \frac{r^2}{2} - \frac{1}{p} \int_\Omega |u|^{p+1}.$$

By Sobolev inequality, since $p + 1 \leq \frac{2n}{n-2}$,

$$\begin{aligned} \mathcal{E}(u) \geq & \frac{r^2}{2} - \frac{C(n,p)}{p} \|\nabla u\|_{L^2(\Omega)}^{p+1} \\ = & \frac{r^2}{2} - \frac{C(n,p)}{p} r^{p+1} \\ = & r^2 \left(\frac{1}{2} - \frac{C(n,p)}{p} r^{p-1} \right). \end{aligned}$$

Observe that $p > 1$, so we can find some very small $r > 0$ such that

$$\frac{1}{2} - \frac{C(n,p)}{p} r^{p-1} > 0.$$

In that case, we set

$$\alpha := r^2 \left(\frac{1}{2} - \frac{C(n,p)}{p} r^{p-1} \right) > 0.$$

and have established (11.19).

(3) Take $v_1 \in C_c^\infty(\Omega)$, $v_1 \not\equiv 0$. Set

$$v_\lambda := \lambda v_1.$$

For λ large enough we have in view of Lemma 11.3,

$$\|v_\lambda\| > r \quad \text{and } \mathcal{E}(v_\lambda) \leq 0.$$

So $w := v_\lambda$ satisfies the third assumption in Theorem 11.23.

According to Theorem 11.23, whose assumptions we now have completely established, there exists some $u \in W_0^{1,2}(\Omega)$ and some $\beta > 0$ such that

$$D\mathcal{E}(u) = 0, \quad \mathcal{E}(u) = \beta.$$

Since $\mathcal{E}(u) = \beta > 0$ we know that $u \neq 0$. The condition $D\mathcal{E}(u) = 0$ implies, according to Example 11.13, more precisely (11.7), that u solves the PDE in question.

We can conclude. $\square$

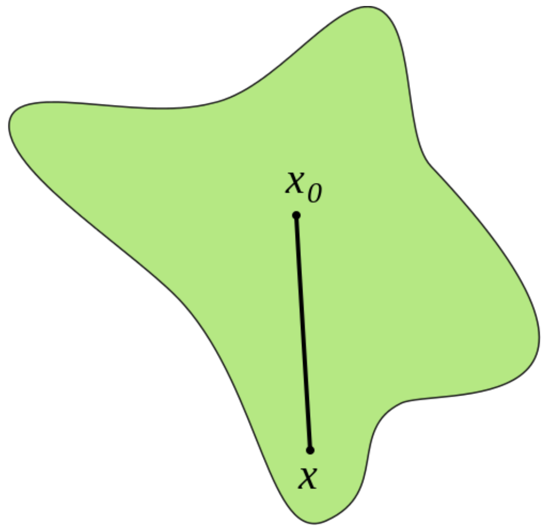


FIGURE 11.3. A star-shaped domain (w.r.t x_0)

11.3. **Nonexistence: Derrick-Pohozaev identity.** We will now prove that in *certain* sets Ω, for $p > \frac{n+2}{n-2}$ there are only trivial (C^2-)solutions to

$$\begin{cases} -\Delta u = |u|^{p-1}u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{11.1}$$

We have already seen that our techniques in existence, Theorem 11.24 and regularity (see Theorem 11.32 below) relied crucially on $p < \frac{n+2}{n-2}$ (because of Sobolev embedding).

Definition 11.26. A set Ω is called *star-shaped* with respect to x_0 if for each $x_1 \in \Omega$ the segment

$$\{(1-\lambda)x_0 + \lambda x_1 - : \quad \lambda \in [0,1]\} \subset \Omega.$$

Cf. Figure 11.3.

Exercise 11.27. *Show that any convex set Ω with $0 \in \Omega$ is star-shaped with respect to 0.*

Exercise 11.28. *Give an example of a star-shaped set that is not convex.*

Lemma 11.29. *Assume that $\Omega \subset\subset \mathbb{R}^n$ is a bounded set which is starshaped w.r.t. 0 and assume that $\partial\Omega \in C^1$.*

For $x \in \partial\Omega$ denote with $\nu(x)$ the outwards facing normal.

Then

$$\langle x, \nu(x)\rangle \geq 0 \quad \forall x \in \partial U.$$

It is easy to check this in a picture

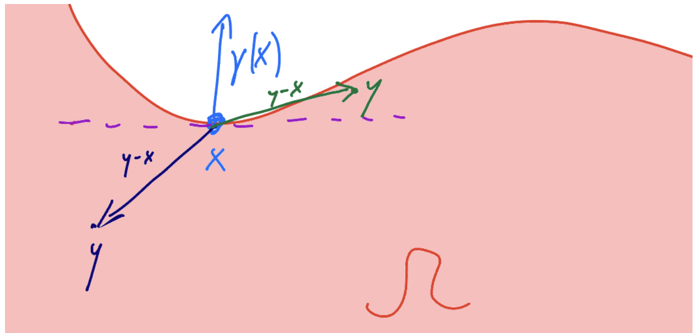


FIGURE 11.4. Let $x \in \partial\Omega$ and $\nu(x)$ be the exterior unit normal. Then for any $\varepsilon > 0$ there exists $\delta > 0$ such that $\nu(x) \cdot \frac{y-x}{|y-x|} \in (-\infty, \varepsilon)$ for all $y \in \Omega$, $|y-x| < \delta$

Sketch of the formal proof of Lemma 11.29. Fix $x \in \partial\Omega$ and let $y \approx x$. We can always decompose

$$\frac{y-x}{|y-x|} = \tau + \lambda\nu(x)$$

where $\tau \in T_\partial\Omega$ and $\lambda \in \mathbb{R}$. The observation is that since $\partial\Omega \in C^1$, the only way that $|\lambda|$ is large is that λ is negative, because only in that direction we move inside the set Ω. In other terms, for any $\varepsilon > 0$ there exists $\delta > 0$ such that $\lambda < \varepsilon$ for all $y \in \Omega$, $|y-x| < \delta$. See Figure 11.4. That is, we have (for any Ω with $\partial\Omega \in C^1$ and any $x \in \partial\Omega$)

$$\limsup_{\Omega \ni y \to x} \nu(x) \cdot \frac{y-x}{|y-x|} \le 0$$

Since Ω is star-shaped, we can choose $y := \lambda x \in \Omega$, $\lambda \in (0,1)$, and have

$$-\nu(x) \cdot \frac{x}{|x|} = \nu(x) \cdot \frac{\lambda x - x}{|\lambda x - x|}.$$

From the above we conclude that

$$-\nu(x) \cdot \frac{x}{|x|} = \limsup_{\lambda \to 1^-} \nu(x) \cdot \frac{\lambda x - x}{|\lambda x - x|} \le 0.$$

This proves the claim. □

Theorem 11.30. *Let $\Omega \subset\subset \mathbb{R}^n$ be a smoothly bounded set such that Ω is star-shaped. Assume that $u \in C^2(\overline{\Omega})$ is a solution to*

$$\begin{cases} -\Delta u = |u|^{p-1}u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{11.1}$$

If $p > \frac{n+2}{n-2}$ then $u \equiv 0$.

Since $u = 0$ on $\partial\Omega$ we can test (11.1) with u itself and obtain

$$\int_\Omega |\nabla u|^2 = \int_\Omega |u|^{p+1}. \tag{11.20}$$

The idea of the proof is *also* to test with $\langle x, \nabla u\rangle$. Testing with this sort of test-function is motivated from certain domain variations in variational functionals, appears a lot in physics as momentum – and is sometimes quite surprising because of its consequences all over PDE and analysis.

We observe that by the regularity theory, Proposition 11.33 below, assuming $u \in W^{1,2}(\Omega) \cap L^\infty(\Omega)$ is enough to conclude the same statement. But if $u \in W^{1,2}(\Omega)$ only, it is not so clear how the argument below goes through if p is too large.

Proof. As discussed we multiply the PDE by $\langle x, Du\rangle = \sum_{\alpha=1}^n x^\alpha \partial_\alpha u$. This is where we use that $u \in W^{2,2}(\Omega) \subset C^2(\overline{\Omega})$.

$$\int_\Omega -\Delta u\ x^\alpha \partial_\alpha u = \int_\Omega |u|^{p-1} u\ x^\alpha \partial_\alpha u. \tag{11.21}$$

We develope both terms further. Firstly, (Einstein's summation!)

$$\begin{aligned}
&\int_\Omega -\Delta u\ x^\alpha \partial_\alpha u \\
=& -\int_\Omega \partial_{\beta\beta} u\ x^\alpha \partial_\alpha u \\
\overset{\text{Integration by Parts}}{=}& + \int_\Omega \partial_\beta u\ \partial_\beta\left(x^\alpha \partial_\alpha u\right) - \int_{\partial\Omega} \partial_\beta u\ \nu^\beta x^\alpha\, \partial_\alpha u \\
\overset{\text{Product rule}}{=}& \int_\Omega \partial_\beta u\ \delta_{\alpha\beta}\partial_\alpha u + \int_\Omega \partial_\beta u\ x^\alpha \partial_{\alpha\beta} u - \int_{\partial\Omega} \partial_\beta u\ \nu^\beta x^\alpha\, \partial_\alpha u \\
=& \int_\Omega |\nabla u|^2 + \int_\Omega \frac{1}{2}\partial_\alpha |\partial_\beta u|^2\ x^\alpha - \int_{\partial\Omega} \partial_\beta u\ \nu^\beta x^\alpha\, \partial_\alpha u \\
\overset{\text{Integration by Parts}}{=}& \int_\Omega |\nabla u|^2 - \int_\Omega \frac{1}{2} \underbrace{|\partial_\beta u|^2}_{\sum_\beta = |\nabla u|^2}\ \underbrace{\partial_\alpha x^\alpha}_{\sum_\alpha = n} + \int_{\partial\Omega} \frac{1}{2} \underbrace{|\partial_\beta u|^2}_{\sum_\beta = |\nabla u|^2}\ \underbrace{x^\alpha\, \nu^\alpha}_{\sum_\alpha = \langle x,\nu\rangle} - \int_{\partial\Omega} \partial_\beta u\ \nu^\beta x^\alpha\, \partial_\alpha u \\
=& \left(1 - \frac{n}{2}\right) \int_\Omega |\nabla u|^2 + \int_{\partial\Omega} \frac{1}{2}|\nabla u|^2\ \langle x,\nu\rangle - \int_{\partial\Omega} \langle \nabla u, \nu\rangle\, \langle x, \nabla u\rangle
\end{aligned}$$

Now we observe: Since $u = 0$ on $\partial\Omega$ (recall that $u \in C^2(\overline{\Omega})$) we have

$$\langle v, \nabla u(x)\rangle = \partial_v u(x) = 0 \quad \text{for any } x \in \partial\Omega \quad \text{and} \quad v \in T_x\partial\Omega$$

By linear algebra, this means that $\nabla u(x)$ is a vector in $(T_x\partial\Omega)^\perp$, that is $\nabla u(x)$ is parallel to $\nu(x)$ for all $x \in \partial\Omega$. This implies

$$\nabla u(x) = \langle \nabla u(x), \nu(x)\rangle \nu(x) \quad \forall x \in \partial\Omega,$$

and in particular

$$|\nabla u(x)|^2 = |\langle \nabla u(x), \nu(x)\rangle|^2.$$

Consequently,

$$\langle x, \nabla u(x)\rangle = \langle x, \nu\rangle \, \langle \nabla u(x), \nu(x)\rangle \quad \forall x \in \partial\Omega.$$

and thus

$$\langle \nabla u(x), \nu(x)\rangle \, \langle x, \nabla u(x)\rangle = |\langle \nabla u(x), \nu(x)\rangle|^2 \, \langle x, \nu\rangle = |\nabla u(x)|^2 \, \langle x, \nu\rangle.$$

From the computations before we then conclude

$$\begin{aligned}
&\int_\Omega -\Delta u \; x^\alpha \partial_\alpha u\\
=&\left(1-\frac{n}{2}\right)\int_\Omega |\nabla u|^2 + \int_{\partial\Omega} \frac{1}{2}|\nabla u|^2 \, \langle x,\nu\rangle - \int_{\partial\Omega} \langle \nabla u, \nu\rangle \, \langle x, \nabla u\rangle\\
=&\left(1-\frac{n}{2}\right)\int_\Omega |\nabla u|^2 + \int_{\partial\Omega} \frac{1}{2}|\nabla u|^2 \, \langle x,\nu\rangle - \int_{\partial\Omega} |\nabla u(x)|^2 \, \langle x,\nu\rangle\\
=&\left(1-\frac{n}{2}\right)\int_\Omega |\nabla u|^2 - \frac{1}{2}\int_{\partial\Omega} |\nabla u|^2 \, \langle x,\nu\rangle
\end{aligned} \tag{11.22}$$

On the other hand we have (using again $|u| = 0$ on $\partial\Omega$)

$$\begin{aligned}
&\int_\Omega |u|^{p-1}u \; x^\alpha \partial_\alpha u\\
=&\int_\Omega |u|^{p-1}u \; \partial_\alpha u \, x^\alpha\\
=&\int_\Omega \frac{1}{p+1}\partial_\alpha \left(|u|^{p+1}\right) \, x^\alpha\\
\overset{\text{Integration by Parts}}{=}&-\int_\Omega \frac{1}{p+1} \left(|u|^{p+1}\right) \, \underbrace{\partial_\alpha x^\alpha}_{\sum_\alpha = n}\\
=&-\frac{n}{p+1}\int_\Omega |u|^{p+1}.
\end{aligned} \tag{11.23}$$

Combining (11.21), (11.22), (11.23) we obtain what is known as the *Derrick-Pohozaev identity*:

$$\left(1-\frac{n}{2}\right)\int_\Omega |\nabla u|^2 - \frac{1}{2}\int_{\partial\Omega} |\nabla u|^2 \, \langle x,\nu\rangle = -\frac{n}{p+1}\int_\Omega |u|^{p+1}.$$

Multiplying by -1 we find

$$\left(\frac{n-2}{2}\right)\int_\Omega |\nabla u|^2 + \frac{1}{2}\int_{\partial\Omega} |\nabla u|^2 \, \langle x,\nu\rangle = +\frac{n}{p+1}\int_\Omega |u|^{p+1}.$$

Plugging our first identity from testing with u, (11.20), we conclude

$$\left(\frac{n-2}{2}\right)\int_\Omega |u|^{p+1} + \frac{1}{2}\int_{\partial\Omega} |\nabla u|^2 \, \langle x,\nu\rangle = \frac{n}{p+1}\int_\Omega |u|^{p+1}.$$

and thus

$$\frac{1}{2}\int_{\partial\Omega} |\nabla u|^2 \, \langle x,\nu\rangle = \left(\frac{n}{p+1} - \frac{n-2}{2}\right)\int_\Omega |u|^{p+1}.$$

Now if $p > \frac{n+2}{n-2}$ then $p+1 > \frac{2n}{n-2}$ and thus

$$\frac{n}{p+1} - \frac{n-2}{2} < 0.$$

On the other hand, since $\langle x, \nu\rangle \geq 0$ we then have

$$0 \leq \frac{1}{2}\int_{\partial\Omega} |\nabla u|^2 \, \langle x,\nu\rangle \leq \underbrace{\left(\frac{n}{p+1} - \frac{n-2}{2}\right)}_{<0} \int_\Omega |u|^{p+1}.$$

The only way this is possible is if $\int_\Omega |u|^{p+1} = 0$, i.e. $u = 0$ a.e. in Ω. □

Remark 11.31. In our argument above we left out the *critical case* $p = \frac{n+2}{n-2}$. This one is quite more delicate (and very interesting). Our existence argument fails, since the Palais-Smale condition fails. Indeed, Pohozaev, [Pohozaev, 1965] proved that for a star-shaped Ω, there is *no* solution to (11.1) under the additional assumption that $u \geq 0$ in Ω (observe that if $u \geq 0$ the strong maximum principle immediately kicks in and tells us that $u > 0$ in Ω). On the other hand it is in no way clear, why the critical case works or doesn't work (whatever we define as work: existence, or non-existence). A famous result by Brezis and Nirenberg, [Brézis and Nirenberg, 1983] discusses slight variations of this problem. This question is very important, as many *geometric* equations tend to be critical in nature, so that existence questions (also regularity questions) become very challenging. Related equations also appear e.g. for standing waves for wave equations, which is where Derrick [Derrick, 1964] discovered a similar result to Pohozaev.

11.4. **Regularity theory.** The goal of this section is to prove the following theorem.

Theorem 11.32. *Let $p \in (1, \frac{n+2}{n-2})$ (strict inequality!) and $u \in W^{1,2}_0(\Omega)$ be a solution to*

$$-\Delta u = |u|^{p-1}u \quad \text{in } \Omega.$$

Then $u \in C^2(\Omega)$ (we don't discuss the boundary regularity).

The main step for Theorem 11.32 is to obtain boundedness for u. Then the theorem is a consequence of Schauder and L^p-theory: Indeed we prove easily the following result.

Proposition 11.33. *Let $p \in (1,\infty)$ and $u \in W^{1,2}_0(\Omega) \cap L^\infty_{loc}(\Omega)$ be a solution to*

$$-\Delta u = |u|^{p-1}u \quad \text{in } \Omega.$$

Then $u \in C^2(\Omega)$ (we don't discuss the boundary regularity).

Proof. Since $u \in L^\infty_{loc}$ we have that if we set $g := |u|^{p-1}u$ that $g \in L^q$ for any $q \in (1,\infty)$ and

$$-\Delta u = g$$

From Calderón-Zygmund/L^q-theory, Theorem 7.10, or Theorem 7.16, we find that $u \in W^{1,q}_{loc}(\Omega)$. By Sobolev-Morrey embedding (taking any $q > n$) we find that $u \in C^\alpha_{loc}$ for

$\alpha = 1 - \frac{n}{q}$. Since $p > 1$ we have that $|u|^{p-1}$ is still Hölder continuous (maybe with a smaller Hölder constant), and thus we have that $g \in C^\beta$ for some $\beta > 0$. Schauder theory, Corollary 8.32, then implies $u \in C^{2,\beta}_{loc}$ □

So, in order to obtain the proof of Theorem 11.32, we need to show $u \in L^\infty$. We obtain this boundedness of u from the following adaptation of De Giorgi's method. Observe that for $p < \frac{n+2}{n-2}$, and $u \in W^{1,2}$, we have $|u|^{p-1} \in L^r$ for $r = \frac{2n}{(n-2)(p-1)} > \frac{n}{2}$, by Sobolev embeddding.

Proposition 11.34. *Let $u \in W^{1,2}(\Omega)$ be a distributional solution to*

$$-\Delta u = fu \quad \text{in } \Omega$$

If $f \in L^r(\Omega)$ with $r > \frac{n}{2}$ then $u \in L^\infty_{loc}(\Omega)$.

Proof. As discussed with the De Giorgi Method, see Proposition 10.19 and Exercise 10.20, it suffices to assume $\Omega = B(0,1)$ and show boundedness in $B(0,1/2)$ – by a covering and scaling argument. We also may assume (otherwise consider $\tilde{u} := \delta \frac{u}{\|u\|_{W^{1,2}(\Omega)}}$ wich solves the same equation)

$$\|u\|_{W^{1,2}(B(0,1))} < \delta, \tag{11.24}$$

where δ is small, and will be chosen later.

We first start with the Cacciopoli type observation: Fix $k \in \mathbb{R}$ and set $v := (u-k)_+$, let $\eta \in C_c^\infty(B(0,1))$. Using the argument from Lemma 10.18 we find

$$\begin{aligned}\int_{B(0,1)} |\nabla(\eta v)|^2 \leq& C\|\nabla \eta\|_{L^\infty}^2 \int_{\operatorname{supp}\eta} |v|^2 + C\int_{B(0,1)} f\,\eta^2 uv\\ \leq& C\|\nabla \eta\|_{L^\infty}^2 \int_{\operatorname{supp}\eta} |v|^2 + C\left(\int_{B(0,1)} |f|^r\right)^{\frac{1}{r}} \left(\int_{\operatorname{supp}\eta} (|u||v|)^{r'}\right)^{\frac{1}{r'}},\end{aligned}$$

Observe that if $n \geq 3$ then $2r' < \frac{2n}{n-2}$ since

$$\frac{1}{2}\left(1 - \frac{1}{r}\right) \overset{r > \frac{n}{2}}{>} \frac{1}{2}\left(1 - \frac{2}{n}\right) = \frac{n-2}{2n}.$$

So, from Sobolev inequality on the left-hand side we find for $q = \frac{2n}{n-2}$ if $n \geq 3$, or any $q \in (2r', \infty)$ if $n = 2$,

$$\left(\int_{B(0,1)} |\eta v|^q\right)^{\frac{2}{q}} \leq C\|\nabla \eta\|_{L^\infty}^2 \int_{\operatorname{supp}\eta} |v|^2 + C\left(\int_{B(0,1)} |f|^r\right)^{\frac{1}{r}} \left(\int_{\operatorname{supp}\eta} (|u||v|)^{r'}\right)^{\frac{1}{r'}},$$

and thus,

$$\left(\int_{B(0,1)} |\eta v|^q\right)^{\frac{1}{q}} \leq C\|\nabla \eta\|_{L^\infty} \left(\int_{\operatorname{supp}\eta} |v|^2\right)^{\frac{1}{2}} + C\left(\int_{B(0,1)} |f|^r\right)^{\frac{1}{2r}} \left(\int_{\operatorname{supp}\eta} (|u||v|)^{r'}\right)^{\frac{1}{2r'}},$$

As in the proof of Proposition 10.19 we apply this inequality on a layered decomposition of $B(0,1) \setminus B(0,1/2)$. Set

$$\rho_i := \frac{1}{2} + 2^{-i-1},$$

and

$$\mu_i := 1 - 2^{-i}$$

and take $\eta_i \in C_c^\infty(B(0,\rho_{i-1}))$, $\eta_i \equiv 1$ in $B(0,\rho_i)$ nonnegative bump functions with $|\nabla \eta_i| \lesssim_n 2^i$. Then for

$$\Lambda := \left(\int_{B(0,1)} |f|^r\right)^{\frac{1}{2r}} < \infty$$

we have

$$\left(\int_{B(0,1)} |\eta_i (u-\mu_i)_+|^q\right)^{\frac{1}{q}} \leq C\, 2^i \left(\int_{\operatorname{supp}\eta_i} |(u-\mu_i)_+|^2\right)^{\frac{1}{2}} + C\,\Lambda \left(\int_{\operatorname{supp}\eta_i} (|u||(u-\mu_i)_+|)^{r'}\right)^{\frac{1}{2r'}},$$

As in (10.4), since $\mu_i > \mu_{i-1}$ we have

$$(u-\mu_i)_+ \leq (u-\mu_{i-1})_+ \tag{11.25}$$

and combined with the fact that $\eta_{i-1} \equiv 1$ in $\operatorname{supp}\eta_i$ we find

$$\left(\int_{B(0,1)} |\eta_i (u-\mu_i)_+|^q\right)^{\frac{1}{q}} \leq C\, 2^i \left(\int_{B(0,1)} |\eta_{i-1} (u-\mu_{i-1})_+|^2\right)^{\frac{1}{2}} + C\,\Lambda \left(\int_{B(0,1)} \left(\eta_{i-1}^2 |u|\, (u-\mu_i)_+\right)^{r'}\right)^{\frac{1}{2r'}},$$

We want to do a similar trick for the term $|u||(u-\mu_i)_+$ above. We claim that

$$|u(x)|\,(u(x)-\mu_i)_+ \leq \left(2^i - 1\right) ((u(x)-\mu_{i-1})_+)^2. \tag{11.26}$$

Indeed, if $u(x) \leq \mu_i$ then $(u(x) - \mu_i)_+ = 0$ and there is nothing to show. Assume now $u(x) > \mu_i$. Then

$$
\begin{aligned}
&u(x) > \mu_i\\
\Leftrightarrow &\frac{\mu_{i-1}}{\mu_i - \mu_{i-1}} u(x) > \frac{\mu_i}{\mu_i - \mu_{i-1}} \mu_{i-1}\\
\Leftrightarrow &\frac{\mu_{i-1}}{\mu_i - \mu_{i-1}} u(x) - \frac{\mu_i}{\mu_i - \mu_{i-1}} \mu_{i-1} > 0\\
\Leftrightarrow &\left(\frac{\mu_{i-1}}{\mu_i - \mu_{i-1}} + 1\right) u(x) - \frac{\mu_i}{\mu_i - \mu_{i-1}} \mu_{i-1} > u(x)\\
\Leftrightarrow &\frac{\mu_i}{\mu_i - \mu_{i-1}} u(x) - \frac{\mu_i}{\mu_i - \mu_{i-1}} \mu_{i-1} > u(x)\\
\Leftrightarrow &\frac{\mu_i}{\mu_i - \mu_{i-1}} \left(u(x) - \mu_{i-1}\right) > u(x)\\
\Leftrightarrow &\frac{1 - 2^{-i}}{1 - 2^{-i} - 1 + 2^{1-i}} \left(u(x) - \mu_{i-1}\right) > u(x)\\
\Leftrightarrow &\frac{2^i - 1}{-1 + 2} \left(u(x) - \mu_{i-1}\right) > u(x)\\
\Leftrightarrow &\left(2^i - 1\right) \left(u(x) - \mu_{i-1}\right) > u(x)
\end{aligned}
$$

In particular we have $|u(x)| = u(x)$ and thus

$$
|u(x)| \leq \left(2^i - 1\right) \left(u(x) - \mu_{i-1}\right)_+
$$

This combined with (11.25) establishes (11.26).

So we arrive at

$$
\left(\int_{B(0,1)} |\eta_i (u - \mu_i)_+|^q\right)^{\frac{1}{q}} \leq C\, 2^i \left(\int_{B(0,1)} |\eta_{i-1} (u - \mu_{i-1})_+|^2\right)^{\frac{1}{2}} + C \left(2^i - 1|\right)^{\frac{1}{2}} \Lambda \left(\int_{B(0,1)} |\eta_{i-1} (u - \mu_{i-1})_+|^{2r'}\right)
$$

With Hölder's inequality (we have $r' \geq 1$) we then find

$$
\|\eta_i (u - \mu_i)_+\|_{L^q(B(0,1))} \leq C(B(0,1), r', \Lambda)\, 2^i\, \|\eta_{i-1} (u - \mu_{i-1})_+\|_{L^{2r'}(B(0,1))}.
$$

Since $q > 2r'$ we can now follow again the argument in Proposition 10.19: from Hölder's inequality

$$
\|\eta_i (u - \mu_i)_+\|_{L^{2r'}(B(0,1))} \leq C(B(0,1), r', \Lambda)\, 2^i\, |\{\eta_i (u - \mu_i)_+ > 0\}|^{\frac{1}{2r'} - \frac{1}{q}}\, \|\eta_{i-1} (u - \mu_{i-1})_+\|_{L^{2r'}(B(0,1))}.
$$

As in the proof of Proposition 10.19, (10.7), we have

$$
\{\eta_i (u - \mu_i)_+ > 0\} \subset \{\eta_{i-1} (u - \mu_{i-1})_+ > 2^{-i}\}
$$

and thus by Chebychev inequality,

$$
|\{\eta_i (u - \mu_i)_+ > 0\}| \leq 2^{2r' i} \int_{B(0,1)} \left(\eta_{i-1} (u - \mu_{i-1})_+\right)^{2r'}
$$

and thus

$$\|\eta_i(u-\mu_i)_+\|_{L^{2r'}(B(0,1))} \leq C(B(0,1),r',\Lambda)\,2^{i(2-\frac{2r'}{q})}\,\|\eta_{i-1}(u-\mu_{i-1})_+\|^{2-\frac{2r'}{q}}_{L^{2r'}(B(0,1))}.$$

That is, for $\gamma := 1-\frac{2r'}{q} > 0$,

$$\|\eta_i(u-\mu_i)_+\|_{L^{2r'}(B(0,1))} \leq \Gamma^i \|\eta_{i-1}(u-\mu_{i-1})_+\|^{1+\gamma}_{L^{2r'}(B(0,1))}$$

We observe that

$$\|\eta_0(u-\mu_0)_+\|_{L^{2r'}(B(0,1))} \leq \|u\|_{L^{2r'}(B(0,1))} \overset{\text{Sobolev, } 2r'<\frac{2n}{n-2}}{\precsim} \|u\|_{W^{1,2}(B(0,1))} \overset{(11.24)}{<} \delta.$$

By Exercise 10.21, choosing $\delta > 0$ small enough (depending on Γ and on γ) we have

$$\lim_{i\to\infty} \|\eta_i(u-\mu_i)_+\|_{L^{2r'}(B(0,1))} = 0$$

and with the same argument as in Proposition 10.19 $\|(u-1)_+\|_{L^{2r'}(B(0,1/2))} = 0$ and thus

$$\sup_{B(0,1/2)} u \leq 1.$$

Recall that we had the normalization chosen in (11.24), so we expect ≤ 1 for the renormalized u (δ depends on $\|f\|_{L^r(\Omega)}$ and $\|u\|_{W^{1,2}(B(0,1))}$), so in general we have found

$$\sup_{B(0,1/2)} u \leq C(r, \|f\|_{L^r(B(0,1))}, \|u\|_{W^{1,2}(B(0,1))}).$$

Applying the same argument to $-u$ instead of u (which solves again the same equation) we find

$$\sup_{B(0,1/2)} |u| \leq C(r, \|f\|_{L^r(B(0,1))}, \|u\|_{W^{1,2}(B(0,1))}).$$

Consequently, $u \in L^\infty(B(0,1/2))$, and we can conclude. □

Theorem 11.32 is a direct consequence of applying first Proposition 11.34 and then Proposition 11.33 with $f := |u|^{p-1}$.

12. Navier–Stokes equation: partial regularity theory

The proof of regularity theory that we present here is due to Caffarelli–Kohn–Nirenberg, [Caffarelli et al., 1982], and the simplification by Lin [Lin, 1998]. We follow the presentations in Seregin's Book [Seregin, 2015], and Tsai's lecture notes [Tsai, 2018]. For more background on Navier-Stokes equation we refer e.g. to the recent [Bedrossian and Vicol, 2022], or to the monograph by (University of Pittsburgh's) Galdi [Galdi, 1994a, Galdi, 1994b].

Euler- and Navier Stokes equation describe the motion of a fluid. We will discuss only the Navier-Stokes equations. We will follow a convention often seen in mathematical physics that bold face symbols, like $\mathbf{f}$, $\mathbf{u}$ denote vectors, whereas usually printed letters f, u are scalars.

The (incompressible) Navier-Stokes equation are given for (unknown) $\mathbf{u} : \Omega \times (T_1, T_2) \to \mathbb{R}^3$ (the *velocity vector*) and $p : \Omega \times (T_1, T_2) \to \mathbb{R}$ (the *pressure*) which solve (Einstein summation!)

$$\begin{cases} \partial_t \mathbf{u} - \Delta \mathbf{u} + u_\alpha \partial_\alpha \mathbf{u} = -\nabla p & \text{in } \Omega \times (T_1, T_2) \\ \operatorname{div}(\mathbf{u}) = 0 & \text{in } \Omega \times (T_1, T_2) \end{cases}$$

subject to initial condition $\mathbf{u}(x, 0) = \mathbf{u}_0(x)$. Another way to write (in coordinates) the first equation is

$$\partial_t u^\beta - \Delta u^\beta + u_\alpha \partial_\alpha u^\beta = -\partial_\beta p \quad \text{in } \Omega \times (T_1, T_2),\ \beta = 1, \ldots, n$$

Observe that since $\operatorname{div} \mathbf{u} = 0$ we can equivalently write

$$\begin{cases} \partial_t \mathbf{u} - \Delta \mathbf{u} + \partial_\alpha (u_\alpha \mathbf{u}) = -\nabla p & \text{in } \Omega \times (T_1, T_2) \\ \operatorname{div}(\mathbf{u}) = 0 & \text{in } \Omega \times (T_1, T_2) \end{cases}$$

The term $u_\alpha \partial_\alpha \mathbf{u}$ is often referred to as *the nonlineratity*. In the literature, this nonlinearity can be written in various ways

$$u_\alpha \partial_\alpha \mathbf{u} = \operatorname{div}(\mathbf{u} \otimes \mathbf{u}) = \nabla \cdot (\mathbf{u} \otimes \mathbf{u}) = \mathbf{u} \cdot \nabla \mathbf{u}$$

Sometimes there is a ν in front of $\Delta \mathbf{u}$ which is called the *viscosity*. If $\nu = 0$ then Navier-Stokes equation becomes the *Euler-equation* – otherwise mathematicians tend to normalize $\nu = 1$.

Sometimes one also adds a *forcing term* $\mathbf{f} : \Omega \times (T_1, T_2) \to \mathbb{R}^3$ (we won't here).

The equation above is called the "incompressible" Navier-Stokes equation, because of the assumption $\operatorname{div}(\mathbf{u}) = 0$.

A (slight reformulation) of the *unsolved* Navier-Stokes Millenium problem[30] is:

Exercise 12.1. *If $\Omega = \mathbb{R}^3$ and $u_0 \in C^\infty(\mathbb{R}^n)$ with strong decay to zero at infinity, is there a smooth solutions $(\mathbf{u}, p)$ to the Navier-Stokes equation with initial data u_0?*

Let me know if you solve it. For $\mathbb{R}^2$ this is comparatively easy.

At first it may seem that the pressure p has essentially no conditions. But take the divergence on both sides (using that $\operatorname{div}(\mathbf{u}) = 0$) we find

$$-\Delta p = \operatorname{div}(u_\alpha \partial_\alpha \mathbf{u}) = \partial_\beta \partial_\alpha (u_\alpha u_\beta).$$

So p is very much related to "something like" u^2.

We will use the notation of space-time Sobolev spaces such tas

$$L^p((0, T); L^q(\Omega)), \quad L^p((0, T); W^{1,q}(\Omega))$$

[30] See https://www.claymath.org/millennium-problems/navier-stokes-equation

and similar. We don't want to go into too much detail here, but this can be defined by the *Bochner integral*, essentially a generalization of the Riemann or Lebesgue integral to vector-valued maps.

Generally, we equip the space $L^p((T_1,T_2),X)$ (that we haven't defined) with the norm

$$\|f\|_{L^p((T_1,T_2);X)} := \left(\int_{(T_1,T_2)} (\|f(t)\|_X)\, dt\right)^{\frac{1}{p}}.$$

What is important for us is the following characterization, which is essentially Fubini's theorem

Definition 12.2. A $\mathcal{L}^{n+1}$-measurable map $f : \Omega \times (T_1,T_2)$

- belongs to $L^p((T_1,T_2),L^q(\Omega))$ if for $\mathcal{L}^1$-a.e. $t \in (T_1,T_2)$ we have $f(t,\cdot) \in L^q(\Omega)$, and we have
$$\|f\|_{L^p((T_1,T_2),L^q(\Omega))} := \left(\int_{(T_1,T_2)} \|f(t,\cdot)\|^p_{L^q(\Omega)}\right)^{\frac{1}{p}}.$$
- belongs to $L^p((T_1,T_2),W^{1,q}(\Omega))$ if for $\mathcal{L}^1$-a.e. $t \in (T_1,T_2)$ we have $f(t,\cdot) \in W^{1,q}(\Omega)$, and we have
$$\|f\|_{L^p((T_1,T_2),W^{1,q}(\Omega))} := \left(\int_{(T_1,T_2)} \|f(t,\cdot)\|^p_{W^{1,q}(\Omega)}\right)^{\frac{1}{p}}.$$

12.1. Suitable weak solutions.

Definition 12.3 (Suitable weak solution)**.** Let $\Omega \subset \mathbb{R}^n$ be an open set and $T_1 < T_2$. A pair $\mathbf{u} : \Omega \times (T_1,T_2) \to \mathbb{R}^3$, $p : \Omega \times (T_1,T_2) \to \mathbb{R}$ is called a *suitable weak solution* to the Navier-Stokes equation in $\Omega \times (T_1,T_2)$ if

- $\mathbf{u} \in L^\infty((T_1,T_2),L^2(\Omega)) \cap L^2((T_1,T_2),W^{1,2}(\Omega))$
- $p \in L^{\frac{3}{2}}(\Omega \times (T_1,T_2))$
- The Navier-Stokes equation holds in the sense of distributions in $\Omega \times (T_1,T_2)$, that is
$$\int_{\Omega\times(T_1,T_2)} -\mathbf{u}\partial_t\varphi + \partial_\alpha \mathbf{u}\partial_\alpha\varphi + u_\alpha\partial_\alpha\mathbf{u}\varphi = \int_{\Omega\times(T_1,T_2)} p\nabla\varphi \quad \forall \varphi \in C_c^\infty(\Omega\times(T_1,T_2)),$$
- $\operatorname{div}\mathbf{u} = 0$ holds a.e. in $\Omega \times (T_1,T_2)$.
- We have the *local energy inequality* for all *nonnegative* $\eta \in C_c^\infty(\Omega \times (T_1,\infty))$ and for $\mathcal{L}^1$-a.e. $t \in (T_1,T_2)$,

$$\begin{aligned}&\int_\Omega \eta(x,t)|\mathbf{u}(x,t)|^2dx + 2\int_{\Omega\times(T_1,t)} \eta(x,t')|\nabla\mathbf{u}|^2\,dx\,dt'\\ \le&\int_{\Omega\times(T_1,t)} |\mathbf{u}(x,t')|^2(\Delta\eta+\partial_t\eta) + \mathbf{u}(x,t')\cdot\nabla\eta(x,t')\ \left(|\mathbf{u}(x,t')|^2 + 2p(x,t')\right)dx\,dt'\end{aligned} \tag{12.1}$$

The local energy inequality comes from the following *formal* computation (precise if $u \in C^2$)

Lemma 12.4. *If $u \in C^2(\mathbb{R}^n \times \mathbb{R})$ is solution to incompressible Navier-Stokes, then the local energy inequality holds as an equality, i.e. we have*

$$\begin{aligned}
&\int_\Omega \eta(x,t)|\mathbf{u}(x,t)|^2 dx + 2\int_{\Omega\times(T_1,t)} \eta(x,t')|\nabla \mathbf{u}(x,t')|^2\, dx\, dt' \\
= &\int_{\Omega\times(T_1,t)} |\mathbf{u}(x,t')|^2(\Delta\eta + \partial_t\eta) + \mathbf{u}(x,t')\cdot\nabla\eta(x,t')\ \Big(|\mathbf{u}(x,t')|^2 + 2p(x,t')\Big)\, dx\, dt'
\end{aligned}$$

Proof.

$$\begin{aligned}
&\int_\Omega \eta(x,t)|\mathbf{u}(x,t)|^2 dx - \int_\Omega \underbrace{\eta(x,T_1)}_{\equiv 0}|\mathbf{u}(x,T_1)|^2 dx \\
=&\int_{\Omega\times(T_1,t)} \frac{d}{dt'}\Big(\eta(x,t')|\mathbf{u}(x,t')|^2\Big)\, dxdt' \\
=&\int_{\Omega\times(T_1,t)} \partial_t\eta(x,t')\ |\mathbf{u}(x,t')|^2 dxdt' + 2\int_{\Omega\times(T_1,t)} \eta(x,t')\ \langle \mathbf{u}(x,t'), \partial_t\mathbf{u}(x,t')\rangle dxdt' \\
\overset{\text{N.S.}}{=}&\int_{\Omega\times(T_1,t)} \partial_t\eta(x,t')\ |\mathbf{u}(x,t')|^2 dxdt' + 2\int_{\Omega\times(T_1,t)} \eta(x,t')\ \langle \mathbf{u}(x,t'), \Delta\mathbf{u}\rangle dxdt' \\
&+ 2\int_{\Omega\times(T_1,t)} \eta(x,t')\ \langle \mathbf{u}(x,t'), -\partial_\alpha(u_\alpha\mathbf{u})\rangle dxdt' \\
&+ 2\int_{\Omega\times(T_1,t)} \eta(x,t')\ \langle \mathbf{u}(x,t'), -\nabla p(x,t')\rangle dxdt'
\end{aligned}$$

Now we have

$$\begin{aligned}
&2\int_{\Omega\times(T_1,t)} \eta\ \langle \mathbf{u}, \Delta\mathbf{u}\rangle dxdt' \\
=&2\int_{\Omega\times(T_1,t)} \eta\, u^\alpha\ \Delta u^\alpha dxdt' \\
=&2\int_{\Omega\times(T_1,t)} u^\alpha\ \mathrm{div}\,(\eta\nabla u^\alpha) dxdt' - 2\int_{\Omega\times(T_1,t)} u^\alpha\ \langle\nabla\eta, \nabla u^\alpha\rangle dxdt' \\
=&2\int_{\Omega\times(T_1,t)} u^\alpha\ \mathrm{div}\,(\eta\nabla u^\alpha) dxdt' - 1\int_{\Omega\times(T_1,t)} \langle\nabla\eta, \nabla|\mathbf{u}|^2\rangle dxdt' \\
\overset{\text{P.I}}{=}&-2\int_{\Omega\times(T_1,t)} \eta|\nabla\mathbf{u}|^2 dxdt' + \int_{\Omega\times(T_1,t)} \Delta\eta\ |\mathbf{u}|^2 dxdt'
\end{aligned}$$

For the term involving the nonlinearity we have

$$
\begin{aligned}
&2\int_{\Omega\times(T_1,t)} \eta\ \langle \mathbf{u}, -\partial_\alpha(u_\alpha \mathbf{u})\rangle dx dt' \\
&= -\,2\int_{\Omega\times(T_1,t)} \eta\ u_\beta \partial_\alpha(u_\alpha u_\beta) dx dt' \\
&\overset{\text{div}=0}{=} -\,2\int_{\Omega\times(T_1,t)} \eta\ u_\beta u_\alpha \partial_\alpha\left(u_\beta\right) dx dt' \\
&= -\,1\int_{\Omega\times(T_1,t)} \eta\ u_\alpha \partial_\alpha |\mathbf{u}|^2 dx dt' \\
&\overset{\text{P.I}}{=} +\int_{\Omega\times(T_1,t)} \partial_\alpha(\eta\ u_\alpha)|\mathbf{u}|^2 dx dt' \\
&\overset{\text{div}=0}{=} \int_{\Omega\times(T_1,t)} \partial_\alpha \eta\ u_\alpha |\mathbf{u}|^2 dx dt' \\
&\overset{\text{div}=0}{=} \int_{\Omega\times(T_1,t)} \langle \nabla\eta, \mathbf{u}\rangle |\mathbf{u}|^2 dx dt'
\end{aligned}
$$

Lastly we consider the pressure term

$$
\begin{aligned}
&2\int_{\Omega\times(T_1,t)} \eta\ \langle \mathbf{u}, -\nabla p\rangle dx dt' \\
&= -\,2\int_{\Omega\times(T_1,t)} \eta\, u_\alpha\, \partial_\alpha p dx dt' \\
&\overset{\text{div}=0}{=} -\,2\int_{\Omega\times(T_1,t)} \eta\, \partial_\alpha\left(u_\alpha\, p\right) dx dt' \\
&\overset{\text{P.I}}{=} +\,2\int_{\Omega\times(T_1,t)} \partial_\alpha \eta\ \left(u_\alpha\, p\right) dx dt' \\
&\overset{\text{P.I}}{=} 2\int_{\Omega\times(T_1,t)} \langle \nabla\eta, \mathbf{u}\rangle\, p\ dx dt'
\end{aligned}
$$

We plug all this computations together and have

$$
\begin{aligned}
&\int_\Omega \eta(x,t)|\mathbf{u}(x,t)|^2 dx \\
&= \int_{\Omega\times(T_1,t)} \partial_t \eta(x,t')\ |\mathbf{u}(x,t')|^2 dx dt' - 2\int_{\Omega\times(T_1,t)} \eta |\nabla \mathbf{u}|^2 dx dt' + \int_{\Omega\times(T_1,t)} \Delta\eta\ |\mathbf{u}|^2 dx dt' \\
&\quad + \int_{\Omega\times(T_1,t)} \langle \nabla\eta, \mathbf{u}\rangle |\mathbf{u}|^2 dx dt' \\
&\quad + 2\int_{\Omega\times(T_1,t)} \langle \nabla\eta, \mathbf{u}\rangle\, p\ dx dt'
\end{aligned}
$$

□

12.2. **Existence of suitable weak solutions.** We are not going to cover the *existence* of suitable weak solutions, but focus on the *regularity*. We refer to [Seregin, 2015] or [Tsai, 2018] for the argument for existence.

12.3. **ε-regularity, and consequence for regularity.** The regularity proof we present here has yet a different conceptional flavor than the proofs we see before. Instead of freezing coefficients, or attacking the PDE directly, we instead use a *blowup argument* to relate the PDE to a linear PDE of which we know the solution's behavior. This sort of argument is very popular and powerful, it can also be used e.g. for reducing boundary estimates to estimates on the half-space, to replace the freezing argument for coefficients A in the regularity theory $\operatorname{div}(A\nabla u) = f$ and many more.

scaling is very important for the theory of the Navier-Stokes equation – and the scaling is the reason regularity is a famous open problem.

Let us illustrate this. Denote the *parabolic cylinder*

$$Q(R) := B(0,R) \times (-R^2, 0).$$

We denote the *mean value*

$$(f)_{Q(R)} := \fint_{Q(R)} f = \frac{1}{|Q(R)|} \int_{Q(R)} f = R^{2-n} \frac{1}{Q(1)} \int_{Q(R)} f$$

If

$$\partial_t \mathbf{u} - \Delta \mathbf{u} + u_\alpha \partial_\alpha \mathbf{u} = -\nabla p \quad \text{in } Q(\rho)$$

Then

$$\mathbf{u}_\lambda(x,t) := \lambda \mathbf{u}(\lambda x, \lambda^2 t), \quad p_\lambda := \lambda^2 p(\lambda x, \lambda^2 t) \tag{12.2}$$

solves

$$\partial_t \mathbf{u}_\lambda - \Delta \mathbf{u}_\lambda + (u_\lambda)_\alpha \partial_\alpha \mathbf{u}_\lambda = -\nabla p_\lambda \quad \text{in } Q(\frac{\rho}{\lambda})$$

We set the scaling-invariant energy[31]

$$E_\rho(\mathbf{u},p) := \rho \left(|Q(\rho)|^{-1} \int_{Q(\rho)} |\mathbf{u} - (\mathbf{u})_{Q(\rho)}|^3 \right)^{\frac{1}{3}} + \rho^2 \left(|Q(\rho)|^{-1} \int_{Q(\rho)} |p(x,t) - (p)_{B(0,\rho)\times\{t\}}|^{\frac{3}{2}} \right)^{\frac{2}{3}}.$$

We can check,

$$E_\rho(\mathbf{u}_\lambda, p_\lambda) = E_{\lambda\rho}(\mathbf{u}, p),$$

The regularity argument is based on the following *decay estimate*. We will work in $\mathbb{R}^n$ – the physically relevant dimensions are $n = 2$ (for which the following argument is somewhat a technical overblow) and $n = 3$ (for which the following argument is the current state of the art – improving it means to solve one of the Millenium problems, if you can do it, you win USD 1 Million).

You can find the following in [Seregin, 2015, Proposition 1.1, p.135] [Tsai, 2018, Lemma 6.5, p.99]

[31] The L^3-norm could be replaced by an L^q-norm for any $q > 2$ (because we want to avoid L^1), but scaling would need to be adapted

Proposition 12.5 (Decay Estimate)**.** *For a constant Υ we have the following: For any $\theta \in (0, \frac{1}{2}]$ there exists $\varepsilon \in (0,1)$ such that the following holds for any $\rho \in (0,1]$.*

Denote

$$Q_\rho := Q(0,\rho) = B(0,\rho) \times (-\rho^2, 0)$$

Assume $\mathbf{u} : Q_\rho \to \mathbb{R}^n$, $p : Q_\rho \to \mathbb{R}$ satisfy

$$\operatorname{div} \mathbf{u} = 0 \quad \text{in } Q_\rho$$

and $\mathbf{u}$ a suitable solution to the Navier-Stokes equation, as defined in Definition 12.3. Assume also that we have the following bounds on Q_ρ:

$$\rho \left| \fint_{Q_\rho} \mathbf{u} \right| \leq 1 \tag{12.3}$$

and

$$E_\rho(\mathbf{u}, p) < \varepsilon \tag{12.4}$$

where we recall

$$E_\rho(\mathbf{u},p) := \rho \left(|Q_\rho|^{-1} \int_{Q_\rho} |\mathbf{u} - (\mathbf{u})_{Q_\rho}|^3 \right)^{\frac{1}{3}} + \rho^2 \left(|Q_\rho|^{-1} \int_{Q_\rho} |p(x,t) - (p)_{B(0,\rho)\times\{t\}}|^{\frac{3}{2}} \right)^{\frac{2}{3}}.$$

Then

$$(\theta\rho) \left| \fint_{Q_{\theta\rho}} \mathbf{u} \right| \leq 1 \tag{12.5}$$

and

$$E_{\theta\rho}(\mathbf{u},p) \leq \Upsilon \theta^{1+\frac{2}{3}} E_\rho(\mathbf{u},p). \tag{12.6}$$

Before we begin with the (length, but conceptionally beautiful) proof of the decay estimate, Proposition 12.5, we show that it implies *partial regularity*:

Namely first we have as a consequence of Proposition 12.5 the following *ε-regularity* theorem

Theorem 12.6 (ε-regularity theorem for Navier-Stokes)**.** *There exists $\varepsilon > 0$ and $\theta > 0$, $\alpha > 0$ such that the following holds:*

If $\mathbf{u}$ is a suitable solution to the Navier Stokes equation in Q_R and

$$R^{-2} \left(\int_{Q(R)} |\mathbf{u}|^3 + \int_{Q(R)} |p(x,t)|^{\frac{3}{2}} \right) < \varepsilon_1$$

then $\mathbf{u}$ is continuous in $Q_{\theta R}$, actually even Hölder continuous.

Proof. Again, by rescaling we may assume $R = 1$, and we see that

$$E_1(\mathbf{u}, p) \lesssim \varepsilon_1$$

and

$$\left| \fint_{Q_1} \mathbf{u} \right| \lesssim \left(\int_{Q(1)} |\mathbf{u}|^3 \right)^{\frac{1}{3}} < \varepsilon_1 \lesssim 1.$$

We plan to iteratively apply the decay estimate, Proposition 12.5. Fix any $\alpha < \frac{2}{3}$.

Choose $\theta \in (0, \frac{1}{2})$ small enough so that $\theta^{2-3\alpha}\Upsilon < 1$. Take the corresponding ε from Proposition 12.5. If we take $\varepsilon_1 \ll \varepsilon$ then we have

$$E_1(\mathbf{u}, p) \lesssim \varepsilon_1 < \varepsilon$$

and

$$\left| \fint_{Q_1} \mathbf{u} \right| \lesssim \left(\int_{Q(1)} |\mathbf{u}|^3 \right)^{\frac{1}{3}} < \varepsilon_1 < \varepsilon,$$

so we can apply Proposition 12.5.

We then have

$$\theta \left| \fint_{Q_{\theta\rho}} \mathbf{u} \right| \leq 1$$

and

$$E_\theta(\mathbf{u}, p) \leq \underbrace{\Upsilon\theta^{1+\frac{2}{3}}}_{<1} \underbrace{E_1(\mathbf{u}, p)}_{<\varepsilon}.$$

In particular we have $E_\theta(\mathbf{u}, p) < \varepsilon$, so we can reapply Proposition 12.5, and obtain

$$E_{\theta^2}(\mathbf{u}, p) \leq \Upsilon\theta^{1+\frac{2}{3}} E_\theta(\mathbf{u}, p) \leq \left(\Upsilon\theta^{1+\frac{2}{3}}\right)^2 E_1(\mathbf{u}, p).$$

Again, we repeat and after doing this k times we have

$$E_{\theta^k}(\mathbf{u}, p) \leq \left(\Upsilon\theta^{1+\frac{2}{3}}\right)^k \underbrace{E_1(\mathbf{u}, p)}_{<\varepsilon\leq 1}$$

We now discuss what this implies for $\mathbf{u}$:

$$\theta^k \left(|Q_{\theta^k}|^{-1} \int_{Q_{\theta^k}} |\mathbf{u} - (\mathbf{u})_{Q_{\theta^k}}|^3 \right)^{\frac{1}{3}} \leq \left(\Upsilon\theta^{1+\frac{2}{3}}\right)^k.$$

That is

$$|Q_{\theta^k}|^{-1} \int_{Q_{\theta^k}} |\mathbf{u} - (\mathbf{u})_{Q_{\theta^k}}|^3 \leq \left(\Upsilon\theta^2\right)^k.$$

Pick any $\rho \in (0, 1)$, then there exists exactly one $k \in \{0, 1, \ldots\}$ such that

$$\theta^{k+1} \leq \rho < \theta^k.$$

Then we have

$$\rho^{-5} \int_{Q_\rho} |\mathbf{u} - (\mathbf{u})_{Q_{\theta^k}}|^3 \lesssim_\theta |Q_{\theta^k}|^{-1} \int_{Q_{\theta^k}} |\mathbf{u} - (\mathbf{u})_{Q_{\theta^k}}|^3 \leq \left(\Upsilon \theta^2\right)^k.$$

Since $\Upsilon \theta^{2-3\alpha} < 1$ we can write this as

$$\rho^{-5} \int_{Q_\rho} |\mathbf{u} - (\mathbf{u})_{Q_{\theta^k}}|^3 \lesssim_\theta |Q_{\theta^k}|^{-1} \int_{Q_{\theta^k}} |\mathbf{u} - (\mathbf{u})_{Q_{\theta^k}}|^3 \leq \left(\theta^{3\alpha}\right)^k.$$

Here comes the trick, we can compare the right-hand side to ρ:

$$(\theta^{3\alpha})^k = (\theta^k)^{3\alpha} = \theta^{-3\alpha}(\theta^{k+1})^{3\alpha} \leq \theta^{-3\alpha}\rho^{3\alpha}.$$

That is, we have shown for all $\rho \in (0,1)$

$$\sup_{\rho \in (0,1)} \rho^{-3\alpha}\rho^{-5} \int_{Q_\rho} |\mathbf{u} - (\mathbf{u})_{Q_{\theta^k}}|^3 \leq C(\theta).$$

Now we observe that since $\theta^k \approx \rho$, we have

$$\begin{aligned}
&\int_{Q_\rho} |\mathbf{u} - (\mathbf{u})_{Q_\rho}|^3 \\
\lesssim &|Q_\rho|^{-1} \int_{Q_\rho}\int_{Q_\rho} |\mathbf{u}(z_1) - \mathbf{u}(z_2)|^3 dz_1 dz_2 \\
\lesssim &|Q_\rho|^{-1} \int_{Q_\rho}\int_{Q_\rho} |\mathbf{u}(z_1) - (\mathbf{u})_{Q_{\theta^k}}|^3\, dz_1\, dz_2 + |Q_\rho|^{-1} \int_{Q_\rho}\int_{Q_\rho} |\mathbf{u}(z_2) - (\mathbf{u})_{Q_{\theta^k}}|^3\, dz_1\, dz_2 \\
= &2 \int_{Q_\rho} |\mathbf{u}(z) - (\mathbf{u})_{Q_{\theta^k}}|^3\, dz
\end{aligned}$$

We conclude that we have shown That is, we have shown for all $\rho \in (0,1)$

$$\sup_{\rho \in (0,1)} \rho^{-3\alpha}\rho^{-5} \int_{Q_\rho} |\mathbf{u} - (\mathbf{u})_{Q_\rho}|^3 \leq C(\theta).$$

This does not yet imply Hölder continuity, since we are not varying the points of Q_ρ.

Thus, we observe the following:

If $\mathbf{u}$ is a suitable solution to the Navier Stokes equation in Q_R and

$$R^{-2} \int_{Q(R)} |\mathbf{u}|^3 + R^{-2} \int_{Q(R)} |p(x,t)|^{\frac{3}{2}} < \varepsilon_1$$

then $\mathbf{u}$ is a suitable solution in any $B(x_0, \frac{R}{2}) \times (-\frac{R^2}{4}, 0]$ for any $x_0 \in B(0, \frac{R}{2})$, and we have

$$\begin{aligned}
&\left(\frac{R}{2}\right)^{-2} \int_{B(x_0,\frac{R}{2})\times(-\frac{R^2}{4},0]} |\mathbf{u}|^3 + \left(\frac{R}{2}\right)^{-2} \int_{B(x_0,\frac{R^2}{4})\times(-\frac{1}{4},0]} |p(x,t)|^{\frac{3}{2}} \\
< &4 R^{-2} \int_{Q(R)} |\mathbf{u}|^3 + R^{-2} \int_{Q(R)} |p(x,t)|^{\frac{3}{2}} \\
< &4\varepsilon_1
\end{aligned}$$

So, if we take ε_1 in the assumptions just smaller, we can apply the above argument on all $B(x_0, \frac{R}{2}) \times (-\frac{R^2}{4}, 0]$ for any $x_0 \in B(0, \frac{R}{2})$.

Thus we have

$$\sup_{x_0 \in B(0,R/2)} \sup_{\rho \in (0,R)} \rho^{-3\alpha} \rho^{-5} \int_{B(x_0,\rho)\times(-\rho^2,0]} |\mathbf{u} - (\mathbf{u})_{B(x_0,\rho)\times(-\rho^2,0]}|^3 \leq C(\theta, R).$$

This indeed is by Campanato's theorem (see my Analysis lecture notes) equivalent to saying $\mathbf{u} \in C^{\alpha,\frac{\alpha}{2}}(B(0,R/2))$. $\square$

The corollary of the ε-regularity theory, Theorem 12.6, is that solutions to Navier-Stokes are *almost everywhere* relatively nice.

Corollary 12.7. *Let $\Omega \subset \mathbb{R}^n$ be an open set and assume that $(\mathbf{u}, p))$ we a suitable weak solution on $\Omega \times (0,T)$, as defined in Definition 12.3.*

Then there exists a singular set *$\Sigma \subset \Omega \times (0,T)$, namely a relatively closed*[32] *Σ*

$$\mathcal{H}^2(\Sigma) < \infty \quad \text{for any compact set } K \subset \Omega$$

such that for any $(x_0, t_0) \in \Omega \times (0,T) \setminus \Sigma$ we have u is Hölder continuous in a small neighborhood around (x_0, t_0).

Observe that in particular Σ is a $\mathcal{L}^4$-zero-set, thus a (suitable) solution to the Navier-Stokes equations is almost everywhere continuous. However there are examples of other flows of similar spirit (e.g. harmonic map heat flow) that develop singularities in finite time. That is, from what we have proven, there is no reason that $\Sigma = \emptyset$. However that is the conjecture to prove or disprove. Observe tht the Hausdorff dimension of the singular set (in our case we prove that Σ has at most Hausdorff dimension 2) is not optimal, and can be improved (you will see below that we make a lazy estimate).

The proof of Corollary 12.7, besides Theorem 12.6, is the following measure theoretical result, sometimes (incorrectly) referred to as Frostman's lemma.

Lemma 12.8. *Let $p \in [1, \infty)$ and $\alpha \in [0, d)$. Assume that $f \in L^p(\mathbb{R}^d)$ and define the set*

$$E := \{x : \quad \limsup_{r \to 0} r^{-\alpha} \int_{B(x,r)} |f|^p dx > 0\}$$

Then

$$\mathcal{H}^\alpha(E) = 0,$$

where $\mathcal{H}^\alpha$ denotes the Hausdorff measure.

[32] Closedness is important: otherwise we could have dense singular sets, e.g. $\Sigma = \mathbb{Q}$, then the solution is pretty bad – but $\mathcal{H} - dim(\Sigma) = 0$

Remark 12.9.
- Let us try to appreciate, get a feeling of this result: If $f \in L^p(\mathbb{R}^d)$ by the Lebesgue differentiation Lemma, we know
$$\lim_{r\to 0} r^{-d} \int_{B(x,r)} |f|^p = |f(x)|^p \quad \text{a.e. } x \in \mathbb{R}^d$$
and thus for $\alpha \in [0,d)$ we have
$$\lim_{r\to 0} r^{-\alpha} \int_{B(x,r)} |f|^p = 0 \quad \text{a.e. } x \in \mathbb{R}^d$$
Now "a.e." simply means that
$$\mathcal{H}^d(E) = 0.$$
So Lemma 12.8 is a refinement of Lebesgue differentiation theorem.
- For the proof, see [Ziemer, 1989, Corollary 3.2.3.].

Proof of Corollary 12.7. Take ε and θ from Theorem 12.6.

Set

$\Sigma := \{(x,t) \in \Omega \times (T_1,T_2) : \quad \mathbf{u}$ is *not* Hölder continuous in a small neighborhood around $(x,t)\}$

It is clear that Σ is relatively closed in $\Omega \times (T_1,T_2)$ since if $\mathbf{u}$ is Hölder continuous around a small neighborhood (x_0,t_0) that neighborhood also covers $(x_0,t_0) \approx (x_0,t_0)$.

Now we want to estimate the size of Σ.

Define

$$E_1 := \left\{(x,t) \in \Omega \times (T_1,T_2) : \limsup_{R\to 0} R^{-2} \left(\int_{B(x_0,R)\times(t-R^2,t)} |\mathbf{u}|^3 + \int_{B(x_0,R)\times(t-R^2,t)} |p(x,t)|^{\frac{3}{2}}\right) > \varepsilon\right\}$$

From Theorem 12.6 we obtain that whenever $(x,t) \notin E_1$ then $\mathbf{u}$ is Hölder continuous around (x,t). So we have
$$\Omega \times (T_1,T_2) \setminus E_1 \subset \Omega \times (T_1,T_2) \setminus \Sigma,$$
i.e.
$$\Sigma \subset E_1.$$
To apply Lemma 12.8 we do a (lazy!) trick. Set
$$E_2 := \{(x,t) \in \Omega \times (T_1,T_2) : \limsup_{R\to 0} R^{-2} \left(\int_{B^4((x_0,t_0),2R)} |\mathbf{u}|^3 + \int_{B^4((x_0,t_0),2R)} |p(x,t)|^{\frac{3}{2}}\right) > \varepsilon\}$$
Clearly, $E_1 \subset E_2$.

And clearly $E_2 \subset E_u \cup E_p$ where
$$E_u := \left\{(x,t) \in \Omega \times (T_1,T_2) : \limsup_{R\to 0} R^{-2} \int_{B^4((x_0,t_0),2R)} |\mathbf{u}|^3 > 0\right\}$$

and

$$E_p := \left\{(x,t) \in \Omega \times (T_1,T_2) : \limsup_{R \to 0} R^{-2} \int_{B^4((x_0,t_0),2R)} |p(x,t)|^{\frac{3}{2}} > 0\right\}$$

Since $(\mathbf{u}, p)$ are suitable solutions we know $\mathbf{u} \in L^2(\Omega \times (T_1,T_2)) \subset L^2((T_1,T_2), W^{1,2}(\Omega))$ and $p \in L^{\frac{3}{2}}(\Omega \times (T_1,T_2))$

By Lemma 12.8 we know

$$\mathcal{H}^2(E_p) = 0, \quad \mathcal{H}^2(E_u) = 0,$$

and thus we conclude $\mathcal{H}^2(\Sigma) = 0$, which is what we wanted. We can conclude. □

The way of proving or disproving the regularity question for the Navier-Stokes question is now to further analyze Σ. Either with the goal of showing that Σ is empty, or with the hope of constructing an example where Σ is nonempty. The basic hope is to do this (as is done for similar problems in other equations) via a blow-up analysis around singular points. See e.g. the recent work [Seregin, 2023].

12.4. **Proof of the decay estimate: Proposition 12.5.** For simplicity we recall the statement of Proposition 12.5

Proposition (Decay Estimate (recall))**.** *For a constant Υ we have the following: For any $\theta \in (0, \frac{1}{2}]$ there exists $\varepsilon \in (0,1)$ such that the following holds for any $\rho \in (0,1]$.*

Denote

$$Q_\rho := Q(0,\rho) = B(0,\rho) \times (-\rho^2, 0)$$

Assume $\mathbf{u} : Q_\rho \to \mathbb{R}^n$, $p : Q_\rho \to \mathbb{R}$ satisfy

$$\operatorname{div} \mathbf{u} = 0 \quad \text{in } Q_\rho$$

and $\mathbf{u}$ a suitable solution to the Navier-Stokes equation, as defined in Definition 12.3. Assume also that we have the following bounds on Q_ρ:

$$\rho \left| \fint_{Q_\rho} \mathbf{u} \right| \le 1 \tag{12.7}$$

and

$$E_\rho(\mathbf{u}, p) < \varepsilon \tag{12.8}$$

where we recall

$$E_\rho(\mathbf{u},p) := \rho \left(|Q_\rho|^{-1} \int_{Q_\rho} |\mathbf{u} - (\mathbf{u})_{Q_\rho}|^3 \right)^{\frac{1}{3}} + \rho^2 \left(|Q_\rho|^{-1} \int_{Q_\rho} |p(x,t) - (p)_{B(0,\rho)\times\{t\}}|^{\frac{3}{2}} \right)^{\frac{2}{3}}.$$

Then

$$(\theta\rho) \left| \fint_{Q_{\theta\rho}} \mathbf{u} \right| \le 1 \tag{12.9}$$

and

$$E_{\theta\rho}(\mathbf{u},p) \leq \Upsilon \theta^{1+\frac{2}{3}}\, E_\rho(\mathbf{u},p). \tag{12.10}$$

A first observation is that (12.9) follows from (12.7) and (12.8) (this has nothing to do with Navier-Stokes equation)

Lemma 12.10. *For any $\theta \in (0,1)$ there exists some $\varepsilon > 0$ (depending also on the dimension n) such that if (12.8) and (12.7) holds then (12.9) holds.*

Proof. We have

$$\begin{aligned}
\left|\left|\fint_{Q_{\theta\rho}} \mathbf{u}\right| - \left|\fint_{Q_\rho} \mathbf{u}\right|\right| \leq & \frac{1}{|Q_{\theta\rho}|}\int_{Q_{\theta\rho}} \left|\mathbf{u} - \fint_{Q_\rho}\mathbf{u}\right| \\
\leq & \theta^{-n-2}\frac{1}{|Q_\rho|}\int_{Q_\rho}\left|\mathbf{u} - \fint_{Q_\rho}\mathbf{u}\right| \\
\overset{\text{Jensen}}{\leq} & \theta^{-n-2}\left(\frac{1}{|Q_\rho|}\int_{Q_\rho}\left|\mathbf{u} - (\mathbf{u})_{Q_\rho}\right|^3\right)^{\frac{1}{3}} \\
\leq & \rho^{-1}\theta^{-n-2}E_\rho(\mathbf{u},p) \\
\overset{(12.8)}{\leq} & \rho^{-1}\theta^{-n-2}\varepsilon.
\end{aligned}$$

Thus,

$$\begin{aligned}
\theta\rho\left|\fint_{Q_{\theta\rho}}\mathbf{u}\right| \leq & \theta\rho\left|\fint_{Q_\rho}\mathbf{u}\right| + \theta^{-n-1}\varepsilon \\
\overset{(12.7)}{\leq} & \theta + \theta^{-n-1}\varepsilon.
\end{aligned}$$

We can take ε small enough so that $\theta + \theta^{-n-1}\varepsilon < 1$ and conclude. $\square$

Proof of Proposition 12.5. As we have discussed at the beginning of the section, we know the scaling behavior of the equation. We observe that the equation, and assumptions (12.8)(12.7) scale, i.e. otherwise replacing $\mathbf{u}$ by and p as in Equation (12.2) (for $\lambda = \frac{1}{\rho}$) we may assume $\rho = 1$.

Assume now the claim is false.

Then there must be a $\theta \in (0, \frac{1}{2}]$, such that the claim does not hold for any $\varepsilon > 0$. So for each $k \in \mathbb{N}$ there exists a sequence of suitable weak solutions $(\mathbf{u}_k, p_k)_{k\in\mathbb{N}}$ to the Navier-Stokes equation

$$\begin{gathered}
\mathbf{u}_k \in L^\infty((-1,0), L^2(B(0,1))) \cap L^2((-1,0), W^{1,2}(B(0,1))) \\
p_k \in L^{\frac{3}{2}}(B(0,1)\times(-1,0))
\end{gathered}$$

with

$$E_1(\mathbf{u}_k, p_k) < \frac{1}{k}$$

and

$$\left|\fint_{Q_1} \mathbf{u}_k\right| \leq 1$$

however (where Υ can still be chosen)

$$E_\theta(\mathbf{u}_k, p_k) > \Upsilon\theta^{1+\frac{2}{3}} E_1(\mathbf{u}_k, p_k).$$

We set

$$\varepsilon_k := E_1(\mathbf{u}_k, p_k),$$

then we have the following properties

- $\varepsilon_k \xrightarrow{k\to\infty} 0$
- $\varepsilon_k = E_1(\mathbf{u}_k, p_k)$
- $E_\theta(\mathbf{u}_k, p_k) > \Upsilon\theta^{1+\frac{2}{3}} \varepsilon_k.$

Set

$$\mathbf{v}_k(x,t) := (\varepsilon_k)^{-1} \left(\mathbf{u}_k(x,t) - (\mathbf{u}_k)_{Q_1}\right)$$

$$q_k(x,t) := (\varepsilon_k)^{-1} \left(p(x,t) - (p)_{B(0,1)\times\{t\}}\right)$$

We then have

$$\begin{aligned}
\partial_t \mathbf{v}_k - \Delta \mathbf{v}_k =&(\varepsilon_k)^{-1}(\partial_t \mathbf{u}_k - \Delta \mathbf{u}_k)\\
=&(\varepsilon_k)^{-1}\left(-u_\alpha \partial_\alpha \mathbf{u}_k - \nabla p_k\right)\\
=&-u_\alpha \partial_\alpha \mathbf{v}_k - \nabla q_k\\
=&-((\mathbf{u}_k)_{Q_1} + \varepsilon_k \mathbf{v}_k)_\alpha \partial_\alpha \mathbf{v}_k - \nabla q_k
\end{aligned}$$

That is for

$$\mathbf{b}_k := (\mathbf{u}_k)_{Q_1}$$

we have

$$\begin{cases} \partial_t \mathbf{v}_k - \Delta \mathbf{v}_k + (\mathbf{b}_k + \varepsilon_k \mathbf{v}_k)_\alpha \partial_\alpha \mathbf{v}_k = -\nabla q_k & \text{in } Q_1 \\ \operatorname{div} \mathbf{v}_k = 0 & \text{in } Q_1 \end{cases} \tag{12.11}$$

with the following properties k

- $|\mathbf{b}_k| \leq 1$ by assumption (12.7)
- $\int_{Q_1} v_k = 0$
- $\int_{B(0,1)\times\{t\}} q_k = 0$ for all $t \in (-1, 0)$.

Up to passing to a subsequence (not relabelled) we may assume that there exists some $\mathbf{b} \in \mathbb{R}^3$, $|\mathbf{b}| \leq 1$ such that

$$\mathbf{b}_k \xrightarrow{k\to\infty} \mathbf{b}.$$

We also observe that

$$\varepsilon_k = E_1(\mathbf{u}_k, p_k) = \left(|Q_1|^{-1}\int_{Q_1} |\mathbf{u}_k - (\mathbf{u}_k)_{Q_1}|^3\right)^{\frac{1}{3}} + \left(|Q_1|^{-1}\int_{Q_1} |p_k(x,t) - (p_k)_{B(0,1)\times\{t\}}|^{\frac{3}{2}}\right)^{\frac{2}{3}}$$
$$= \left(|Q_1|^{-1}\int_{Q_1} |\varepsilon_k \mathbf{v}_k|^3\right)^{\frac{1}{3}} + \left(|Q_1|^{-1}\int_{Q_1} |\varepsilon_k q_k|^{\frac{3}{2}}\right)^{\frac{2}{3}}$$

so that (dividing out the ε_k) we find

$$\left(|Q_1|^{-1}\int_{Q_1} |\mathbf{v}_k|^3\right)^{\frac{1}{3}} + \left(|Q_1|^{-1}\int_{Q_1} |q_k|^{\frac{3}{2}}\right)^{\frac{2}{3}} \leq 1 \quad \forall k \in \mathbb{N}. \tag{12.12}$$

Similarly,

$$\begin{aligned}\Upsilon\theta^{1+\frac{2}{3}}\,\varepsilon_k < E_\theta(\mathbf{u}_k, p_k) =&\theta\left(|Q_\theta|^{-1}\int_{Q_\theta} |\mathbf{u}_k - (\mathbf{u}_k)_{Q_\theta}|^3\right)^{\frac{1}{3}} + \theta^2\left(|Q_\theta|^{-1}\int_{Q_\theta} |p_k(x,t) - (p_k)_{B(0,\theta)\times\{t\}}|^{\frac{3}{2}}\right)^{\frac{2}{3}}\\
=&\varepsilon_k\theta\left(|Q_\theta|^{-1}\int_{Q_\theta} |\mathbf{v}_k - (\mathbf{v}_k)_{Q_\theta}|^3\right)^{\frac{1}{3}} + \varepsilon_k\theta^2\left(|Q_\theta|^{-1}\int_{Q_\theta} |q_k(x,t) - (q_k)_{B(0,\theta)\times\{t\}}|^{\frac{3}{2}}\right)^{\frac{2}{3}}\\
=&\varepsilon_k E_\theta(\mathbf{v}_k, q_k),\end{aligned}$$

so that we have

$$E_\theta(\mathbf{v}_k, q_k) > \Upsilon\ \theta^{1+\frac{2}{3}}. \tag{12.13}$$

(12.12) implies in particular

$$\sup_k \|\mathbf{v}_k\|_{L^3(Q_1)} + \|q_k\|_{L^{\frac{3}{2}}(Q_1)} < \Gamma_1, \tag{12.14}$$

where Γ_1 is a uniform constant (depending only on the size of Q_1).

Up to subsequence we then find some $\mathbf{v} \in L^3(Q_1)$, $q \in L^{\frac{3}{2}}(Q_1)$ (here we use $\frac{3}{2} > 1$) such that

$$\mathbf{v}_k \rightharpoonup \mathbf{v} \quad \text{weakly in } L^3(Q_1, \mathbb{R}^3)$$

$$q_k \rightharpoonup q \quad \text{weakly in } L^{\frac{3}{2}}(Q_1)$$

We now would like to pass to the limit in the PDE of $\mathbf{v}_k$, (12.11). However the nonlinearity creates an obstacly if we only have *weak* convergence (since the product of two weakly convergent functions may not be weakly convergent).

To overcome this we need $W^{1,2}$-control – and for this we use the local energy inequality, (12.1). But we assumed it only for $\mathbf{u}_k$, we need it for $\mathbf{v}_k$.

Claim 12.11. $\mathbf{v}_k$ satisfies the following local energy inequality.

For almost all $t \in (-1,0]$ and for any $\eta \in C_c^\infty(B(0,1)\times(-1,\infty))$,

$$\begin{aligned}
&\int_{B(0,1)} \eta(x,t)|\mathbf{v}_k(x,t)|^2 dx + 2\int_{B(0,1)\times(-1,t)} \eta(x,t')|\nabla \mathbf{v}_k|^2\, dx\, dt' \\
\le &\int_{B(0,1)\times(-1,t)} |\mathbf{v}_k(x,t')|^2(\Delta\eta + \partial_t\eta) \\
&+ \int_{B(0,1)\times(-1,t)} \mathbf{v}_k(x,t')\cdot\nabla\eta(x,t')\ \ 2q_k dx\, dt' \\
&+ \varepsilon_k \int_{B(0,1)\times(-1,t)} \mathbf{v}_k(x,t')\cdot\nabla\eta(x,t')\ |\mathbf{v}_k(x,t')|^2 dx\, dt' \\
&+ \int_{B(0,1)\times(-1,t)} (\mathbf{u}_k)_{Q_1}\cdot\nabla\eta(x,t')\ |\mathbf{v}_k(x,t')|^2 dx\, dt'
\end{aligned}$$

Assuming Claim 12.11 for now (we prove it below) for any a.e. $t \in (-1,0)$,

$$\int_{B(0,1)\times(-1,t)} |\nabla \mathbf{v}_k|^2 dx\, dt' \lesssim_\eta \left(\|\mathbf{v}_k\|^2_{L^2(Q_1)} + \|q_k\|_{L^{\frac32}(Q_1)}\|\mathbf{v}_k\|_{L^3(Q_1)} + \varepsilon_k\|\mathbf{v}_k\|^3_{L^3(Q_1)} + |(\mathbf{u}_k)_{Q_1}|\|\mathbf{v}_k\|^2_{L^2(Q_1)}\right)$$

We observe the right-hand side is uniformly bounded in k in view of Equation (12.14), and thus we have (letting $t \to 1^-$)

$$\sup_{k\in\mathbb{N}} \|\nabla \mathbf{v}_k\|_{L^2(Q_R)} \lesssim_R \Gamma_1 \quad \forall R \in (0,1). \tag{12.15}$$

Observe this does *not* mean $\mathbf{v}_k \in W^{1,2}(Q_R)$, since ∇ is the *spatial* gradient (for now).

By a similar argument, using the first term of the local energy inequality we also obtain

$$\sup_{t\in(-R^2,0)} \|\mathbf{v}_k(\cdot,t)\|_{L^2(B(0,R))} \lesssim \Gamma \quad \forall R \in (0,1). \tag{12.16}$$

While we don't yet have $W^{1,2}$-uniform control, we can however conclude already that, up to taking a subsequence (not relabeled),

$$\nabla \mathbf{v}_k \rightharpoonup \nabla \mathbf{v} \quad \text{weakly in } L^2_{loc}(B(0,1)\times(-1,0],\mathbb{R}^3).$$

As a consequence we have for any $\phi \in C_c^\infty(Q_1)$ control of the nonlinearity,

$$\lim_{k\to\infty} \int_{Q_1} (\mathbf{b}_k + \varepsilon_k \mathbf{v}_k)_\alpha \partial_\alpha \mathbf{v}_k \phi = \int_{Q_1} (\mathbf{b} + 0)_\alpha \partial_\alpha \mathbf{v}\phi$$

Indeed, by trilinearity

$$
\begin{aligned}
&\left|\int_{Q_1} (\mathbf{b}_k + \varepsilon_k \mathbf{v}_k)_\alpha \partial_\alpha \mathbf{v}_k \phi - \int_{Q_1} (\mathbf{b} + 0)_\alpha \partial_\alpha \mathbf{v} \phi\right| \\
\leq &|\mathbf{b}_k - \mathbf{b}| \int_{Q_1} |\nabla \mathbf{v}_k|\, |\phi| + \left|\mathbf{b}_\alpha \int_Q \partial_\alpha (\mathbf{v_k} - \mathbf{v})\phi\right| + \varepsilon_k \int_{Q_1} |\mathbf{v}_k|\, |\nabla \mathbf{v}_k|\ |\phi| \\
\lesssim_\phi &\underbrace{|\mathbf{b}_k - \mathbf{b}|}_{\xrightarrow{k\to\infty} 0} \underbrace{\|\nabla \mathbf{v}_k\|_{L^1(\operatorname{supp}\phi)}}_{\sup_k < \infty} + |\mathbf{b}| \underbrace{\left|\int_Q \partial_\alpha (\mathbf{v}_k - \mathbf{v})\ \phi\right|}_{\xrightarrow{k\to\infty} 0} + \underbrace{\varepsilon_k}_{\xrightarrow{k\to\infty} 0} \underbrace{\|\mathbf{v}_k\|_{L^2(\operatorname{supp}\phi)}}_{\sup_k<\infty} \underbrace{\|\nabla \mathbf{v}_k\|_{L^2(\operatorname{supp}\phi)}}_{\sup_k<\infty} \\
&\xrightarrow{k\to\infty} 0.
\end{aligned}
$$

In view of the PDE for $\mathbf{v}_k$, (12.11), we can now pass to the limit and obtain

$$
\begin{cases}
\partial_t \mathbf{v} - \Delta \mathbf{v} + \mathbf{b}_\alpha \partial_\alpha \mathbf{v} = -\nabla q \qquad & \text{in } Q_1\\
\operatorname{div} \mathbf{v} = 0 & \text{in } Q_1
\end{cases}
\tag{12.17}
$$

Observe that $\mathbf{b}$ is a constant vector, so this is almost a linear constant coefficient equation – if it wasn't for ∇q.

But we can seperate the equation for q and $\vec{v}$: Since $\operatorname{div} \mathbf{v} = 0$ we can take div in the equation (12.17) and find that (observe that $\mathbf{b}$ is constant!)

$$
0 = -\Delta q \quad \text{in } B(0,1) \times \{t\} \text{ for a.e. any } t \in (-1,0)
$$

On the other hand, we can take the curl of the equation, where

$$
\operatorname{curl} \mathbf{f} = \left(\partial_\alpha f^\beta - \partial_\beta f^\alpha\right)_{\alpha,\beta=1,\ldots,3},
$$

and see that then $\operatorname{curl} \nabla q = 0$ and thus

$$
\begin{cases}
\partial_t \operatorname{curl} \mathbf{v} - \Delta \operatorname{curl} \mathbf{v} + \mathbf{b}_\alpha \partial_\alpha \operatorname{curl} \mathbf{v} = 0 \qquad & \text{in } Q_1\\
\operatorname{div} \mathbf{v} = 0 & \text{in } Q_1
\end{cases}
$$

Since $\mathbf{v}$ is completely controlled by $\operatorname{div} \mathbf{v}$ and $\operatorname{curl} \mathbf{v}$ (this will be the so-called *Helmholtz-Hodge decomposition*) we now have three very nice linear PDE that control $\mathbf{v}$ (and later as well q).

For $\mathbf{v}$ we have the following *regularity for the blow up equation*:

Claim 12.12. We have for all $R \in (0,1)$

$$
\sup_{(x,t),(y,t)\in Q_R} \frac{|\mathbf{v}(x,t) - \mathbf{v}(y,s)|}{|x-y|^{\frac{2}{3}} + |s-t|^{\frac{1}{3}}} \lesssim_R \Gamma_1
\tag{12.18}
$$

We give the proof below, but first show how to use this.

Recall our "counterexample radius" θ (whose existence we like to disprove). Applying (12.18) we conclude (recall $Q_\theta = B(0,\theta) \times (-\theta^2, 0)$)

$$|\mathbf{v}(x,t) - \mathbf{v}(y,s)| \lesssim \Gamma_1 \Big(|x-y|^{\frac{2}{3}} + \underbrace{|s-t|}_{\lesssim \theta^2}^{\frac{1}{3}} \Big) \lesssim \Gamma_1 \theta^{\frac{2}{3}}.$$

It is important to note that the constant in $\lesssim$ is independent of θ, since we assumed that $\theta \leq \frac{1}{2}$ so we can apply (12.18) for $R = 1/2$.

We conclude that

$$\fint_{Q_\theta} |\mathbf{v} - (\mathbf{v})_{Q_\theta}|^3 \leq \frac{1}{2} C_2 \theta^2$$

Next we need to transfer this estimate to $\mathbf{v}_k$. For this we need the following compactness result, which is a consequence of *Aubin-Lions-Lemma*, Theorem 12.19 – essentially a Rellich-type theorem.

Claim 12.13. Up to taking a subsequence, $\mathbf{v}_k$ converges *strongly* in $L^3_{loc}(B(0,1) \times (-1,0])$

From the strong L^3-convergence we find that for all *large* $k \gg 1$

$$\theta \left(\fint_{Q_\theta} |\mathbf{v}_k - (\mathbf{v}_k)_{Q_\theta}|^3 \right)^{\frac{1}{3}} \leq (C_2)^{\frac{1}{3}} \, \theta^{1+\frac{2}{3}} \tag{12.19}$$

Observe that this is *almost* a contradiction to (12.13) – if we choose $\Upsilon > (C_2)^{\frac{1}{3}}$ large enough – it wasn't for the q_k-term that we treat next. Recall that q_k and $\mathbf{v}_k$ are related by the PDE, namely we have

$$\begin{cases} \partial_t \mathbf{v}_k - \Delta \mathbf{v}_k + (\mathbf{b}_k + \varepsilon_k \mathbf{v}_k)_\alpha \partial_\alpha \mathbf{v}_k = -\nabla q_k \quad \text{in } Q_1 \\ \operatorname{div} \mathbf{v}_k = 0 \quad \text{in } Q_1 \end{cases} \tag{12.11}$$

Taking the divergence we find

$$\text{div } ((\mathbf{b}_k + \varepsilon_k \mathbf{v}_k)_\alpha \partial_\alpha \mathbf{v}_k) = -\Delta q_k \quad \text{in } Q_1$$

Observe that

$$\begin{aligned}
&\text{div } ((\mathbf{b}_k + \varepsilon_k \mathbf{v}_k)_\alpha \partial_\alpha \mathbf{v}_k) \\
=&\partial_\beta \left((\mathbf{b}_k + \varepsilon_k \mathbf{v}_k)_\alpha \partial_\alpha (\mathbf{v}_k)_\beta \right) \\
\overset{\text{div}=0}{=}&\partial_\beta (\mathbf{b}_k + \varepsilon_k \mathbf{v}_k)_\alpha \; \partial_\alpha (\mathbf{v}_k)_\beta \\
=&\varepsilon_k \partial_\beta (\mathbf{v}_k)_\alpha \; \partial_\alpha (\mathbf{v}_k)_\beta \\
\overset{\text{div}=0}{=}&\varepsilon_k \partial_\beta \left((\mathbf{v}_k)_\alpha \; \partial_\alpha (\mathbf{v}_k)_\beta \right) \\
\overset{\text{div}=0}{=}&\varepsilon_k \partial_\beta \partial_\alpha \left((\mathbf{v}_k)_\alpha \; (\mathbf{v}_k)_\beta \right)
\end{aligned}$$

So we have

$$-\Delta q_k = \varepsilon_k \partial_\beta \partial_\alpha \left((\mathbf{v}_k)_\alpha \; (\mathbf{v}_k)_\beta \right) \quad \text{in } Q_1.$$

Observe that is an elliptic PDE without time control, i.e. we should think of it rather as

$$-\Delta q_k = \varepsilon_k \partial_\beta \partial_\alpha \left((\mathbf{v}_k)_\alpha \ (\mathbf{v}_k)_\beta\right) \quad \text{in } B(0,1) \times \{t\},\ t \in (-1,0).$$

In terms of the Riesz transforms $\mathcal{R}_\alpha$ from Calderon-Zygmund theory, Equation (7.4), we can think of this as *almost*

$$q_k \approx \varepsilon_k \mathcal{R}_\alpha \mathcal{R}_\beta \left((\mathbf{v}_k)_\alpha \ (\mathbf{v}_k)_\beta\right) \quad \text{in } \mathbb{R}^3 \times \{t\}.$$

So the L^3-estimate for $\mathbf{v}_k$ should become a $L^{\frac{3}{2}}$-estimate for q_k.

To make this idea precise, let $\eta \in C_c^\infty(B(0,1))$, $\eta \equiv 1$ in $B(0, \frac{1+R}{2})$ for some $R < 1$. Set

$$\tilde{q}_k := \varepsilon_k \mathcal{R}_\alpha \mathcal{R}_\beta \left(\eta \, (\mathbf{v}_k)_\alpha \ (\mathbf{v}_k)_\beta\right) \text{ in } \mathbb{R}^3 \times \{t\}.$$

Then we observe two things:

Firstly, from Calderon-Zygmund theory, we have

$$\begin{aligned}
\|\tilde{q}_k(t)\|_{L^{\frac{3}{2}}(\mathbb{R}^3)} \lesssim & \varepsilon_k \left\|\eta \, (\mathbf{v}_k)_\alpha \ (\mathbf{v}_k)_\beta\right\|_{L^{\frac{3}{2}}(\mathbb{R}^3)} \\
\lesssim & \varepsilon_k \left\|(\mathbf{v}_k)_\alpha \ (\mathbf{v}_k)_\beta\right\|_{L^{\frac{3}{2}}(B(0,1))} \\
\lesssim & \varepsilon_k \|\mathbf{v}_k\|^2_{L^3(B(0,1))}
\end{aligned}$$

So we have in particular

$$\|\tilde{q}_k(t)\|_{L^{\frac{3}{2}}(Q_1)} \lesssim \varepsilon_k \|\mathbf{v}_k\|^2_{L^3(Q_1)} \overset{(12.14)}{\leq} \varepsilon_k \ (\Gamma_1)^2. \tag{12.20}$$

Now we consider the difference of $\tilde{q}_k$ and q_k: By construction with the Riesz transform we have

$$\Delta(\tilde{q}_k - q_k) = \varepsilon_k \partial_\alpha \partial_\beta \left((\eta - 1)(\mathbf{v}_k)_\alpha \ (\mathbf{v}_k)_\beta\right) \quad \text{in } B(0,1) \times \{t\}$$

In particular, since $\eta \equiv 1$ in $B(\frac{1+R}{2})$ we have

$$\Delta(\tilde{q}_k - q_k) = 0 \quad \text{in } B(0, \frac{1+R}{2}) \times (-1,0).$$

That is, $h := \tilde{q}_k - q_k$ is harmonic, and thus we have from Cauchy estimates, Exercise 2.43,

$$\begin{aligned}
\int_{B(0,\rho)} |h - \fint_{B(0,\rho)} h|^{\frac{3}{2}} \lesssim & \left(\frac{\rho}{R}\right)^{\frac{3}{2}+3} \int_{B(0,R)} |h|^{\frac{3}{2}} \\
\lesssim & \left(\frac{\rho}{R}\right)^{\frac{3}{2}+3} \int_{B(0,R)} \left(|\tilde{q}_k|^{\frac{3}{2}} + |q_k|^{\frac{3}{2}}\right).
\end{aligned}$$

Using again the definition of h on the left-hand side, we have

$$\begin{aligned}
\int_{B(0,\rho)} |q_k - \fint_{B(0,\rho)} q_k|^{\frac{3}{2}} \lesssim & \left(\frac{\rho}{R}\right)^{\frac{3}{2}+3} \int_{B(0,R)} \left(|\tilde{q}_k|^{\frac{3}{2}} + |q_k|^{\frac{3}{2}}\right) \\
& + \int_{B(0,1)} |\tilde{q}_k(t)|^{\frac{3}{2}}
\end{aligned}$$

Integrating this in time, we have

$$\begin{aligned}\int_{Q_\rho} |q_k - \fint_{B(0,\rho)\times\{t\}} q_k|^{\frac{3}{2}} \lesssim & \left(\frac{\rho}{R}\right)^{\frac{3}{2}+3} \int_{Q_R} \left(|\tilde{q}_k|^{\frac{3}{2}} + |q_k|^{\frac{3}{2}}\right)\\ & + \int_{Q_1} |\tilde{q}_k(t)|^{\frac{3}{2}}\\ \overset{(12.20),(12.14)}{\lesssim} & \left(\frac{\rho}{R}\right)^{\frac{3}{2}+3} \left((\varepsilon_k)^{\frac{3}{2}}\Gamma_1^3 + \Gamma_1^{\frac{3}{2}}\right)\\ & + (\varepsilon_k)^{\frac{3}{2}}\Gamma_1^3\end{aligned}$$

We apply this inequality for $\rho = \theta$ and $R = \frac{99}{100}$. For $k \gg 1$ we have that $(\varepsilon_k)^{\frac{3}{2}} \ll \theta^{\frac{3}{2}+3}$,

$$\int_{Q_\theta} |q_k - \fint_{B(0,\theta)\times\{t\})} q_k|^{\frac{3}{2}} \lesssim (\theta)^{\frac{3}{2}+3} \left(2\Gamma_1^3 + \Gamma_1^{\frac{3}{2}}\right)$$

.

and thus

$$\theta^2 \left(\fint_{Q_\theta} |q_k - \fint_{B(0,\theta)\times\{t\})} q_k|^{\frac{3}{2}}\right)^{\frac{2}{3}} \lesssim (\theta)^{1+\frac{2}{3}} \left(2\Gamma_1^3 + \Gamma_1^{\frac{3}{2}}\right)^{\frac{2}{3}}.$$

So, combining this with (12.19), for all $k \gg 1$ we have found

$$\begin{aligned}E_\theta(\mathbf{v}_k, q_k) =& \theta \left(|Q_\theta|^{-1} \int_{Q_\theta} |\mathbf{v}_k - (\mathbf{v}_k)_{Q_\theta}|^3\right)^{\frac{1}{3}} + \theta^2 \left(|Q_\theta|^{-1} \int_{Q_\theta} |q_k(x,t) - (q_k)_{B(0,\theta)\times\{t\}}|^{\frac{3}{2}}\right)^{\frac{2}{3}}\\ \leq& \theta^{1+\frac{2}{3}}\ C(\Gamma_1,\Gamma_2).\end{aligned}$$

Thus, taking $\Upsilon > C(\Gamma_1,\Gamma_2)$ we have found a contradiction to (12.13).

We finally can conclude – up to proving all the claims we made before. □

Exercise 12.14. *Prove the Cauchy estimates from Exercise 2.43*

12.5. **Proof of the energy inequality, Claim 12.11.** Above we used the energy inequality for $\mathbf{v}_k$, the proof is a bit messy, but it follows essentially from a combination of the locally energy inequality of $\mathbf{u}$ combined with the fact that we actually can test the equation of $\mathbf{v}_k$ with $\eta(\mathbf{u})_k$ which removes appearing cross-terms.

Proof of the energy inequality, Claim 12.11.

(12.21)

$$\begin{aligned}&\int_{B(0,1)} \eta(x,t)|\mathbf{u}_k(x,t)|^2 dx + 2\int_{B(0,1)\times(-1,t)} \eta(x,t')|\nabla \mathbf{u}_k|^2\, dx\, dt'\\ \leq& \int_{B(0,1)\times(-1,t)} |\mathbf{u}_k(x,t')|^2(\Delta\eta + \partial_t\eta) + \mathbf{u}_k(x,t')\cdot\nabla\eta(x,t')\ \left(|\mathbf{u}_k(x,t')|^2 + 2p_k(x,t')\right) dx\, dt'\end{aligned}$$

We now stubbornly replace

$$\mathbf{u}_k = \varepsilon_k \mathbf{v}_k + (\mathbf{u}_k)_{Q_1}, \quad p_k = \varepsilon_k q_k + (p_k)_{B(0,1)\times\{t\}}.$$

Then we find

(12.22)

$$\int_{B(0,1)} \eta(x,t)|\varepsilon_k \mathbf{v}_k(x,t) + (\mathbf{u}_k)_{Q_1}|^2 dx + 2(\varepsilon_k)^2 \int_{B(0,1)\times(-1,t)} \eta(x,t')|\nabla \mathbf{v}_k|^2\, dx\, dt'$$
$$\leq \int_{B(0,1)\times(-1,t)} |\varepsilon_k \mathbf{v}_k(x,t') + (\mathbf{u}_k)_{Q_1}|^2 (\Delta\eta + \partial_t \eta)$$
$$+ (\varepsilon_k \mathbf{v}_k(x,t') + (\mathbf{u}_k)_{Q_1}) \cdot \nabla\eta(x,t') \ \left(|\varepsilon_k \mathbf{v}_k(x,t') + (\mathbf{u}_k)_{Q_1}|^2 + 2\left(\varepsilon_k q_k + (p_k)_{B(0,1)\times\{t\}}\right)\right) dx\, dt'$$

Expanding the quadratic terms on the left-hand side (and bringing them to the right-hand side) we have

$$\int_{B(0,1)} \eta(x,t)|\varepsilon_k \mathbf{v}_k(x,t)|^2 dx + 2(\varepsilon_k)^2 \int_{B(0,1)\times(-1,t)} \eta(x,t')|\nabla \mathbf{v}_k|^2\, dx\, dt'$$
$$\leq \int_{B(0,1)\times(-1,t)} |\varepsilon_k \mathbf{v}_k(x,t') + (\mathbf{u}_k)_{Q_1}|^2 (\Delta\eta + \partial_t \eta)$$
$$- \int_{B(0,1)} \eta(x,t)|(\mathbf{u}_k)_{Q_1}|^2 dx$$
$$- 2\int_{B(0,1)} \eta(x,t)\langle \varepsilon_k \mathbf{v}_k(x,t), (\mathbf{u}_k)_{Q_1}\rangle dx$$
$$+ \int_{B(0,1)\times(-1,t)} (\varepsilon_k \mathbf{v}_k(x,t') + (\mathbf{u}_k)_{Q_1}) \cdot \nabla\eta(x,t') \ \left(|\varepsilon_k \mathbf{v}_k(x,t') + (\mathbf{u}_k)_{Q_1}|^2 + 2\left(\varepsilon_k q_k + (p_k)_{B(0,1)\times\{t\}}\right)\right) dx\, dt'$$

and thus (expanding the first term on the right),

$$\int_{B(0,1)} \eta(x,t)|\varepsilon_k \mathbf{v}_k(x,t)|^2 dx + 2(\varepsilon_k)^2 \int_{B(0,1)\times(-1,t)} \eta(x,t')|\nabla \mathbf{v}_k|^2\, dx\, dt'$$
$$\leq \int_{B(0,1)\times(-1,t)} |\varepsilon_k \mathbf{v}_k(x,t')|^2 (\Delta\eta + \partial_t \eta)$$
$$+ \int_{B(0,1)\times(-1,t)} |(\mathbf{u}_k)_{Q_1}|^2 (\Delta\eta + \partial_t \eta)$$
$$+ 2\int_{B(0,1)\times(-1,t)} \langle \varepsilon_k \mathbf{v}_k(x,t'), (\mathbf{u}_k)_{Q_1}\rangle (\Delta\eta + \partial_t \eta)$$
$$- \int_{B(0,1)} \eta(x,t)|(\mathbf{u}_k)_{Q_1}|^2 dx$$
$$- 2\int_{B(0,1)} \eta(x,t)\langle \varepsilon_k \mathbf{v}_k(x,t), (\mathbf{u}_k)_{Q_1}\rangle dx$$
$$+ \int_{B(0,1)\times(-1,t)} (\varepsilon_k \mathbf{v}_k(x,t') + (\mathbf{u}_k)_{Q_1}) \cdot \nabla\eta(x,t') \ \left(|\varepsilon_k \mathbf{v}_k(x,t') + (\mathbf{u}_k)_{Q_1}|^2 + 2\left(\varepsilon_k q_k + (p_k)_{B(0,1)\times\{t\}}\right)\right) dx\, dt'$$

We observe

$$\int_{B(0,1)\times(-1,t)} |(\mathbf{u}_k)_{Q_1}|^2 \Delta\eta = 0$$

and

$$\int_{B(0,1)\times(-1,t)} |(\mathbf{u}_k)_{Q_1}|^2 \partial_t \eta = \int_{B(0,1)} |(\mathbf{u}_k)_{Q_1}|^2 \eta - 0.$$

So

$$\int_{B(0,1)\times(-1,t)} |(\mathbf{u}_k)_{Q_1}|^2 (\Delta\eta + \partial_t \eta) - \int_{B(0,1)} \eta(x,t) |(\mathbf{u}_k)_{Q_1}|^2 dx = 0$$

Then we arrive at

$$\begin{aligned}
&(\varepsilon_k)^2 \left(\int_{B(0,1)} \eta(x,t)|\mathbf{v}_k(x,t)|^2 dx + 2\int_{B(0,1)\times(-1,t)} \eta(x,t')|\nabla \mathbf{v}_k|^2\, dx\, dt' \right)\\
\leq &(\varepsilon_k)^2 \int_{B(0,1)\times(-1,t)} |\mathbf{v}_k(x,t')|^2 (\Delta\eta + \partial_t\eta)\\
&+ 2\int_{B(0,1)\times(-1,t)} \langle \varepsilon_k \mathbf{v}_k(x,t'), (\mathbf{u}_k)_{Q_1}\rangle (\Delta\eta + \partial_t\eta) - 2\int_{B(0,1)} \eta(x,t) \langle \varepsilon_k \mathbf{v}_k(x,t), (\mathbf{u}_k)_{Q_1}\rangle dx\\
&+ \int_{B(0,1)\times(-1,t)} (\varepsilon_k \mathbf{v}_k(x,t') + (\mathbf{u}_k)_{Q_1}) \cdot \nabla\eta(x,t') \ \left(|\varepsilon_k \mathbf{v}_k(x,t') + (\mathbf{u}_k)_{Q_1}|^2 + 2\left(\varepsilon_k q_k + (p_k)_{B(0,1)\times\{t\}}\right)\right) dx\, dt'
\end{aligned}$$

We now try to simplify the third term.

$$\begin{aligned}
&\int_{B(0,1)\times(-1,t)} (\varepsilon_k \mathbf{v}_k(x,t') + (\mathbf{u}_k)_{Q_1}) \cdot \nabla\eta(x,t') \ \left(|\varepsilon_k \mathbf{v}_k(x,t') + (\mathbf{u}_k)_{Q_1}|^2 + 2\left(\varepsilon_k q_k + (p_k)_{B(0,1)\times\{t\}}\right)\right) dx\, dt'\\
=&\int_{B(0,1)\times(-1,t)} (\varepsilon_k \mathbf{v}_k(x,t') + (\mathbf{u}_k)_{Q_1}) \cdot \nabla\eta(x,t') \ |\varepsilon_k \mathbf{v}_k(x,t') + (\mathbf{u}_k)_{Q_1}|^2 dx\, dt'\\
&+ 2\int_{B(0,1)\times(-1,t)} (\varepsilon_k \mathbf{v}_k(x,t') + (\mathbf{u}_k)_{Q_1}) \cdot \nabla\eta(x,t') \ \left(\varepsilon_k q_k + (p_k)_{B(0,1)\times\{t\}}\right) dx\, dt'\\
=&\int_{B(0,1)\times(-1,t)} \varepsilon_k \mathbf{v}_k(x,t') \cdot \nabla\eta(x,t') \ |\varepsilon_k \mathbf{v}_k(x,t') + (\mathbf{u}_k)_{Q_1}|^2 dx\, dt'\\
&+ \int_{B(0,1)\times(-1,t)} (\mathbf{u}_k)_{Q_1} \cdot \nabla\eta(x,t') \ |\varepsilon_k \mathbf{v}_k(x,t') + (\mathbf{u}_k)_{Q_1}|^2 dx\, dt'\\
&+ 2\int_{B(0,1)\times(-1,t)} \varepsilon_k \mathbf{v}_k(x,t') \cdot \nabla\eta(x,t') \ \left(\varepsilon_k q_k + (p_k)_{B(0,1)\times\{t\}}\right) dx\, dt'\\
&+ 2\int_{B(0,1)\times(-1,t)} (\mathbf{u}_k)_{Q_1} \cdot \nabla\eta(x,t') \ \left(\varepsilon_k q_k + (p_k)_{B(0,1)\times\{t\}}\right) dx\, dt'
\end{aligned}$$

We notice that the term

$$2\int_{B(0,1)\times(-1,t)} (\mathbf{u}_k)_{Q_1} \cdot \nabla\eta(x,t')\, (p_k)_{B(0,1)\times\{t\}} dx\, dt' = 0$$

and since $\operatorname{div} \mathbf{v}_k = 0$ also

$$2\int_{B(0,1)\times(-1,t)} \varepsilon_k \mathbf{v}_k(x,t') \cdot \nabla\eta(x,t') \ (p_k)_{B(0,1)\times\{t\}} dx\, dt' = 0.$$

So

$$
\begin{aligned}
&\int_{B(0,1)\times(-1,t)} (\varepsilon_k \mathbf{v}_k(x,t') + (\mathbf{u}_k)_{Q_1}) \cdot \nabla\eta(x,t') \ \left(|\varepsilon_k \mathbf{v}_k(x,t') + (\mathbf{u}_k)_{Q_1}|^2 + 2\left(\varepsilon_k q_k + (p_k)_{B(0,1)\times\{t\}}\right)\right) dx\, dt' \\
=&\int_{B(0,1)\times(-1,t)} (\varepsilon_k \mathbf{v}_k(x,t') + (\mathbf{u}_k)_{Q_1}) \cdot \nabla\eta(x,t') \ |\varepsilon_k \mathbf{v}_k(x,t') + (\mathbf{u}_k)_{Q_1}|^2 dx\, dt' \\
&+ 2\int_{B(0,1)\times(-1,t)} (\varepsilon_k \mathbf{v}_k(x,t') + (\mathbf{u}_k)_{Q_1}) \cdot \nabla\eta(x,t') \ \left(\varepsilon_k q_k + (p_k)_{B(0,1)\times\{t\}}\right) dx\, dt' \\
=&\int_{B(0,1)\times(-1,t)} \varepsilon_k \mathbf{v}_k(x,t') \cdot \nabla\eta(x,t') \ |\varepsilon_k \mathbf{v}_k(x,t') + (\mathbf{u}_k)_{Q_1}|^2 dx\, dt' \\
&+ \int_{B(0,1)\times(-1,t)} (\mathbf{u}_k)_{Q_1} \cdot \nabla\eta(x,t') \ |\varepsilon_k \mathbf{v}_k(x,t') + (\mathbf{u}_k)_{Q_1}|^2 dx\, dt' \\
&+ {\varepsilon_k}^2 \int_{B(0,1)\times(-1,t)} \mathbf{v}_k(x,t') \cdot \nabla\eta(x,t') \ 2q_k dx\, dt' \\
&+ 2\int_{B(0,1)\times(-1,t)} (\mathbf{u}_k)_{Q_1} \cdot \nabla\eta(x,t') \ \varepsilon_k q_k dx\, dt'
\end{aligned}
$$

For the first term in this equation we observe

$$
\begin{aligned}
&\int_{B(0,1)\times(-1,t)} \varepsilon_k \mathbf{v}_k(x,t') \cdot \nabla\eta(x,t') \ |\varepsilon_k \mathbf{v}_k(x,t') + (\mathbf{u}_k)_{Q_1}|^2 dx\, dt' \\
=&\int_{B(0,1)\times(-1,t)} (\varepsilon_k)^3 \mathbf{v}_k(x,t') \cdot \nabla\eta(x,t') \ |\mathbf{v}_k(x,t')|^2 dx\, dt' \\
&+ \underbrace{\int_{B(0,1)\times(-1,t)} \varepsilon_k \mathbf{v}_k(x,t') \cdot \nabla\eta(x,t') \ |(\mathbf{u}_k)_{Q_1}|^2 dx\, dt'}_{\overset{\operatorname{div} \mathbf{v}_k=0}{=} 0} \\
&+ 2(\varepsilon_k)^2 \int_{B(0,1)\times(-1,t)} \mathbf{v}_k(x,t') \cdot \nabla\eta(x,t') \ \langle \mathbf{v}_k(x,t'), (\mathbf{u}_k)_{Q_1}\rangle dx\, dt'
\end{aligned}
$$

Thus

$$
\begin{aligned}
&\int_{B(0,1)\times(-1,t)} (\varepsilon_k \mathbf{v}_k(x,t') + (\mathbf{u}_k)_{Q_1}) \cdot \nabla\eta(x,t') \ \left(|\varepsilon_k \mathbf{v}_k(x,t') + (\mathbf{u}_k)_{Q_1}|^2 + 2\left(\varepsilon_k q_k + (p_k)_{B(0,1)\times\{t\}}\right)\right) dx\, dt' \\
=&\int_{B(0,1)\times(-1,t)} (\varepsilon_k)^3 \mathbf{v}_k(x,t') \cdot \nabla\eta(x,t') \ |\mathbf{v}_k(x,t')|^2 dx\, dt' \\
&+ \int_{B(0,1)\times(-1,t)} (\mathbf{u}_k)_{Q_1} \cdot \nabla\eta(x,t') \ |\varepsilon_k \mathbf{v}_k(x,t') + (\mathbf{u}_k)_{Q_1}|^2 dx\, dt' \\
&+ 2(\varepsilon_k)^2 \int_{B(0,1)\times(-1,t)} \mathbf{v}_k(x,t') \cdot \nabla\eta(x,t') \ \langle \mathbf{v}_k(x,t'), (\mathbf{u}_k)_{Q_1}\rangle dx\, dt' \\
&+ {\varepsilon_k}^2 \int_{B(0,1)\times(-1,t)} \mathbf{v}_k(x,t') \cdot \nabla\eta(x,t') \ 2q_k dx\, dt' \\
&+ 2\int_{B(0,1)\times(-1,t)} (\mathbf{u}_k)_{Q_1} \cdot \nabla\eta(x,t') \ \varepsilon_k q_k dx\, dt'
\end{aligned}
$$

Treating the second term in this equation similarly

$$
\begin{aligned}
&\int_{B(0,1)\times(-1,t)} (\mathbf{u}_k)_{Q_1} \cdot \nabla\eta(x,t')\ |\varepsilon_k \mathbf{v}_k(x,t') + (\mathbf{u}_k)_{Q_1}|^2 dx\, dt' \\
=&\int_{B(0,1)\times(-1,t)} (\mathbf{u}_k)_{Q_1} \cdot \nabla\eta(x,t')\ |\varepsilon_k \mathbf{v}_k(x,t')|^2 dx\, dt' \\
&+ \underbrace{\int_{B(0,1)\times(-1,t)} (\mathbf{u}_k)_{Q_1} \cdot \nabla\eta(x,t')\ |(\mathbf{u}_k)_{Q_1}|^2 dx\, dt'}_{=0} \\
&+ 2\varepsilon_k \int_{B(0,1)\times(-1,t)} (\mathbf{u}_k)_{Q_1} \cdot \nabla\eta(x,t')\ \langle \mathbf{v}_k(x,t'), (\mathbf{u}_k)_{Q_1}\rangle dx\, dt'
\end{aligned}
$$

Thus

$$
\begin{aligned}
&\int_{B(0,1)\times(-1,t)} \left(\varepsilon_k \mathbf{v}_k(x,t') + (\mathbf{u}_k)_{Q_1}\right) \cdot \nabla\eta(x,t')\ \left(|\varepsilon_k \mathbf{v}_k(x,t') + (\mathbf{u}_k)_{Q_1}|^2 + 2\left(\varepsilon_k q_k + (p_k)_{B(0,1)\times\{t\}}\right)\right) dx\, dt' \\
=&\int_{B(0,1)\times(-1,t)} (\varepsilon_k)^3 \mathbf{v}_k(x,t') \cdot \nabla\eta(x,t')\ |\mathbf{v}_k(x,t')|^2 dx\, dt' \\
&+ \int_{B(0,1)\times(-1,t)} (\mathbf{u}_k)_{Q_1} \cdot \nabla\eta(x,t')\ |\varepsilon_k \mathbf{v}_k(x,t')|^2 dx\, dt' \\
&+ 2\varepsilon_k \int_{B(0,1)\times(-1,t)} (\mathbf{u}_k)_{Q_1} \cdot \nabla\eta(x,t')\ \langle \mathbf{v}_k(x,t'), (\mathbf{u}_k)_{Q_1}\rangle dx\, dt' \\
&+ 2(\varepsilon_k)^2 \int_{B(0,1)\times(-1,t)} \mathbf{v}_k(x,t') \cdot \nabla\eta(x,t')\ \langle \mathbf{v}_k(x,t'), (\mathbf{u}_k)_{Q_1}\rangle dx\, dt' \\
&+ {\varepsilon_k}^2 \int_{B(0,1)\times(-1,t)} \mathbf{v}_k(x,t') \cdot \nabla\eta(x,t')\ \ 2q_k dx\, dt' \\
&+ 2\int_{B(0,1)\times(-1,t)} (\mathbf{u}_k)_{Q_1} \cdot \nabla\eta(x,t')\ \varepsilon_k q_k dx\, dt'
\end{aligned}
$$

So, for our inequality we arrive at

$$\begin{aligned}
&(\varepsilon_k)^2 \left(\int_{B(0,1)} \eta(x,t) |\mathbf{v}_k(x,t)|^2 dx + 2 \int_{B(0,1)\times(-1,t)} \eta(x,t') |\nabla \mathbf{v}_k|^2 \, dx\, dt' \right)\\
\leq &(\varepsilon_k)^2 \int_{B(0,1)\times(-1,t)} |\mathbf{v}_k(x,t')|^2 (\Delta \eta + \partial_t \eta)\\
&+ 2(\varepsilon_k)^2 \int_{B(0,1)\times(-1,t)} \mathbf{v}_k(x,t') \cdot \nabla \eta(x,t') \ \langle \mathbf{v}_k(x,t'), (\mathbf{u}_k)_{Q_1} \rangle dx\, dt'\\
&+ {\varepsilon_k}^2 \int_{B(0,1)\times(-1,t)} \mathbf{v}_k(x,t') \cdot \nabla \eta(x,t') \ \ 2 q_k dx\, dt'\\
&+ (\varepsilon_k)^3 \int_{B(0,1)\times(-1,t)} \mathbf{v}_k(x,t') \cdot \nabla \eta(x,t') \ |\mathbf{v}_k(x,t')|^2 dx\, dt'\\
&+ (\varepsilon_k)^2 \int_{B(0,1)\times(-1,t)} (\mathbf{u}_k)_{Q_1} \cdot \nabla \eta(x,t') \ |\mathbf{v}_k(x,t')|^2 dx\, dt'\\
&+ 2\varepsilon_k \int_{B(0,1)\times(-1,t)} \langle \mathbf{v}_k(x,t'), (\mathbf{u}_k)_{Q_1} \rangle (\Delta \eta + \partial_t \eta)\\
&+ 2\varepsilon_k \int_{B(0,1)\times(-1,t)} (\mathbf{u}_k)_{Q_1} \cdot \nabla \eta(x,t') \ \langle \mathbf{v}_k(x,t'), (\mathbf{u}_k)_{Q_1} \rangle dx\, dt'\\
&+ 2\varepsilon_k \int_{B(0,1)\times(-1,t)} (\mathbf{u}_k)_{Q_1} \cdot \nabla \eta(x,t') \ q_k dx\, dt'\\
&- 2\varepsilon_k \int_{B(0,1)} \eta(x,t) \langle \mathbf{v}_k(x,t), (\mathbf{u}_k)_{Q_1} \rangle dx
\end{aligned}$$

Formally[33] we have (since $\eta \equiv 0$ at -1, this holds for a.e. t)

$$\begin{aligned}
&2\varepsilon_k \int_{B(0,1)\times(-1,t)} \langle \mathbf{v}_k(x,t'), (\mathbf{u}_k)_{Q_1} \rangle (\Delta \eta + \partial_t \eta)\\
= &- 2\varepsilon_k \int_{B(0,1)\times(-1,t)} \langle (\partial_t - \Delta) \mathbf{v}_k(x,t'), (\mathbf{u}_k)_{Q_1} \rangle \eta\\
&+ 2\varepsilon_k \int_{B(0,1)\times\{t\}} \langle \mathbf{v}_k(x,t'), (\mathbf{u}_k)_{Q_1} \rangle \eta\\
&- 2\varepsilon_k \int_{B(0,1)\times\{-1\}} \langle \mathbf{v}_k(x,t'), (\mathbf{u}_k)_{Q_1} \rangle \underbrace{\eta}_{\equiv 0}
\end{aligned}$$

Using the equation

$$\partial_t \mathbf{v}_k - \Delta \mathbf{v}_k + ((\mathbf{u}_k)_{Q_1} + \varepsilon_k \mathbf{v}_k)_\alpha \partial_\alpha \mathbf{v}_k = -\nabla q_k$$

[33]This needs to be made precise, $(\mathbf{u}_k)_{Q_1}\eta$ is a permissible test function but not zero on the time-boundary

we have

$$
\begin{aligned}
&2\varepsilon_k \int_{B(0,1)\times(-1,t)} \langle \mathbf{v}_k(x,t'), (\mathbf{u}_k)_{Q_1}\rangle (\Delta\eta + \partial_t \eta)\\
=&+2\varepsilon_k \int_{B(0,1)\times(-1,t)} \langle ((\mathbf{u}_k)_{Q_1} + \varepsilon_k \mathbf{v}_k)_\alpha \partial_\alpha \mathbf{v}_k, (\mathbf{u}_k)_{Q_1}\rangle \eta\\
&-2\varepsilon_k \int_{B(0,1)\times(-1,t)} q_k \langle (\mathbf{u}_k)_{Q_1}\rangle \nabla\eta\\
&+2\varepsilon_k \int_{B(0,1)\times\{t\}} \langle \mathbf{v}_k(x,t'), (\mathbf{u}_k)_{Q_1}\rangle \eta
\end{aligned}
$$

$$
\begin{aligned}
\overset{\operatorname{div}\mathbf{v}_k=0}{=}&-2\varepsilon_k \int_{B(0,1)\times(-1,t)} \langle ((\mathbf{u}_k)_{Q_1} + \varepsilon_k \mathbf{v}_k)_\alpha \mathbf{v}_k, (\mathbf{u}_k)_{Q_1}\rangle \partial_\alpha\eta\\
&-2\varepsilon_k \int_{B(0,1)\times(-1,t)} q_k \langle (\mathbf{u}_k)_{Q_1}\rangle \nabla\eta\\
&+2\varepsilon_k \int_{B(0,1)\times\{t\}} \langle \mathbf{v}_k(x,t'), (\mathbf{u}_k)_{Q_1}\rangle \eta
\end{aligned}
$$

$$
\begin{aligned}
=&-2\varepsilon_k \int_{B(0,1)\times(-1,t)} \langle ((\mathbf{u}_k)_{Q_1})_\alpha \mathbf{v}_k, (\mathbf{u}_k)_{Q_1}\rangle \partial_\alpha\eta\\
&-2(\varepsilon_k)^2 \int_{B(0,1)\times(-1,t)} \langle (\mathbf{v}_k)_\alpha \mathbf{v}_k, (\mathbf{u}_k)_{Q_1}\rangle \partial_\alpha\eta\\
&-2\varepsilon_k \int_{B(0,1)\times(-1,t)} q_k \langle (\mathbf{u}_k)_{Q_1}\rangle \nabla\eta\\
&+2\varepsilon_k \int_{B(0,1)\times\{t\}} \langle \mathbf{v}_k(x,t'), (\mathbf{u}_k)_{Q_1}\rangle \eta
\end{aligned}
$$

That is

$$
\begin{aligned}
&(\varepsilon_k)^2\left(\int_{B(0,1)}\eta(x,t)|\mathbf{v}_k(x,t)|^2dx+2\int_{B(0,1)\times(-1,t)}\eta(x,t')|\nabla\mathbf{v}_k|^2\,dx\,dt'\right)\\
\leq&(\varepsilon_k)^2\int_{B(0,1)\times(-1,t)}|\mathbf{v}_k(x,t')|^2(\Delta\eta+\partial_t\eta)\\
&+2(\varepsilon_k)^2\int_{B(0,1)\times(-1,t)}\mathbf{v}_k(x,t')\cdot\nabla\eta(x,t')\ \langle\mathbf{v}_k(x,t'),(\mathbf{u}_k)_{Q_1}\rangle dx\,dt'\\
&+{\varepsilon_k}^2\int_{B(0,1)\times(-1,t)}\mathbf{v}_k(x,t')\cdot\nabla\eta(x,t')\ \ 2q_k dx\,dt'\\
&+(\varepsilon_k)^3\int_{B(0,1)\times(-1,t)}\mathbf{v}_k(x,t')\cdot\nabla\eta(x,t')\ |\mathbf{v}_k(x,t')|^2dx\,dt'\\
&+(\varepsilon_k)^2\int_{B(0,1)\times(-1,t)}(\mathbf{u}_k)_{Q_1}\cdot\nabla\eta(x,t')\ |\mathbf{v}_k(x,t')|^2dx\,dt'\\
&-2(\varepsilon_k)^2\int_{B(0,1)\times(-1,t)}\langle(\mathbf{v}_k)_\alpha\mathbf{v}_k,(\mathbf{u}_k)_{Q_1}\rangle\partial_\alpha\eta\\
&+2\varepsilon_k\int_{B(0,1)\times(-1,t)}(\mathbf{u}_k)_{Q_1}\cdot\nabla\eta(x,t')\ \langle\mathbf{v}_k(x,t'),(\mathbf{u}_k)_{Q_1}\rangle dx\,dt'-2\varepsilon_k\int_{B(0,1)\times(-1,t)}\langle((\mathbf{u}_k)_{Q_1})_\alpha\mathbf{v}_k,(\mathbf{u}_k)_{Q_1}\rangle\partial_\alpha\eta\\
&+2\varepsilon_k\int_{B(0,1)\times(-1,t)}(\mathbf{u}_k)_{Q_1}\cdot\nabla\eta(x,t')\ q_k dx\,dt'-2\varepsilon_k\int_{B(0,1)\times(-1,t)}q_k\langle(\mathbf{u}_k)_{Q_1}\rangle\nabla\eta\\
&-2\varepsilon_k\int_{B(0,1)}\eta(x,t)\langle\mathbf{v}_k(x,t),(\mathbf{u}_k)_{Q_1}\rangle dx+2\varepsilon_k\int_{B(0,1)\times\{t\}}\langle\mathbf{v}_k(x,t'),(\mathbf{u}_k)_{Q_1}\rangle\eta
\end{aligned}
$$

The last three lines are zero (and the red lines cancel).

So we have shown

$$
\begin{aligned}
&(\varepsilon_k)^2\left(\int_{B(0,1)}\eta(x,t)|\mathbf{v}_k(x,t)|^2dx+2\int_{B(0,1)\times(-1,t)}\eta(x,t')|\nabla\mathbf{v}_k|^2\,dx\,dt'\right)\\
\leq&(\varepsilon_k)^2\int_{B(0,1)\times(-1,t)}|\mathbf{v}_k(x,t')|^2(\Delta\eta+\partial_t\eta)\\
&+{\varepsilon_k}^2\int_{B(0,1)\times(-1,t)}\mathbf{v}_k(x,t')\cdot\nabla\eta(x,t')\ \ 2q_k dx\,dt'\\
&+(\varepsilon_k)^3\int_{B(0,1)\times(-1,t)}\mathbf{v}_k(x,t')\cdot\nabla\eta(x,t')\ |\mathbf{v}_k(x,t')|^2dx\,dt'\\
&+(\varepsilon_k)^2\int_{B(0,1)\times(-1,t)}(\mathbf{u}_k)_{Q_1}\cdot\nabla\eta(x,t')\ |\mathbf{v}_k(x,t')|^2dx\,dt'
\end{aligned}
$$

Dividing by $(\varepsilon_k)^2$ we conclude

$$\begin{aligned}
&\left(\int_{B(0,1)} \eta(x,t)|\mathbf{v}_k(x,t)|^2 dx + 2\int_{B(0,1)\times(-1,t)} \eta(x,t')|\nabla \mathbf{v}_k|^2\, dx\, dt'\right)\\
\leq& \int_{B(0,1)\times(-1,t)} |\mathbf{v}_k(x,t')|^2(\Delta\eta + \partial_t\eta)\\
&+\int_{B(0,1)\times(-1,t)} \mathbf{v}_k(x,t')\cdot\nabla\eta(x,t')\ \ 2q_k dx\, dt'\\
&+\varepsilon_k\int_{B(0,1)\times(-1,t)} \mathbf{v}_k(x,t')\cdot\nabla\eta(x,t')\ |\mathbf{v}_k(x,t')|^2 dx\, dt'\\
&+\int_{B(0,1)\times(-1,t)} (\mathbf{u}_k)_{Q_1}\cdot\nabla\eta(x,t')\ |\mathbf{v}_k(x,t')|^2 dx\, dt'
\end{aligned}$$

This concludes the proof of Claim 12.11. □

12.6. **Regularity for blow-up equation: Proof of Claim 12.12.**

12.6.1. *Regularity theory for global linear estimates.* The following are essentially parabolic Cauchy-type estimates, cf. Theorem 3.24, see also Lemma 2.41 for the elliptic version.

Lemma 12.15. *Let* $\mathbf{b}\in\mathbb{R}^n$ *be a constant vector with* $|\mathbf{b}|\leq 1$, *and assume* $u\in L^2(B(0,1)\times(-1,0))$ *solves in distributional sense*

$$(\partial_t-\Delta)u - b_\alpha\partial_\alpha u = 0 \quad \text{in } B(0,1)\times(-1,0)$$

Then for any $R\in(0,1)$ *and any* $\ell=0,1,\ldots$ *there exists a constant* C *depending only on the dimension* n *and* R *such that*

$$\|\nabla^\ell u\|_{L^\infty(B(0,R)\times(-R^2,0))} \precsim_R \|u\|_{L^2(B(0,1)\times(-1,0))}$$

Proof. Let $\eta\in C_c^\infty(B(0,1)\times(-1,0])$ with $\eta\equiv 1$ in $B(0,\frac{1+R}{2})\times(-(\left(\frac{1+R}{2}\right)^2,0]$ a typical bump function (whose derivative all depend on R, but are otherwise uniform).

Then set $v:=\eta u$ (in particular we have $v\equiv 0$ for $t\approx -1$)

$$(\partial_t-\Delta)v - b_\alpha\partial_\alpha v = g \quad \text{in } \mathbb{R}^n\times(-1,0)$$

where

$$\begin{aligned}
g :=&(\partial_t\eta)u - (\Delta\eta)u - 2\nabla\eta\cdot\nabla u - b_\alpha(\partial_\alpha\eta)u\\
=&(\partial_t\eta)u + (\Delta\eta)u - 2\mathrm{div}\,(\nabla\eta\, u) - b_\alpha(\partial_\alpha\eta)u
\end{aligned}$$

Observe that $2\nabla\eta\cdot\nabla u$ or $\mathrm{div}\,(\nabla\eta\, u)$ make sense only in a distributional sense – since $u\in L^2$. But this will not be a problem. It will be useful to observe that

$$g\equiv 0 \quad \text{in } B(0,\frac{1+R}{2})\times(-(\left(\frac{1+R}{2}\right)^2,0], \tag{12.23}$$

which happens because all terms of g contain derivatives of η and $\eta\equiv 1$ in the above set.

Now we have reduced our situation to an equation on $\mathbb{R}^n \times (-1,0)$,

$$(\partial_t - \Delta)v - b_\alpha \partial_\alpha v = g. \tag{12.24}$$

Observe that since b is constant, $b \cdot \nabla v$ is a sort of transport term, namely

$$\begin{cases} (\partial_t - \Delta)v(x-tb,t) = g(x-tb,t) & \text{in } \mathbb{R}^n \times (-1,0) \\ v(x-tb,t)\Big|_{t=-1} = 0. \end{cases}$$

From the representation formula of the heat equation, Theorem 3.4, we find

$$v(x-tb,t) = c_6 \int_{\mathbb{R}^n} \int_{s=-1}^t \frac{1}{(t-s)^{\frac{n}{2}}} e^{-c_7 \frac{|x-y|^2}{t-s}} g(y-sb,s)ds\, dy.$$

Consequently, with a substitution

$$v(x,t) = c_6 \int_{\mathbb{R}^n} \int_{s=-1}^t \frac{1}{(t-s)^{\frac{n}{2}}} e^{-c_7 \frac{|x-y|^2}{t-s}} g(y+(t-s)b,s)ds\, dy$$

and thus

$$\nabla^\ell v(x,t) = c_6 \int_{\mathbb{R}^n} \int_{s=-1}^t \frac{1}{(t-s)^{\frac{n}{2}}} \nabla^\ell_x e^{-c_7 \frac{|x-y|^2}{t-s}} g(y+(t-s)b,s)ds\, dy.$$

We need to estimate this expression, where from now on *we always assume*

$$(x,t) \in B(0,R) \times (-R^2,0).$$

We discuss two regimes. First, in view of (12.23) we see that there exist constants ε, δ depending on R

$$g(y+(t-s)b,s) = 0 \quad \forall y \in \overline{B(0,R+2\delta)} \quad |s-t| < \varepsilon$$

We split the integral first in space and time,

$$\begin{aligned} \left|\nabla^\ell v(x,t)\right| \lesssim & \left|\int_{\mathbb{R}^n} \int_{s=-1}^t \chi_{|s-t|>\varepsilon} \frac{1}{(t-s)^{\frac{n}{2}}} \nabla^\ell_x e^{-c_7 \frac{|x-y|^2}{t-s}} g(y+(t-s)b,s)ds\, dy\right| \\ & + \left|\int_{\mathbb{R}^n} \int_{s=-1}^t \chi_{|s-t|<\varepsilon} \chi_{|x-y|>2\delta} \frac{1}{(t-s)^{\frac{n}{2}}} \nabla^\ell_x e^{-c_7 \frac{|x-y|^2}{t-s}} g(y+(t-s)b,s)ds\, dy\right| \end{aligned} \tag{12.25}$$

For the first term in (12.25) we observe by Young's convolution inequality in x, $L^\infty \subset L^\infty * L^1$ (we do this computation for a generic $\tilde{g}$, for later use)

$$\begin{aligned}
&\sup_{x\in B(0,R)} \left|\int_{\mathbb{R}^n}\int_{s=-1}^t \chi_{|s-t|>\varepsilon} \frac{1}{(t-s)^{\frac{n}{2}}} \nabla_x^\ell e^{-c_7\frac{|x-y|^2}{t-s}} \tilde{g}(y+(t-s)b,s)ds\,dy\right|\\
\precsim_\varepsilon &\sup_{x\in B(0,R)} \left|\int_{s=-1}^t \chi_{|s-t|>\varepsilon} \left|\int_{\mathbb{R}^n} \nabla_x^\ell e^{-c_7\frac{|x-y|^2}{t-s}} \tilde{g}(y+(t-s)b,s)dy\right| ds\right|\\
=&\sup_{x\in B(0,R)} \int_{s=-1}^t \chi_{|s-t|>\varepsilon} \left|\left(\nabla^\ell e^{-c_7\frac{|\cdot|^2}{t-s}} * \tilde{g}(\cdot+(t-s)b,s)\right)(x)\right| ds\\
\precsim &\int_{s=-1}^t \chi_{|s-t|>\varepsilon} \|\nabla^\ell e^{-c_7\frac{|\cdot|^2}{t-s}}\|_{L^\infty(\mathbb{R}^n)} \|\tilde{g}(\cdot+(t-s)b,s)\|_{L^1(\mathbb{R}^n)}\,ds\\
\overset{\varepsilon<|s-t|\leq 2}{\precsim} &\int_{s=-1}^0 C(\varepsilon)\|\tilde{g}(,s)\|_{L^1(\mathbb{R}^n)}\,ds\\
=&C(\varepsilon)\|\tilde{g}(,s)\|_{L^1(\mathbb{R}^n\times(-1,0))}.
\end{aligned}$$

This is a good estimates for most of the terms appearing in g, namely we observe

$$\|(\partial_t\eta)u\|_{L^1(\mathbb{R}^d\times(-1,0)} + \|(\Delta\eta)u\|_{L^1(\mathbb{R}^d\times(-1,0)} + \|b_\alpha(\partial_\alpha\eta)u\|_{L^1(\mathbb{R}^d\times(-1,0)} \precsim \|u\|_{L^2(B(0,1)\times(-1,0))} \tag{12.26}$$

The only term to deal with is div $(\nabla\eta\, u)$ byt for this we observe similar to the above (recall that $x \in B(0,R)$ by assumption,

$$\begin{aligned}
&\sup_{x\in B(0,R)} \left|\int_{\mathbb{R}^n}\int_{s=-1}^t \chi_{|s-t|>\varepsilon} \frac{1}{(t-s)^{\frac{n}{2}}} \nabla_x^\ell e^{-c_7\frac{|x-y|^2}{t-s}} \operatorname{div}\tilde{G}(y+(t-s)b,s)ds\,dy\right|\\
\precsim_\varepsilon &\sup_{x\in B(0,R)} \left|\int_{s=-1}^t \chi_{|s-t|>\varepsilon} \left|\int_{\mathbb{R}^n} \nabla_x\nabla_x^\ell e^{-c_7\frac{|x-y|^2}{t-s}} \cdot \tilde{G}(y+(t-s)b,s)dy\right| ds\right|\\
=&\int_{s=-1}^t \chi_{|s-t|>\varepsilon} \left|\left(\nabla^{\ell+1} e^{-c_7\frac{|\cdot|^2}{t-s}} * \tilde{G}(\cdot+(t-s)b,s)\right)(x)\right| ds\\
=&C(\varepsilon)\|\tilde{G}(,s)\|_{L^1(\mathbb{R}^n\times(-1,0))}\,ds
\end{aligned}$$

In conclusion we have established control for the first term in (12.25), namely

$$\sup_{(x,t)\in B(0,R)\times(0,-R^2)} \left|\int_{\mathbb{R}^n}\int_{s=-1}^t \chi_{|s-t|>\varepsilon} \frac{1}{(t-s)^{\frac{n}{2}}} \nabla_x^\ell e^{-c_7\frac{|x-y|^2}{t-s}} g(y+(t-s)b,s)ds\,dy\right| \precsim \|u\|_{L^2(B(0,1)\times(-1,0)}.$$

For the second term in (12.25), we argue slightly different. We first discuss the case $\ell = 0$. Using this time Young's inequality in $\mathbb{R}$, $L^\infty(\mathbb{R}) \subset L^\infty(\mathbb{R}) * L^1(\mathbb{R})$ we have

$$
\begin{aligned}
&\left|\int_{\mathbb{R}^n} \int_{s=-1}^t \chi_{|s-t|<\varepsilon} \chi_{|x-y|>2\delta} \frac{1}{(t-s)^{\frac{n}{2}}} e^{-c_7 \frac{|x-y|^2}{t-s}} \tilde{g}(y+(t-s)b,s)ds\,dy\right|\\
\lesssim& \int_{s=-1}^t \frac{1}{(t-s)^{\frac{n}{2}}} e^{-c_7 \frac{\delta^2}{t-s}} \left|\int_{\mathbb{R}^n} \tilde{g}(y+(t-s)b,s)dy\right|\, ds\\
\leq& \int_{s\in\mathbb{R}} \frac{1}{|t-s|^{\frac{n}{2}}} \nabla^\ell e^{-c_7 \frac{\delta^2}{t-s}} \left|\chi_{s\in(-1,0)} \int_{\mathbb{R}^n} \tilde{g}(y,s)dy\right|\, ds\\
=& \left(\frac{1}{|\cdot|^{\frac{n}{2}}} e^{-c_7 \frac{\delta^2}{\cdot}} *_{\mathbb{R}} \left|\int_{\mathbb{R}^n} \tilde{g}(y,\cdot)dy\right|\right)(t)\\
\leq& \|\frac{1}{|\cdot|^{\frac{n}{2}}} e^{-c_7 \frac{\delta^2}{\cdot}}\|_{L^\infty(\mathbb{R})} \|\tilde{g}\|_{L^1(\mathbb{R}^n\times(-1,0))}
\end{aligned}
$$

Here we use that the exponential absorbs the singularity in $t-s$. This again suffices for most terms contained in g, cf. (12.26).

Similarly, by an integration by parts (here we use that for our $\tilde{G} = \nabla\eta\, u$ we know that for $|x-y| = \delta$ we have $\tilde{G}(y+(t-s)b,s) \equiv 0$),

$$
\begin{aligned}
&\left|\int_{\mathbb{R}^n} \int_{s=-1}^t \chi_{|s-t|<\varepsilon} \chi_{|x-y|>2\delta} \frac{1}{(t-s)^{\frac{n}{2}}} e^{-c_7 \frac{|x-y|^2}{t-s}} \operatorname{div} \tilde{G}(y+(t-s)b,s)ds\,dy\right|\\
=& \int_{s=-1}^t \frac{1}{(t-s)^{\frac{n}{2}+1}} e^{-c_7 \frac{\delta^2}{t-s}} \left|\int_{\mathbb{R}^n} \tilde{G}(y+(t-s)b,s)dy\right|\, ds\\
\lesssim& \|\frac{1}{|\cdot|^{\frac{n}{2}+1}} e^{-c_7 \frac{\delta^2}{\cdot}}\|_{L^1(\mathbb{R})} \|\tilde{G}\|_{L^1(\mathbb{R}^n\times(-1,0))}
\end{aligned}
$$

This settles the case for $\ell = 0$.

If ℓ is larger we use that for certain polynomials by repeated product rule,

$$
\nabla_x^\ell e^{-c_7 \frac{|x-y|^2}{t-s}} = \sum_{i=0}^\ell (t-s)^{-i} p_{i,\ell}(x-y) e^{-c_7 \frac{|x-y|^2}{t-s}}.
$$

Then we have

$$\begin{aligned}
&\left|\int_{\mathbb{R}^n}\int_{s=-1}^{t} \chi_{|s-t|<\varepsilon}\chi_{|x-y|>2\delta}\frac{1}{(t-s)^{\frac{n}{2}}}\nabla^\ell e^{-c_7\frac{|x-y|^2}{t-s}}\tilde{g}(y+(t-s)b,s)ds\,dy\right|\\
\lesssim&\sum_{i=1}^{\ell}\int_{s=-1}^{t}\frac{1}{(t-s)^{\frac{n}{2}+i}}e^{-c_7\frac{\delta^2}{t-s}}\left|\int_{\mathbb{R}^n} p_i(x-y)\tilde{g}(y+(t-s)b,s)dy\right| ds\\
\leq&\sum_{i}\int_{s\in\mathbb{R}}\frac{1}{|t-s|^{\frac{n}{2}+i}}\nabla^\ell e^{-c_7\frac{\delta^2}{t-s}}\left|\chi_{s\in(-1,0)}\int_{\mathbb{R}^n} p_i(x-y-(t-s)b)\tilde{g}(y,s)dy\right| ds\\
=&\left(\frac{1}{|\cdot|^{\frac{n}{2}+i}}e^{-c_7\frac{\delta^2}{\cdot}}*_{\mathbb{R}}\left|\int_{\mathbb{R}^n} p_i(x-y-(t-s)b)\tilde{g}(y,\cdot)dy\right|\right)(t)
\end{aligned}$$

We can now conclude as before, observing: If $|s-t|<\varepsilon$ then for $x\in B(0,R)$ and $y\in \operatorname{supp}\tilde{g}(\cdot+(t-s)b,s)$ we know that $|x-y|\lesssim R$. So whether we need to do an integration by parts for $\tilde{g}$ or not, the p_i-term is bounded by a constant depending on R.

We can conclude. □

12.6.2. *(An easy consequence of) Helmholtz-Hodge decomposition.* The following can relatively easily be generalized to other dimensions (the simplest notion then using differential forms), but our focus is on 3 dimensions here.

For a vector field $\mathbf{F}=(F_1,F_2,F_3):\Omega\subset\mathbb{R}^3\to\mathbb{R}^3$ we already are familiar with the operator div

$$\operatorname{div}\mathbf{F}=\partial_\alpha F_\alpha$$

We also define

$$\operatorname{curl}(\mathbf{F})\equiv\nabla\times\mathbf{F}=\begin{pmatrix}\partial_2F_3-\partial_3F_2\\ \partial_3F_1-\partial_1F_3\\ \partial_1F_2-\partial_2F_1\end{pmatrix}$$

The two operations are essentially perpendicular to each other,

$$\operatorname{div}\operatorname{curl}\mathbf{F}=0,\quad \text{and } \operatorname{curl}\nabla f=0$$

Helmholtz-Hodge decomposition tells us that any vector field $\mathbf{F}$ can be split into it divergence-part and curl-part. Namely we can find a and $\mathbf{b}$ such that

$$\mathbf{F}=\nabla a+\operatorname{curl}\mathbf{b}$$

and a is determined by $\operatorname{div}\mathbf{F}$ and b is determined by $\operatorname{curl}\mathbf{F}$.

This can be seen e.g. from the identity

$$\operatorname{curl}\,(\operatorname{curl}\mathbf{F})=\Delta\mathbf{F}-\nabla(\operatorname{div}\mathbf{F})\tag{12.27}$$

This identity (12.27) can be computed by hand from the 3D cross-product rule (where $\times$ is the cross product, and $\cdot$ the scalar product in $\mathbb{R}^3$)

$$\mathbf{a}\times(\mathbf{b}\times\mathbf{c})=(\mathbf{a}\cdot\mathbf{c})\mathbf{b}-\mathbf{c}(\mathbf{a}\cdot\mathbf{b})$$

and replacing $\mathbf{a}$ and $\mathbf{b}$ with symbols $\mathbf{a}=\nabla$, $\mathbf{b}=\nabla$, and noting that then $\nabla\times c=\operatorname{curl}c$.

(Formally) Inverting the Δ in (12.27) we have

$$\mathbf{F} = \operatorname{curl} \Delta^{-1} (\operatorname{curl} \mathbf{F}) + \nabla \Delta^{-1} (\operatorname{div} \mathbf{F}) \tag{12.28}$$

This equation appears at several directions in mathematics. It seems that it tends to be called "Helmholtz-decomposition" in Navier-Stokes equations, whereas in geometric context (working with differential forms this is $\omega = d\alpha + d^*\beta$) it is more often called Hodge decomposition. The details are a bit unpleasant because of boundary data needing to be chosen (e.g. when talking about Δ^{-1} we need to fix what Δ-equation we solve, with what boundary and on what domain). We don't want to diverge in this direction too much here, we only record what is essentially a corollary of the Helmholtz-Hodge decomposition.

Corollary 12.16. *Let $\Omega \subset \mathbb{R}^n$ be any open set.*

Assume that $\mathbf{v} \in W^{1,2}(\Omega, \mathbb{R}^3)$ with

$$\operatorname{div} \mathbf{v} = 0 \quad \text{in } \Omega$$

and assume that for some $r \in [2, \infty)$ we have for some $\ell \in \{0, 1, 2, \ldots\}$

$$\operatorname{curl} \mathbf{v} \in W^{\ell,r}(\Omega, \mathbb{R}^3).$$

Then for any open set $\Omega' \subset\subset \Omega$ we have

$$\|\nabla^{\ell+1} \mathbf{v}\|_{L^r(\Omega')} \precsim_{\Omega,\Omega',r} \|\operatorname{curl} \mathbf{v}\|_{W^{\ell,r}(\Omega)} + \|\mathbf{v}\|_{L^2(\Omega)}.$$

Proof. Observe from (12.27) (this holds for smooth vectors, and by density also in distributional sense)

$$\Delta \mathbf{v} = \operatorname{curl} (\operatorname{curl} \mathbf{v}) \quad \text{in } \Omega$$

Now from the interior Calderon-Zygmund/L^p-theory (applied componentwise!), Theorem 7.10, we have for any $r \in (1, \infty)$

$$\|\nabla \mathbf{v}\|_{L^r(\Omega')} \precsim_{\Omega',\Omega} \|\operatorname{curl} \mathbf{v}\|_{L^r(\Omega)} + \|\mathbf{v}\|_{L^2(\Omega)}.$$

This establishes the case $\ell = 0$.

If $\ell = 1$, let $\Omega' \subset\subset \Omega'' \subset\subset \Omega$ we already have

$$\begin{aligned} \|\nabla \mathbf{v}\|_{L^r(\Omega'')} &\precsim_{\Omega'',\Omega} \|\operatorname{curl} \mathbf{v}\|_{L^r(\Omega)} + \|\mathbf{v}\|_{L^2(\Omega)} \\ &\leq \|\operatorname{curl} \mathbf{v}\|_{W^{1,r}(\Omega)} + \|\mathbf{v}\|_{L^2(\Omega)} \end{aligned} \tag{12.29}$$

Now we apply the previous argument to (if we are unhappy with the distributional sense we can use discrete differentiating, the important thing is that $\partial_\alpha \operatorname{curl} (\operatorname{curl} \mathbf{v}) = \operatorname{curl} \operatorname{curl} (\partial_\alpha \mathbf{v})$)

$$\left\{ \Delta \partial_\alpha \mathbf{v} = \operatorname{curl} (\operatorname{curl} \partial_\alpha \mathbf{v}) \quad \text{in } \Omega' \right.$$

Then from the Calderon-Zygmund theory we obtain

$$\|\nabla \partial_\alpha \mathbf{v}\|_{L^r(\Omega')} \precsim_{\Omega',\Omega'',r} \|\partial_\alpha \operatorname{curl} \mathbf{v}\|_{L^r(\Omega'')} + \|\partial_\alpha \mathbf{v}\|_{L^2(\Omega'')}$$

$$\overset{(12.29)}{\precsim}_{\Omega'',\Omega',r} \|\partial_\alpha \operatorname{curl} \mathbf{v}\|_{L^r(\Omega'')} + \|\mathbf{v}\|_{L^2(\Omega'')}.$$

For general ℓ we simply do this argument ℓ times. $\square$

12.6.3. *Regularity for blow-up equation: Proof of Claim 12.12.* Recall our situation:

We have $\mathbf{v}$ a solution to

$$\begin{cases} \partial_t \mathbf{v} - \Delta \mathbf{v} + \mathbf{b}_\alpha \partial_\alpha \mathbf{v} = -\nabla q & \text{in } Q_1 \\ \operatorname{div} \mathbf{v} = 0 & \text{in } Q_1 \end{cases}$$

where we know (from the energy inequality, cf. (12.15))

$$\sup_{k\in\mathbb{N}} \|\nabla \mathbf{v}_k\|_{L^2(Q_R)} \lesssim_R \Gamma_1 \quad \forall R \in (0,1).$$

We want to show for all $R \in (0,1)$

$$\sup_{(x,t),(y,t)\in Q_R} \frac{|\mathbf{v}(x,t) - \mathbf{v}(y,s)|}{|x-y|^{\frac{2}{3}} + |s-t|^{\frac{1}{3}}} \lesssim_R \Gamma_1$$

Proof of Claim 12.12. Recall that we have that $\nabla \mathbf{v} \in L^2_{loc}(B(0,1)\times(-1,0])$. That is

$$(\operatorname{curl} \mathbf{v}) := \begin{pmatrix} \partial_2 v_3 - \partial_3 v_2 \\ \partial_3 v_1 - \partial_1 v_3 \\ \partial_1 v_2 - \partial_2 v_1 \end{pmatrix} \in L^2_{loc}(B(0,1)\times(-1,0],\mathbb{R}^3).$$

and more precisely we have by (12.15)

$$\|\operatorname{curl} \mathbf{v}\|_{L^2(Q_R)} \lesssim_{R,n} \Gamma_1 \quad \forall R \in (0,1).$$

It is easy to check that $\operatorname{curl}(\nabla q) = 0$, so we can apply curl to the equation (12.17) and find that (componentwise)

$$\partial_t \operatorname{curl} \mathbf{v} - \Delta \operatorname{curl} \mathbf{v} + \mathbf{b}_\alpha \partial_\alpha \operatorname{curl} \mathbf{v} = \mathit{0} \quad \text{in } Q_1$$

By Lemma 12.15 we conclude regularity for $\operatorname{curl} \mathbf{v}$, namely we have for all $\ell = 0,1,2,\ldots$

$$\|\nabla^\ell \operatorname{curl} \mathbf{v}\|_{L^\infty(Q_R)} \lesssim_{R,n} \Gamma_1 \quad \forall R \in (0,1).$$

By Fubini's theorem and Hölder's inequality we conclude that for any $r \in (1,\infty)$,

$$\sup_{t\in(-R^2,0)} \|\nabla^\ell \operatorname{curl} \mathbf{v}\|_{L^r(B(0,R))\times\{t\})} \lesssim_{R,n,r} \Gamma_1 \quad \forall R \in (0,1).$$

Now we employ *Helmholtz-Hodge decomposition*, since we have div and curl control for $\nabla \mathbf{v}$. More precisely, by Corollary 12.16 we have

$$\sup_{t\in(-R^2,0)} \|\nabla^{\ell+1} \mathbf{v}\|_{L^r(B(0,R))\times\{t\}} \lesssim_{R,n,r} \Gamma_1 \quad \forall R \in (0,1),\ r \in (1,\infty) \tag{12.30}$$

Observe, at this stage we only have spatial control but not in time-direction.

We also obtain the same sort of (spatial) control for q: By taking div of the PDE of $\mathbf{v}$ and q, (12.17), observing that div commutes with all differentiable operators in that PDE, in particular that

$$\operatorname{div}\,(b_\alpha \partial_\alpha \mathbf{v}) = b_\alpha \partial_\alpha \operatorname{div} \mathbf{v} = 0,$$

we find that

$$-\Delta q = 0 \quad \text{in } Q_1.$$

In particular (by Fubini's theorem) for a.e. $t \in (-1,0)$ we have $q(\cdot,t) \in L^{\frac{3}{2}}(B(0,1) \times \{t\})$ and

$$-\Delta q(\cdot,t) = 0 \quad \text{in } B(0,1).$$

From the Cauchy estimates, Theorem 3.24, we have for each of those $t \in (-1,0)$

$$\|\nabla^\ell q(\cdot,t)\|_{L^\infty(B(0,R))} \lesssim_{R,\ell} \|q(\cdot,t)\|_{L^{\frac{3}{2}}(B(0,1)\times\{t\}} \quad \forall R \in (0,1).$$

With (12.14) this implies

$$\left(\int_{t=-R^2}^0 \|\nabla^\ell q(\cdot,t)\|^{\frac{3}{2}}_{L^\infty(B(0,R))}\, dt\right)^{\frac{2}{3}} \lesssim_R \Gamma_1 \tag{12.31}$$

It remains to obtain *time-control* of $\mathbf{v}$ and q. We look at the PDE, (12.17), again as an ODE

$$\partial_t \mathbf{v} = \Delta \mathbf{v} - \mathbf{b}_\alpha \partial_\alpha \mathbf{v} - \nabla q$$

we see that

$$|\partial_t \nabla^\ell \mathbf{v}(x,t)| \lesssim |\nabla^{\ell+2}\mathbf{v}(x,t)| + |\nabla^{\ell+1}\mathbf{v}(x,t)| + |\nabla^{\ell+1} q(x,t)|$$

and thus by (12.30) and (12.31)

$$\begin{aligned}
\|\partial_t \nabla^\ell \mathbf{v}\|_{L^{\frac{3}{2}}(Q_R)} \lesssim &\sup_{t\in(-R^2,0)} \|\nabla^{\ell+2}\mathbf{v}(\cdot,t)\|_{L^{\frac{3}{2}}(B(0,R))} + \sup_{t\in(-R^2,0)} \|\nabla^{\ell+1}\mathbf{v}(\cdot,t)\|_{L^{\frac{3}{2}}(B(0,R))}\\
&+ \left(\int_{t=-R^2}^0 \|\nabla^{\ell+1} q(\cdot,t)\|^{\frac{3}{2}}_{L^\infty(B(0,R))}\, dt\right)^{\frac{2}{3}}\\
\lesssim_{R,\ell}&\Gamma_1.
\end{aligned}$$

By spatial Sobolev embedding $W^{\ell,\frac{3}{2}}(B(0,R)) \subset L^\infty(B(0,R))$ for a suitably large ℓ. We conclude in particular that

$$\begin{aligned}
&\left(\int \|\partial_t \mathbf{v}(\cdot,t)\|^{\frac{3}{2}}_{L^\infty(B(0,R))} dt\right)^{\frac{2}{3}}\\
\lesssim &\left(\int \|\nabla^\ell \partial_t \mathbf{v}(\cdot,t)\|^{\frac{3}{2}}_{L^{\frac{3}{2}}(B(0,R))} dt\right)^{\frac{2}{3}}\\
&+ \left(\int \|\partial_t \mathbf{v}(\cdot,t)\|^{\frac{3}{2}}_{L^{\frac{3}{2}}(B(0,R))} dt\right)^{\frac{2}{3}}\\
\lesssim&\Gamma_1.
\end{aligned} \tag{12.32}$$

So for a.e. $(x,t),(y,t) \in Q_R$ we have

$$
\begin{aligned}
|\mathbf{v}(x,t)-\mathbf{v}(y,s)| \leq& \int_{[s,t]} |\partial_t \mathbf{v}(x,\sigma)|\,d\sigma + |\mathbf{v}(x,s)-\mathbf{v}(y,s)|\\
\leq& |s-t|^{\frac{1}{3}} \left(\int_{[s,t]} \|\partial_t \mathbf{v}(x,\sigma)\|_{L^\infty(B(0,R))}^{\frac{3}{2}} d\sigma\right)^{\frac{2}{3}} + |x-y| \|\nabla \mathbf{v}(\cdot,s)\|_{L^\infty(B(0,R))}\\
\overset{(12.32)}{\lesssim}& |s-t|^{\frac{1}{3}}\Gamma_1 + |x-y| \left(\|\mathbf{v}\|_{L^2(B(0,R))} + \|\nabla^{100}\mathbf{v}(\cdot,s)\|_{L^2(B(0,R))}\right)\\
\overset{(12.30)}{\lesssim}& \left(|s-t|^{\frac{1}{3}} + |x-y|\right)\Gamma_1
\end{aligned}
$$

That is, we have a control over Hölder continuity of $\mathbf{v}$ in Q_R. This proves (12.18). □

12.7. **The Aubin-Lions Lemma: Proof of Claim 12.13.** We first recall the Ehrling-Lemma

Theorem 12.17 (Ehrling lemma)**.** *Let $(X,\|\cdot\|_X)$, $(Y,\|\cdot\|_Y)$, $(Z,\|\cdot\|_Z)$ be three Banach spaces which are subspaces of each other $X \subset Y \subset Z$ with the following properties.*

- *X is compactly embedded in Y, that is $X \subset Y$ and every $\|\cdot\|_X$-bounded sequence $(x_k)_k \subset X$, $\sup_k \|x_k\|_X < \infty$, has a strongly $\|\cdot\|_Y$-convergent subsequence $(x_{k_i})_{i\in\mathbb{N}}$, i.e. for some $y \in Y$ and $\|x_{k_i} - y\|_Y \xrightarrow{i\to\infty} 0$.*
- *Y is continuously embedded in Z, that is $Y \subset Z$ and there exists $\Lambda > 0$ such that $\|y\|_Z \leq \Lambda \|y\|_Y$ for all $y \in Y$.*

Then for every $\varepsilon > 0$ there exists a constant $C(\varepsilon) > 0$ such that the following holds

$$\|x\|_Y \leq \varepsilon \|x\|_X + C(\varepsilon)\|x\|_Z \quad \forall x \in X.$$

Exercise 12.18. *Let $(X,\|\cdot\|_X)$, $(Y,\|\cdot\|_Y)$ be Banach spaces. Show that if $X \subset Y$ is compactly embedded, then $X \subset Y$ is continuously embedded.*

Proof of Theorem 12.17. This is once again a typical blow-up proof.

Fix $\varepsilon > 0$. Assume the claim is false, then for any $k \in \mathbb{N}$ there exists a "counterexample" $x_k \in X$ such that

$$\|x_k\|_Y > \varepsilon \|x_k\|_X + k\|x_k\|_Z \quad \forall k$$

Dividing this inequality by $\|x_k\|_Y$ (cannot be zero because of the strict inequality) and otherwise switching over to $\tilde{x}_k := \frac{x_k}{\|x_k\|_Y}$ we may assume w.l.o.g. $\|x_k\|_Y = 1$ for all k and thus

$$1 > \varepsilon \|x_k\|_X + k\|x_k\|_Z \quad \forall k.$$

In particular,

$$\sup_k \|x_k\|_X \leq \frac{1}{\varepsilon}$$

and

$$\lim_{k\to\infty} \|x_k\|_Z \leq \lim_{k\to\infty} \frac{1}{k} = 0.$$

By compactness $X \subset Y$ and since $(x_k)_k$ is $\|\cdot\|_X$-bounded, up to passing to a subsequence $(x_{k_i})_i$, we can assume w.l.o.g. that there exists $y \in Y$ such that $\|x_k - y\|_Y \xrightarrow{k\to\infty} 0$. In particular since $\|x_k\|_Y = 1$ we find that $\|y\|_Y = 1$. By the continuous embedding $Y \subset Z$ we also have $\|x_k - y\|_Z \xrightarrow{k\to\infty} 0$, but since $\|x_k\|_Z \xrightarrow{k\to\infty} 0$ we have $y = 0$, which contradicts $\|y\|_Y = 1$.

So there must have been some $k \in \mathbb{N}$ for which there was no counterexample x_k – and thus the claim is proven. $\square$

Theorem 12.19 (Aubin-Lions Lemma)**.** *Let X, Y, Z be reflexive*[34] *Banach spaces such that*

$$X \hookrightarrow Y \quad \textit{embedds compactly}$$

and

$$Y \hookrightarrow Z \quad \textit{embedds continuously.}$$

Assume that for some $p, q \in (1,\infty)$, $T \in (0,\infty)$

$$(u_k)_{k\in\mathbb{N}} \in L^p([0,T];X), \quad (\partial_t u_k)_{k\in\mathbb{N}} \in L^q([0,T];Z)$$

is bounded, i.e.

$$\sup_k \int_0^T (\|u_k(t)\|_X)^p\, dt + \sup_k \int_0^T (\|\partial_t u_k(t)\|_Z)^q\, dt < \infty$$

Then there exists a subsequence $(u_{k_i})_{i\in\mathbb{N}}$ and $u \in L^p([0,T];Y)$ such that

$$\int_0^T \|u_{k_i}(t) - u(t)\|_Y^p dt \xrightarrow{k\to\infty} 0.$$

Sounds like a combination of Rellich's theorem, Theorem 5.19, and Ehrling Lemma, Theorem 12.17? This is, because it is pretty much that.

Proof of Theorem 12.19. Let $\eta \in C_c^\infty((-1,1))$ be the typical nonnegative bump function with $\int_{\mathbb{R}} \eta = 1$. Set $\eta_\varepsilon(x) := \varepsilon^{-n}\eta(x/\varepsilon)$.

W.l.o.g. we may assume that $u_k : (-T, 2T) \to X, Y, Z$, and assume

$$\sup_k \int_{-T}^{2T} (\|u_k(t)\|_X)^p\, dt + \sup_k \int_{-T}^{2T} (\|\partial_t u_k(t)\|_Z)^q\, dt < \infty \tag{12.33}$$

Otherwise we can work with a reflection that matches zero-th and first derivative:

$$\tilde{u}_k(t) := \begin{cases} -3u_k(T-(t-T)) + 4u_k(T - \frac{t-T}{2}) & t \in (T, 2T) \\ u_k(t) & t \in [0,T] \\ -3u_k(-t) + 4u_k(-\frac{t}{2}) & t \in (-T, 0) \end{cases}$$

[34] one can remove this assumption, then its called the *Aubin-Lions-Simons lemma*

Observe that then we'd still have

$$\sup_k \int_{-T}^{2T} (\|\tilde{u}_k(t)\|_X)^p \, dt + \sup_k \int_{-T}^{2T} (\|\partial_t \tilde{u}_k(t)\|_Z)^q \, dt < \infty$$

So we will assume (12.33) from now on.

For the typical bump function $\eta \in C_c^\infty(-1,1)$, $\int_{(-1,1)} \eta = 1$ and $\eta_\varepsilon(x) := \varepsilon^{-1}\eta_(x/\varepsilon)$ we set

$$u_{k;\varepsilon}(t) := u_k *_t \eta_\varepsilon = \int_{\mathbb{R}} \eta_\varepsilon(t-s)\, u_k(s)\, ds.$$

This makes sense for $t \in [0,T]$ if we (as we shall from now on) assume $\varepsilon < T$. Observe that

$$\|u_{k;\varepsilon}(t)\|_X \le \int_{\mathbb{R}} \eta_\varepsilon(t-s)\|u_k(s)\|_X \, ds \quad \forall t \in [0,T].$$

so that from (12.33) we obtain

$$\sup_{k,\varepsilon\in(0,1)} \int_0^T (\|u_{k;\varepsilon}(t)\|_X)^p \, dt < \infty, \tag{12.34}$$

$$\sup_{t\in[0,T]} \|u_{k;\varepsilon}(t)\|_X \le C(\varepsilon)$$

and

$$\sup_{t\in[0,T]} \|\partial_t u_{k;\varepsilon}(t)\|_X \le C(\varepsilon),$$

and thus

$$\sup_{t,s\in[0,T]} \|u_{k;\varepsilon}(t) - u_{k;\varepsilon}(s)\|_X \le C(\varepsilon)\ |s-t|. \tag{12.35}$$

So if we fix $\varepsilon > 0$ then

$$\{u_{k;\varepsilon}(t), t \in [0,T]\} \quad \text{is a bounded set in } X$$

and thus, by the compact embedding $X \hookrightarrow Y$,

$$\{u_{k;\varepsilon}(t), t \in [0,T]\} \subset \text{is contained in a } \textit{compact} \text{ set in } Y$$

From (12.35) we find that the sequene

$$u_{k;\varepsilon}(\cdot) : [0,T] \to Y$$

is equicontinuous (as a sequence in k, ε is fixed!). Then we use *Arzela-Ascoli*, Exercise 12.20, to obtain that up to taking a subsequence $u_{k_i;\varepsilon}$ converges uniformly to some $u_{;\varepsilon} : [0,T] \to Y$,

$$\sup_{t\in[0,T]} \|u_{k_i;\varepsilon}(t) - u_{;\varepsilon}(t)\|_Y \xrightarrow{i\to\infty} 0.$$

In particular we have for any $\varepsilon > 0$ a subsequence $u_{k_i;\varepsilon}$ and some $u_{;\varepsilon}$ such that

$$\left(\int_{[0,T]} \|u_{k_i;\varepsilon}(t) - u_{;\varepsilon}(t)\|_Y^p\right)^{\frac{1}{p}} \xrightarrow{i\to\infty} 0.$$

We now need to let $\varepsilon \to 0$ (we pretend that ε is a discrete sequence with $\varepsilon \to 0$). By a diagonal argument we can assume that the subsequence $u_{k_i;\varepsilon}$ is the same for all ε as $\varepsilon \to 0$). We express this in terms of Cauchy sequences:

$$\forall \varepsilon > 0, \forall \sigma > 0\ \exists L = L(\varepsilon, \sigma):\ \sup_{i,j \geq L} \left(\int_{[0,T]} \|u_{k_i;\varepsilon}(t) - u_{k_j;\varepsilon}(t)\|_Y^p \right)^{\frac{1}{p}} < \sigma. \tag{12.36}$$

We now claim:

$$\forall \sigma > 0, \quad \exists \varepsilon > 0 \text{ s.t. } \sup_i \left(\int_{[0,T]} \|u_{k_i;\varepsilon}(t) - u_{k_i}(t)\|_Y^p \right)^{\frac{1}{p}} \leq \sigma. \tag{12.37}$$

Once we have this, we conclude that also u_{k_i} is a Cauchy sequence in $L^p([0,T], Y)$ (and thus convergent by completeness), i.e. that

$$\forall \varepsilon > 0, \forall \sigma > 0\ \exists L = L(\varepsilon, \sigma):\ \sup_{i,j \geq L} \left(\int_{[0,T]} \|u_{k_i}(t) - u_{k_j}(t)\|_Y^p \right)^{\frac{1}{p}} < \sigma.$$

By completeness of $L^p([0,T], Y)$ we conclude.

So it remains to obtain (12.37).

Recall that by (12.34) and assumption

$$\Gamma := \sup_{k, \varepsilon \in (0,1)} \left(\int_0^T (\|u_{k;\varepsilon}(t)\|_X)^p\, dt \right)^{\frac{1}{p}} + \sup_k \left(\int_0^T (\|u_k(t)\|_X)^p\, dt \right)^{\frac{1}{p}} + \sup_k \left(\int_{-T}^{2T} (\|\partial_t u_k(t)\|_Z)^q\, dt \right)^{\frac{1}{q}} < \infty$$

By Theorem 12.17, for any $\delta > 0$

$$\|u_{k_i;\varepsilon}(t) - u_{k_i}(t)\|_Y \leq \delta \|u_{k_i;\varepsilon}(t) - u_{k_i}(t)\|_X + C(\delta) \|u_{k_i;\varepsilon}(t) - u_{k_i}(t)\|_Z \quad \forall x \in X.$$

So that we have

$$\left(\int_{[0,T]} \|u_{k_i;\varepsilon}(t) - u_{k_i}(t)\|_Y^p dt \right) \leq \delta \Gamma + C(\delta) \left(\int_{[0,T]} \|u_{k_i;\varepsilon}(t) - u_{k_i}(t)\|_Z^p \right)^{\frac{1}{p}}.$$

Now we have for $t \in [0,T]$

$$\begin{aligned} u_{k_i;\varepsilon}(t) - u_{k_i}(t) &= \int_{-T}^{2T} (u_{k_i}(s) - u_{k_i}(t))\, \eta_\varepsilon(t-s)\, ds \\ &= \int_{-T}^{2T} \int_s^t \partial_s u_{k_i}(r) dr \eta_\varepsilon(t-s)\, ds. \end{aligned}$$

Thus,

$$
\begin{aligned}
\|u_{k_i;\varepsilon}(t) - u_{k_i}(t)\|_Z \leq & \int_{-T}^{2T} \int_s^t \|\partial_s u_{k_i}(r)\|_Z dr \eta_\varepsilon(t-s)\, ds \\
\leq & \int_{-T}^{2T} \left(\int_s^t \|\partial_s u_{k_i}(r)\|_Z^q dr\right)^{\frac{1}{q}} |s-t|^{1-\frac{1}{q}} \eta_\varepsilon(t-s)\, ds \\
\leq & \int_{-T}^{2T} \left(\int_{-T}^{2T} \|\partial_s u_{k_i}(r)\|_Z^q dr\right)^{\frac{1}{q}} |s-t|^{1-\frac{1}{q}} \eta_\varepsilon(t-s)\, ds \\
\leq & \Gamma \int_{\mathbb{R}} |s-t|^{1-\frac{1}{q}} \eta_\varepsilon(t-s)\, ds \\
\lesssim_{\eta,q} & \Gamma\, \varepsilon^{1-\frac{1}{q}}.
\end{aligned}
$$

That is, we have shown

$$
\left(\int_{[0,T]} \|u_{k_i;\varepsilon}(t) - u_{k_i}(t)\|_Y^p dt\right) \leq \left(C_1\delta + \varepsilon^{1-\frac{1}{q}} C_2(\delta, q, \eta) T\right) \Gamma.
$$

We first pick $\delta > 0$ so that $\delta C_1 \Gamma < \frac{\sigma}{2}$. Then we pick $\varepsilon > 0$ so that $\varepsilon^{1-\frac{1}{q}} C(\delta, q, \eta) T\Gamma < \frac{\sigma}{2}$, then we have shown (12.37).

We can conclude. □

Exercise 12.20. *Let $K \subset (X,d)$ be a compact set in a metric space (X,d). Assume we have a sequence of functions $f_k : [0,1] \to K$ which is equicontinuous, i.e.*

$$
\forall \varepsilon > 0 \exists \delta > 0 : \quad d(f_k(x), f_k(y)) < \varepsilon \quad \forall k \in \mathbb{N}\ \forall x, y \in [0,1] \text{ with } |x-y| < \delta.
$$

Show that there exists $\bar{f} \in C^0([0,1], K)$ and a subsequence $(f_{k_i})_{i\in\mathbb{N}}$ such that f_{k_i} uniformly converges to $\bar{f}$, i.e.

$$
\lim_{i\to\infty} \sup_{x\in[0,1]} d(f(x), f_{k_i}(x)) = 0.
$$

Hint: If you only learned the proof of the Arzela-Ascoli theorem for $(X,d) = \mathbb{R}^n$ (with additionally boundedness), simply inspect that proof and repeat it to get this result. As always: we should care about proofs, not theorems.

Exercise 12.21. *Set*

$$
Z := \left(W_0^{1,3}(B(0,1))\right)^*
$$

and

$$
X := W^{1,2}(B(0,1))
$$

and

$$
Y := L^2(B(0,1)).
$$

Show that X, Y, Z satisfy the embedding assumptions of Theorem 12.19.

We are now ready to prove Claim 12.13

Proof of Claim 12.13. Recall that the goal is to show (up to subsequence):

$$\mathbf{v}_k \text{ converges } \textit{strongly} \text{ in } L^3_{loc}(B(0,1)\times(-1,0])$$

From Aubin-Lion's we have the following: Recall that from the PDE of $\mathbf{v}_k$,

$$\begin{cases}\partial_t \mathbf{v}_k = \Delta \mathbf{v}_k - (\mathbf{b}_k + \varepsilon_k \mathbf{v}_k)_\alpha \partial_\alpha \mathbf{v}_k - \nabla q_k \quad \text{in } Q_1\\ \operatorname{div} \mathbf{v}_k = 0 \quad \text{in } Q_1.\end{cases}$$

From the estimates (12.14) and (12.15) we have

$$\sup_k \|\mathbf{v}_k\|_{L^3(Q_1)} + \|q_k\|_{L^{\frac{3}{2}}(Q_1)} < \Gamma_1,$$

$$\sup_{k\in\mathbb{N}} \|\nabla \mathbf{v}_k\|_{L^2(Q_R)} \lesssim_R \Gamma_1 \quad \forall R \in (0,1).$$

Observe that $q \in L^{\frac{3}{2}}$ so $\nabla q \in W^{-1,\frac{3}{2}} = (W^{1,3})^*$. We conclude that

$$\sup_k \int_{(-R^2,0)} \|\partial_t \mathbf{v}_k\|^{\frac{3}{2}}_{\left(W_0^{1,3}(B(0,R))\right)^*} < \infty.$$

Also,

$$\sup_k \left(\int_{-R^2}^0 \left(\|\mathbf{v}_k(\cdot,t)\|_{W^{1,2}(B(0,R))}\right)^2\right)^{\frac{1}{2}} dt \lesssim_R \Gamma_1,$$

Set

$$Z := \left(W_0^{1,3}(B(0,R))\right)^*$$

and

$$X := W^{1,2}(B(0,R))$$

and

$$Y := L^2(B(0,R)).$$

Then from *Aubin-Lions lemma*, Theorem 12.19, Exercise 12.21 we find (using a diagonal argument in R) that for a subsequence (not relabeled) and any $R \in (0,1)$ we have that $\mathbf{v_k}$ converges strongly in $L^{\mathit{2}}(Q_R)$.

However we want strong convergence in $L^{\mathit{3}}(Q_R)$! But we can conclude this now from what we already know:

Recall, from (12.14) and (12.15)

$$\sup_k \left(\int_{-R^2}^0 \left(\|\mathbf{v}_k(\cdot,t)\|_{L^2(B(0,R))} + \|\nabla \mathbf{v}_k(\cdot,t)\|_{L^2(B(0,R))}\right)^2\right)^{\frac{1}{2}} dt \lesssim_R \Gamma_1, \tag{12.38}$$

By Sobolev embedding in space $(1-\frac{3}{2} = 0 - \frac{3}{6})$ we conclude

$$\sup_k \|\mathbf{v}_k\|_{L^2_t L^6_x(Q_R)} \equiv \sup_k \left(\int_{-R^2}^0 \|\mathbf{v}_k(\cdot,t)\|^2_{L^{\mathit{6}}(B(0,R))}\right)^{\frac{1}{2}} dt \lesssim_R \Gamma_1,$$

On the other hand, from the first term of the local energy identity, Claim 12.11, namely (12.16) we have

$$\sup_k \|\mathbf{v}_k\|_{L^\infty_t L^2_x(Q_R)} \equiv \sup_k \sup_{t\in(-R^2,0)} \|\mathbf{v}_k(\cdot,t)\|_{L^2(B(0,R))} \lesssim_R \Gamma_1.$$

We can combine the $L^2_t L^6_x$ and $L^\infty_t L^2_x$ to an $L^{\frac{10}{3}}_{x,t}$-estimate: We apply twice Hölder's inequality, first in space $L^5 \cdot L^{10} \subset L^{\frac{10}{3}}$, then in time, we get

$$\begin{aligned}
\|\mathbf{v}_k\|_{L^{\frac{10}{3}}(Q_R)} &= \left(\int_{(-R^2,0)} \|\mathbf{v}_k(\cdot,t)\|^{\frac{10}{3}}_{L^{\frac{10}{3}}(B(0,R))}\right)^{\frac{3}{10}}\\
&= \left(\int_{(-R^2,0)} \|\,|\mathbf{v}_k(\cdot,t)|^{\frac{2}{5}}\ |\mathbf{v}_k(\cdot,t)|^{\frac{3}{5}}\,\|^{\frac{10}{3}}_{L^{\frac{10}{3}}(B(0,R))}\right)^{\frac{3}{10}}\\
&\leq \left(\int_{(-R^2,0)} \left(\|\,|\mathbf{v}_k(\cdot,t)|^{\frac{2}{5}}\,\|_{L^5(B(0,R))}\,\|\,|\mathbf{v}_k(\cdot,t)|^{\frac{3}{5}}\,\|_{L^{10}(B(0,R))}\right)^{\frac{10}{3}}\right)^{\frac{3}{10}}\\
&= \left(\int_{(-R^2,0)} \|\mathbf{v}_k(\cdot,t)\|^{\frac{4}{3}}_{L^2(B(0,R))}\,\|\mathbf{v}_k(\cdot,t)\|^2_{L^6(B(0,R))}\right)^{\frac{3}{10}}\\
&\lesssim_R (\Gamma_1)^{\frac{2}{5}} \left(\int_{(-R^2,0)} \|\mathbf{v}_k(\cdot,t)\|^2_{L^6(B(0,R))}\right)^{\frac{3}{10}}\\
&\lesssim_R (\Gamma_1)^{\frac{2}{5}}\,(\Gamma_1)^{\frac{3}{5}}.
\end{aligned}$$

That is, up to subsequence we know that for any $R \in (0,1)$

$$\mathbf{v}_k \rightharpoonup \mathbf{v} \quad \text{weakly in } L^{\frac{10}{3}}(Q_R).$$

We also know (from Aubin-Lions Lemma)

$$\mathbf{v}_k \to \mathbf{v} \quad \text{strongly in } L^2(Q_R).$$

Combining these two facts, measure theory (this a consequence of Vitali's convergence theorem) we conclude now that

$$\mathbf{v}_k \to \mathbf{v} \quad \text{strongly in } L^r(Q_R).$$

for any $r < \frac{10}{3}$. In particular we can choose $r = 3 < \frac{10}{3}$ and conclude. □

13. $C^{1,\alpha}$-regularity theory for the p-Laplacian

A standard references for the *p-Laplace* is the book by Lindqvist, [Lindqvist, 2006]. More specific to regularity theory, cf. Maly-Ziemer [Malý and Ziemer, 1997], Lieberman [Lieberman, 1996], and DiBenedetto [DiBenedetto, 2010].

The Laplacian equation can be seen as *Euler-Lagrange equation* of the Dirichlet energy

$$\mathcal{E}_2(u) := \int_\Omega |\nabla u|^2.$$

For example we have shown in what is sometimes called *Dirichlet principle*, see Theorem 2.44, that u solves

$$\begin{cases} -\Delta u = 0 & \text{in } \Omega \\ u = g & \text{on } \partial\Omega \end{cases}$$

If and only if u minimizes $\mathcal{E}_2$ in the set $X := \{u \in W^{1,2}(\Omega) : \quad u - g \in W_0^{1,2}(\Omega)\}$.

What happens when we replace 2 by a general p? I.e. what happens if we consider

$$\mathcal{E}_p(u) := \int_\Omega |\nabla u|^p.$$

Then the Euler-Lagrange equation is

$$\operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) = 0 \quad \text{in } \Omega \tag{13.1}$$

Sometimes people write $\Delta_p u \equiv \operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right)$ for the *p-Laplace*.

We will not discuss the role of the p-Laplace in applications (for this we refer to the above mentioned references), but treat it as an analytical object of interest.

If $p > 2$, we quickly observe that the p-Laplace is *degenerate* elliptic: We have that $|\nabla u|^{p-2} \geq 0$, but it can actually vanish, in which case the PDE does not measure anything. However $|\nabla u| = 0$ on any *open* set means that u is piecewise constant (and in particular pretty regular).

So the behavior of a solution of a p-Laplace equation is a governed by a dichotomy of $|\nabla u|$ being small (where this is the only information we can use) and $|\nabla u|$ not small (where the equation is uniformly elliptic).

It turns out, for $p > 2$, solutions to (13.1) may not be C^2, but they are $C^{1,\alpha}$ for some $\alpha > 0$ (there are cases in which the optimal α is known, and cases where it is unknown). Cf. Lewis and Iwaniec-Manfredi [Lewis, 1980, Iwaniec and Manfredi, 1989] and references within.

On the other hand the proof of $C^{1,\alpha}$-regularity is *de facto* a clever combination of De Giorgi-Nash-Moser theory.

In the following we plan to prove

Theorem 13.1. *Assume $u \in W^{1,p}(\Omega)$*

$$\operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) = 0 \quad \text{in } \Omega$$

Then $u \in C^{1,\alpha}_{loc}(\Omega)$.

Unlike in the case of the Laplace-equation, the so-called systems, i.e. the case of $u \in W^{1,p}(\Omega, \mathbb{R}^m)$ solving

$$\operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) = 0 \quad \text{in } \Omega$$

is quite more complicated – while for the Laplacian-case $p = 2$ we can simply consider $\Delta u = 0$ componentwise as the scalar case $m = 1$, for $p \neq 2$ this does not hold, since $|\nabla u|^{p-2}$ mixes the different components. However the argument below can be adapted to the systems case for $C^{1,\alpha}$-theory, and to what is referred to Uhlenbeck-type equations.

13.1. **Boundedness.** Solutions to

$$\mathrm{div}\,(|\nabla u|^{p-2}\nabla u) = 0 \quad \text{in } \Omega$$

are bounded, which follows from essentially the usual Moser-iteration argument, Theorem 10.3.

13.2. **Lipschitz continuity.** Solutions to

$$\mathrm{div}\,(|\nabla u|^{p-2}\nabla u) = 0 \quad \text{in } \Omega$$

are Lipschitz continuous which follows from a type of Moser iteration testing the equation with powers of $\partial_{x^i} u$, cf. [Malý and Ziemer, 1997]

From now on we will always assume that solutions are already Lipschitz.

13.3. $C^{1,\alpha}$**-regularity theory.** The $C^{1,\alpha}$-regularity has been proven by Ural'tseva [Ural'ceva, 1968], Uhlenbeck [Uhlenbeck, 1977], [Lewis, 1983] (for $p < 2$), see also [Evans, 1982].

In the following we discuss the $C^{1,\alpha}$-regularity theory under the assumptions of Lipschitz continuity of the solutions – as explained to me by Adimurthi, see [Adimurthi and Banerjee, 2022].

We have to consider two cases: the *nondegenerate case* ("$|\nabla u| \neq 0$") where the equation behaves essentially linear:

Proposition 13.2 (nondegenerate case)**.** *There exists $\gamma, \alpha \in (0,1)$ and $C > 1$ depending only on p and n such that the following holds:*

Assume $u \in W^{1,p}(B(x_0,R))$ and $\lambda > 0$ are such that

- *u is a solution to*

$$\mathrm{div}\,(|\nabla u|^{p-2}\nabla u) = 0 \quad \text{in } B(x_0,R)$$

- *we have*

$$|\nabla u| \leq \lambda \quad \text{in } B(x_0,R) \tag{13.2}$$

- *$|\nabla u|$ is not too small too often in $B(R)$, more precisely there exists some $k \in \{1,\dots,n\}$ such that*

$$|\{x \in B(x_0,R) : \quad \partial_{x_k} u \leq \frac{1}{2}\lambda\}| \leq \gamma |B(R)|$$

or

$$|\{x \in B(x_0,R) : \quad \partial_{x_k} u \geq -\frac{1}{2}\lambda\}| \leq \gamma |B(R)|$$

- *x_0 is a Lebesgue point of ∇u*

Then we have

$$|\nabla u(x_0) - \fint_{B(\rho)} \nabla u| \leq C\,\lambda\, \left(\frac{\rho}{R}\right)^{\alpha} \quad \forall \rho \in (0,R]$$

The second case is the *degenerate case* "$|\nabla u| \ll 1$"

Proposition 13.3. *For any $\gamma \in (0,1)$ there exist $\sigma, \theta \in (0,1)$ depending only on p and n such that the following holds:*

Assume $u \in W^{1,p}(B(R))$ and $\lambda > 0$ are such that

- *u is a solution to*
$$\operatorname{div}(|\nabla u|^{p-2}\nabla u) = 0 \quad \text{in } B(R)$$
- *we have*
$$|\nabla u| \leq \lambda \quad \text{in } B(R)$$
- *$|\nabla u|$ is small quite often in $B(R)$, more precisely:*
$$|\{x \in B(R) : \quad \partial_{x_i} u \leq \frac{1}{2}\lambda\}| \geq \gamma |B(R)| \quad \forall i \in \{1,\ldots,n\}$$
and
$$|\{x \in B(R) : \quad \partial_{x_i} u \geq -\frac{1}{2}\lambda\}| \geq \gamma |B(R)| \quad \forall i \in \{1,\ldots,n\}$$

Then

$$\sup_{B(\sigma R)} |\nabla u| \leq \theta \lambda$$

Combining Proposition 13.2 and Proposition 13.3

Theorem 13.4. *There exists $\alpha \in (0,1)$ depending only on n and p such that the following holds:*

Assume $u \in W^{1,p}(B(x_0,R))$ and $\lambda > 0$ are such that

- *u is a solution to*
$$\operatorname{div}(|\nabla u|^{p-2}\nabla u) = 0 \quad \text{in } B(x_0,R)$$
- *we have*
$$|\nabla u| \leq \lambda \quad \text{in } B(x_0,R)$$

Then if x_0 is a Lebesgue point of ∇u we have

$$|\nabla u(x_0) - (\nabla u)_{B(x_0,\rho)}| \leq \lambda \left(\frac{\rho}{R}\right)^{\alpha} \quad \forall \rho \in (0,R).$$

From Campanato's theorem we immediately obtain from Theorem 13.4 the $C^{1,\alpha}$-regularity theory

Corollary 13.5. *Under the assumption of the theorem, if for $\Omega \subset \mathbb{R}^n$ open $u \in W^{1,p}(\Omega) \cap \mathrm{Lip}$ solves*

$$\operatorname{div}(|\nabla u|^{p-2}\nabla u) = 0 \quad \text{in } \Omega$$

then $u \in C^{1,\alpha}(\Omega)$, and we have for any $\Omega' \subset\subset \Omega$

$$[\nabla u]_{C^\alpha(\Omega')} \precsim_{\Omega',\Omega} \|\nabla u\|_{L^\infty(\Omega)}.$$

Proof of Theorem 13.4. We take $\gamma, \alpha_1 \in (0,1)$ and C from Proposition 13.2 and $\sigma, \theta \in (0,1)$ from Proposition 13.3.

For simplicity of notation we assume that all balls below are centered at x_0 and that x_0 is a Lebesgue point of ∇u.

We observe that if $F = (F_1, F_2, \ldots, F_n) : B(R) \to \mathbb{R}^n$ is a vectorial function then one of the following must be true

(1) $|\{x \in B(R) : \quad F_k(x) \leq \frac{\lambda}{2}\}| \leq \gamma |B(R)|$ for at least one $k \in \{1, \ldots, n\}$, or
(2) $|\{x \in B(R) : \quad F_k(x) \geq -\frac{\lambda}{2}\}| \leq \gamma |B(R)|$ for at least one $k \in \{1, \ldots, n\}$, or
(3) $|\{x \in B(R) : \quad F_k(x) \leq \frac{\lambda}{2}\}| \geq \gamma |B(R)|$ *and* $|\{x \in B(R) : \quad F_k(x) \geq \frac{\lambda}{2}\}| \geq \gamma |B(R)|$ for all $k \in \{1, \ldots, n\}$.

Thus we can fix $i_0 \in \{0, 1, \ldots, \infty\}$ such that

- for $i < i_0$ we have

(13.3)
$$\min\left\{|\{B(\sigma^i R) : \partial_k u \leq \theta^i \frac{\lambda}{2}\}|, |\{B(\sigma^i R) : \partial_k u > -\theta^i \frac{\lambda}{2}\}|\right\} > \gamma |B(\sigma^i R)| \quad \forall k \in \{1, \ldots, n\}$$

- for i_0 we have

(13.4)
$$\min\left\{|\{B(\sigma^{i_0} R) : \partial_k u \leq \theta^{i_0} \frac{\lambda}{2}\}|, |\{B(\sigma^{i_0} R) : \partial_k u \geq -\theta^{i_0} \frac{\lambda}{2}\}|\right\} \leq \gamma |B(\sigma^{i_0} R)| \quad \text{for at least one } k \in \{1, \ldots, n\}$$

It is important to note that $i_0 = 0$ (i.e. only the second case happens) or $i_0 = \infty$ (only the first case happens) are included in our analysis. We have no information what value i_0 is, and we will use what is sometimes called a *stopping time argument* to show that it does not matter.

We now apply Proposition 13.2 and Proposition 13.3, and have

- By induction we have for any $i < i_0$ we have

$$\sup_{B(\sigma^i R)} |\nabla u| \leq \theta^i \lambda$$

and thus from (13.3) and Proposition 13.3

$$\sup_{B(\sigma\sigma^i R)} |\nabla u| \leq \theta^{i+1}\lambda$$

that is in particular

$$\sup_{B(\sigma^{i+1} R)} |\nabla u| \leq \theta^{i+1}\lambda \quad \forall i < i_0. \tag{13.5}$$

- for i_0 (assuming its finite) we have from (13.5)

$$\sup_{B(\sigma^{i_0} R)} |\nabla u| \leq \theta^{i_0}\lambda \tag{13.6}$$

and thus from (13.4) and Proposition 13.2 for some $\alpha_1 > 0$

$$|\nabla u(x_0) - \fint_{B(\rho)} \nabla u| \leq C(n)\, \theta^{i_0}\lambda \left(\frac{\rho}{\sigma^{i_0} R}\right)^{\alpha_1} \quad \forall \rho \in (0, \sigma^{i_0} R]$$

Now for $\alpha_2 = \log_\sigma \theta$ we have

$$\theta^{i_0} = \sigma^{i_0 \alpha_2}$$

so that the above is

$$|\nabla u(x_0) - \fint_{B(\rho)} \nabla u| \leq C(n)\, \sigma^{i_0\alpha_2}\lambda \left(\frac{\rho}{\sigma^{i_0} R}\right)^{\alpha_1} \quad \forall \rho \in (0, \sigma^{i_0} R]$$

Now let $\alpha_3 := \min\{\alpha_1, \alpha_2\}$ and observing that $\sigma < 1$ and $\frac{\rho}{\sigma^{i_0} R} < 1$, we have

$$|\nabla u(x_0) - \fint_{B(\rho)} \nabla u| \leq C(n)\, \sigma^{i_0\alpha_3}\lambda \left(\frac{\rho}{\sigma^{i_0} R}\right)^{\alpha_3} \quad \forall \rho \in (0, \sigma^{i_0} R]$$

That is we have

$$|\nabla u(x_0) - \fint_{B(\rho)} \nabla u| \leq C(n)\, \lambda \left(\frac{\rho}{R}\right)^{\alpha_3} \quad \forall \rho \in (0, \sigma^{i_0} R] \tag{13.7}$$

Yet again observe that if $i_0 = \infty$ this is a completely trivial statement.

Since we have no information on i_0 we are not done yet, we have to consider the case $\rho \in (\sigma^{i_0} R, R)$:

So let now $\rho \in (\sigma^{i_0} R, R)$. Then take $\bar{i} \leq i_0$ such that $\sigma^{\bar{i}+1} R \leq \rho \leq \sigma^{\bar{i}} R$. Then

$$\begin{aligned}
|\nabla u(x_0) - (\nabla u)_{B(\rho)}| \leq& 2 \sup_{\sigma^{\bar{i}} R} |\nabla u| \\
\overset{(13.5)}{\leq}& 2\theta^{\bar{i}}\lambda \\
=& 2\sigma^{\bar{i}\alpha_1}\lambda \\
\leq& 2\sigma^{-\alpha_1}\lambda \left(\frac{\rho}{R}\right)^{\alpha_1} \\
\leq& 2\sigma^{-\alpha_1}\lambda \left(\frac{\rho}{R}\right)^{\alpha_3}
\end{aligned}$$

Combining this with (13.7) we have shown

$$|\nabla u(x_0) - \fint_{B(\rho)} \nabla u| \leq C(\kappa, \sigma, n, p)\, \lambda \left(\frac{\rho}{R}\right)^{\alpha_3} \quad \forall \rho \in (0, R)$$

The proof of Theorem 13.4 is now complete. □

13.3.1. *The nondegenerate case: Proof of Proposition 13.2.* Taking a directional derivative of the equation we find that

$$\operatorname{div}\left(|\nabla u|^{p-2} \nabla\left(\partial_{x_i} u\right)\right) = 0.$$

We first prove the following seemingly weaker result than Proposition 13.2, cf. Proposition 10.19.

Lemma 13.6 (nondegenerate case)**.** *There exists $\gamma \in (0,1)$ depending only on p and n such that the following holds:*

Assume $u \in W^{1,p}(B(R))$ and $\lambda > 0$ and $k \in \{1, \ldots, n\}$ are such that

- *u is a solution to*

$$-\operatorname{div}\left(|\nabla u|^{p-2} \nabla\left(\partial_{x_k} u\right)\right) \leq 0. \tag{13.8}$$

- *we have*

$$|\nabla u| \leq \lambda \quad \text{in } B(R)$$

- $$|\{x \in B(R) : \quad \partial_{x_k} u \leq \frac{1}{2}\lambda\}| \leq \gamma |B(R)|$$

Then

$$\partial_{x_k} u > \frac{\lambda}{4} \quad \text{on } B(R/2).$$

Proof of Lemma 13.6. As usual we test the PDE (13.8) with $(\partial_{x_k} u - \mu)_- \eta^2$ for some $\mu > 0$ and $\eta \in C_c^\infty(B(R))$, $\eta \geq 0$, to find

$$\int |\nabla u|^{p-2} |\nabla(\partial_{x_k} u - \mu)_-|^2 \eta^2 \leq -\int |\nabla u|^{p-2} \nabla(\partial_{x_k} u - \mu)_- \,(\partial_{x_k} u - \mu)_- 2\eta \nabla \eta$$

Which by Young's inequality and absorption readily leads to

$$\int |\nabla u|^{p-2} |\nabla(\partial_{x_k} u - \mu)_-|^2 \eta^2 \precsim \int |\nabla u|^{p-2} |(\partial_{x_k} u - \mu)_-|^2 |\nabla \eta|^2$$

If $p \geq 2$, since $|\nabla u| \leq \lambda$ we find

$$\int |\nabla u|^{p-2} |\nabla(\partial_{x_k} u - \mu)_-|^2 \eta^2 \leq C\, \lambda^{p-2} \int |(\partial_{x_k} u - \mu)_-|^2 |\nabla \eta|^2$$

If $0 < \nu < \mu$ we can estimate this further, setting

$$f_{\nu,\mu} := \min\{(f - \mu)_-, \mu - \nu\},$$

or, in other words,

$$f_{\nu,\mu} := \begin{cases} 0 & \text{if } f \geq \mu, \\ \mu - f & \text{if } \mu \geq f \geq \nu, \\ \mu - \nu & \text{if } f \leq \nu, \end{cases}$$

then

$$\begin{aligned} &\nu^{p-2} \int |\nabla (\partial_{x_k} u)_{\nu,\mu}|^2 \eta^2 \\ =&\nu^{p-2} \int |\nabla (\partial_{x_k} u - \mu)_-|^2 \eta^2 \chi_{\{\partial_{x_k} u > \nu\}} \\ \approx&\lambda^{p-2} \int |(\partial_{x_k} u - \mu)_-|^2 |\nabla \eta|^2 \\ \leq&\lambda^{p-2} \int |(\partial_{x_k} u - \mu)_-|^2 |\nabla \eta|^2 \end{aligned}$$

And thus, from Sobolev embedding we have

$$\begin{aligned} &\nu^{p-2} \left(\int |\eta (\partial_{x_k} u)_{\nu,\mu})|^{\frac{2n}{n-2}} \right)^{\frac{n-2}{n}} \\ \lesssim&\nu^{p-2} \int |\nabla (\eta (\partial_{x_k} u)_{\nu,\mu})|^2 \\ \lesssim&\lambda^{p-2} \int |(\partial_{x_k} u - \mu)_-|^2 |\nabla \eta|^2 + \nu^{p-2} \int |\nabla \eta|^2 (\partial_{x_k} u)^2_{\nu,\mu} \\ \leq& \left(\lambda^{p-2} + \nu^{p-2} \right) \int |(\partial_{x_k} u - \mu)_-|^2 |\nabla \eta|^2 \end{aligned}$$

Which readily leads to

$$\nu^{\frac{p-2}{2}} \|\eta (\partial_{x_k} u)_{\nu,\mu}\|_{L^2} \lesssim |\{\eta (\partial_{x_k} u)_{\nu,\mu} > 0\}|^{\frac{1}{n}} \ \left(\lambda^{\frac{p-2}{2}} + \nu^{\frac{p-2}{2}} \right) \ \|(\partial_{x_k} u - \mu)_- \, |\nabla \eta|\|_{L^2} \tag{13.9}$$

Again, this holds only if $p \geq 2$. If $p \leq 2$ we have instead

$$\lambda^{\frac{p-2}{2}} \|\eta (\partial_{x_k} u)_{\nu,\mu}\|_{L^2} \lesssim |\{\eta (\partial_{x_k} u)_{\nu,\mu} > 0\}|^{\frac{1}{n}} \ \left(\mu^{\frac{p-2}{2}} + \nu^{\frac{p-2}{2}} \right) \ \|(\partial_{x_k} u - \mu)_- \, |\nabla \eta|\|_{L^2} \tag{13.10}$$

As usual we apply this to an iteration.

Set

$$\bar{\mu} := \sup_B (u_{x_k} - \frac{\lambda}{2})_- \overset{|\nabla u| \leq \lambda}{\leq} \frac{3}{2} \lambda.$$

If $\bar{\mu} < \frac{\lambda}{4}$ we have $u_{x_k} > \frac{\lambda}{4}$ in all of B, and we can conclude. Thus we may assume w.l.o.g.

$$\frac{\lambda}{4} \leq \bar{\mu} \leq \frac{3}{2} \lambda \tag{13.11}$$

We will also assume w.l.o.g. $B = B(0,1)$. We define a decreasing set of balls with $B_0 = B(0,1)$ and $B_\infty = B(0,1/2)$, namely

$$B_m := B(0, \frac{1}{2} + \frac{1}{2^{m+1}}),$$

and fix nonnegative cutoff-functions $\eta_m \in C_c^\infty(B_m)$, $\eta_m \equiv 1$ in B_{m+1}, and $\|\nabla \eta_m\|_{L^\infty} \precsim 2^m$.

We also define a decreasing sequence of μ_m

$$\mu_m := \frac{\lambda}{2} - \frac{\bar{\mu}}{16}(1-\frac{1}{2^m}),$$

and we observe that with $\mu_0 = \frac{\lambda}{2}$ and $\mu_\infty = \frac{\lambda}{2} - \frac{\bar{\mu}}{16} \geq \frac{\lambda}{4}$, so $|\mu_m| \approx \lambda$. Moreover,

$$\mu_m - \mu_{m+1} = \frac{\bar{\mu}}{16}\frac{1}{2^{m+1}} \overset{(13.11)}{\approx} \frac{\lambda}{2^m}.$$

Then we want to use (13.9) (assume $p \geq 2$ for the moment) to find an iterative estimate for

$$A_m := \{x \in B_m : u_{x_k} \leq \mu_m\},$$

and indeed from (13.9) we obtain

$$\mu_{m+1}^{\frac{p-2}{2}} \|(\partial_{x_k} u)_{\mu_{m+1},\mu_m}\|_{L^2(B_{m+1})} \precsim |A_m|^{\frac{1}{n}} \ \left(\lambda^{\frac{p-2}{2}} + \mu_{m+1}^{\frac{p-2}{2}}\right) \ 2^m \|(\partial_{x_k} u - \mu_m)_- \|_{L^2(B_m)}$$

and thus

$$\begin{aligned}
\mu_{m+1}^{\frac{p-2}{2}} \ (\mu_m - \mu_{m+1}) \, |A_{m+1}|^{\frac{1}{2}} =& \mu_{m+1}^{\frac{p-2}{2}} \|(\partial_{x_k} u)_{\mu_{m+1},\mu_m}\|_{L^2(A_{m+1})} \\
&\precsim |A_m|^{\frac{1}{n}} \ \left(\lambda^{\frac{p-2}{2}} + \mu_{m+1}^{\frac{p-2}{2}}\right) \ 2^m \|(\partial_{x_k} u - \mu_m)_- \|_{L^2(A_m)} \\
&\precsim |A_m|^{\frac{1}{2}+\frac{1}{n}} \ \left(\lambda^{\frac{p-2}{2}} + \mu_{m+1}^{\frac{p-2}{2}}\right) \ 2^m \, (\lambda + \mu_m)
\end{aligned}$$

With the estimates for μ_m this becomes

$$\lambda^{\frac{p-2}{2}} \ \frac{\lambda}{2^m} |A_{m+1}|^{\frac{1}{2}} \precsim |A_m|^{\frac{1}{2}+\frac{1}{n}} \ \lambda^{\frac{p-2}{2}+1} 2^m$$

The same holds true if $p \leq 2$ and we use we use (13.10) instead of (13.9).

So, for any $p \in (1,\infty)$, with a constant independent of λ we have

$$\frac{1}{2^m} |A_{m+1}|^{\frac{1}{2}} \leq C \, \Gamma^m \, |A_m|^{\frac{1}{2}+\frac{1}{n}}$$

for some constant $C > 0$ and $\Gamma > 1$, both independent of λ. With the usual argument, Exercise 10.21, we conclude that if

$$|A_0| = \left|\{x \in B(0,1) : u_{x_k} \leq \frac{\lambda}{2}\}\right| \ll 1$$

then

$$0 = |A_\infty| \geq \left|\{x \in B(0,1/2) : u_{x_k} \leq \frac{\lambda}{4}\}\right|$$

and thus $u_{x_k} \leq \frac{\lambda}{4}$ in $B(0,1/2)$. We can conclude.

□

With the help of Lemma 13.6 and linear DeGiorgi-Nash-Moser theory we now obtain the proof of Proposition 13.2. It is worth observing that argument below also works for *systems*, i.e. if u is vectorial (this leads to the so-called *Uhlenbeck*-structure), because the system

$$\operatorname{div}(|\nabla u|^{p-2}\partial_{x_\ell} u) = 0$$

can be considered componentwise.

Proof of Proposition 13.2. Substituting u by $-u$ we may assume

$$|\{x \in B(R) : \quad \partial_{x_k} u \leq \frac{1}{2}\lambda\}| \leq \gamma |B(R)|.$$

If γ is taken from Lemma 13.6, then we have

$$|\nabla u| \geq |\partial_{x_k} u| \geq \frac{\lambda}{4} \quad \text{in } B(R/2).$$

Thus, for any $\ell \in \{1, \dots, n\}$ we have

$$\operatorname{div}(|\nabla u|^{p-2}\partial_{x_\ell} u) = 0 \quad \text{in } B(R/2)$$

is a uniformly elliptic equation with bounded measurable data $|\nabla u|^{p-2} \in (\frac{\lambda}{4}, \lambda)$.

By DeGiorgi-Nash-Moser regularity theory, Theorem 10.1 (observe that upper and lower ellipticity constants are both comparable to λ, meaning the constant from DeGiorgi-Nash-Moser is independent of λ) we have some $\alpha > 0$ such that

$$\sup_{x,y \in B(R/4)} \frac{|\partial_{x_\ell} u(x) - |\partial_{x_\ell} u(y)|}{|x-y|^\alpha} \leq C(R,n) \|\partial_{x_\ell} u\|_{L^2(B(R/2)} \leq \tilde{C}(R,n)\, \lambda$$

By scaling arguments we find out the scaling of the constant $C(R,n)$ and have

$$\sup_{x,y \in B(R/4)} \frac{|\partial_{x_\ell} u(x) - \partial_{x_\ell} u(y)|}{|x-y|^\alpha} \leq \tilde{C}(n)\, R^\alpha \lambda.$$

In particular we have

$$|\partial_{x_\ell} u(x_0) - \fint_{B(x_0,\rho)} \partial_{x_\ell} u| \leq C\, \lambda \left(\frac{\rho}{R}\right)^\alpha \quad \forall \rho \in (0, \frac{R}{4}]$$

For $\rho \in (\frac{R}{4}, R)$ this estimate follows from the Lipschitz bound $|\nabla u| \leq \lambda$,

$$|\partial_{x_\ell} u(x_0) - \fint_{B(x_0,\rho)} \partial_{x_\ell} u| \leq 2\lambda \leq 2\lambda \left(\frac{\rho}{R}\right)^\alpha 4^\alpha \quad \forall \rho \in (\frac{R}{4}, R].$$

In conclusion we have shown

$$|\partial_{x_\ell} u(x_0) - \fint_{B(x_0,\rho)} \partial_{x_\ell} u| \leq C\, \lambda \left(\frac{\rho}{R}\right)^\alpha \quad \forall \rho \in (0, R]$$

This holds for every $\ell \in \{1, \dots, n\}$ so we can conclude.

$\square$

13.3.2. *The degenerate case: Proof of Proposition 13.3.* Proposition 13.3 is a consequence of the following lemma applied to u and $-u$.

Lemma 13.7. *Given any $\mu > 0$ there exists $\theta \in (0,1)$ such that for any $i \in \{1,\dots,n\}$, $R > 0$ and $\lambda > 0$ we have the following:*

If

- *we have*
$$|\nabla u| \leq \lambda \quad \text{in } B(0,R)$$
- *u is a solution to the equation*
$$\operatorname{div}\left(|\nabla u|^{p-2}\nabla \partial_{x_i} u\right) = 0 \quad \text{in } B(0,R)$$
- *we have*
$$|\{\partial_{x_i} u \leq \frac{\lambda}{2}\} \cap B(0,R)| \geq \mu |B(0,R)|$$

Then
$$\sup_{B(0,R/4)} \partial_{x_i} u \leq \theta\lambda.$$

Proof. We may assume that $\lambda = 1$ and $R = 2$.

Set
$$\tilde{v} := 2\left(\partial_{x_i} u - \frac{1}{2}\right)$$
Then we have

- we have
$$\tilde{v} \leq 1 \quad \text{in } B(0,2)$$
- If $\tilde{v} \geq \mu$ and $\mu > -1$ then
$$\partial_{x_i} u \geq \frac{1}{2}(1+\mu)$$
and thus
$$|\nabla u| \geq \frac{1}{2}(1+\mu). \tag{13.12}$$
- u is a solution to the equation
$$\operatorname{div}\left(|\nabla u|^{p-2}\nabla \tilde{v}\right) = 0 \quad \text{in } B(0,2)$$
- we have and
$$|\{\tilde{v} \leq 0\} \cap B(0,1)| \geq \delta$$

For $k \in \mathbb{N}$ set
$$w_k := 2^k(\tilde{v} - (1-2^{-k}))_+.$$
We first collect some properties of w_k

- we still have $w_k \leq 1$ in $B(0,2)$ (since $\tilde{v} \leq 1$ in $B(0,2)$)
- we also have
$$\{w_k \leq 0\} \cap B(0,1) = \{\tilde{v} \leq (1-2^{-k})\} \cap B(0,1) \supset \{\tilde{v} \leq 0\} \cap B(0,1)$$
so that
$$|\{w_k \leq 0\} \cap B(0,1)| \geq \mu. \tag{13.13}$$
- We can write $w_k = (2^k\tilde{v} - 2^k + 1)_+$, and conclude that
$$\begin{aligned} &w_k \geq \frac{1}{2}\\ \Leftrightarrow &2w_k - 1 \geq 0\\ \Leftrightarrow &w_{k+1} \geq 0, \end{aligned}$$
i.e.
$$\{w_k \geq \frac{1}{2}\} = \{w_{k+1} \geq 0\} \tag{13.14}$$
- We have $w_{k+1} = (2w_k - 1)_+$. Hence,
$$w_k(x) < \frac{1}{2} \quad \Rightarrow w_{k+1}(x) = 0.$$
This implies that
$$\{0 < w_k(x) < \frac{1}{2}\} \cap \{0 < w_j(x) < \frac{1}{2}\} = \emptyset \quad k \neq j \tag{13.15}$$
- For any $\mu \geq 0$, taking a cutoff-function $\eta \in C_c^\infty(B(0,2))$, $\eta \equiv 1$, and testing
$$\operatorname{div}(|\nabla u|^{p-2}\nabla\tilde{v}) = 0 \quad \text{in } B(0,2)$$
with $\eta^2(\tilde{v}-\mu)_+$ we obtain, if $p \geq 2$,
$$\left(\frac{1}{2}(1+\mu)\right)^{p-2} \int_{B(0,2)} |\nabla(\tilde{v}-\mu)_+|^2\eta^2 \overset{(13.12)}{\leq} \int_{B(0,2)} |\nabla u|^{p-2}|\nabla(\tilde{v}-\mu)_+|^2\eta^2 \precsim \int_{B(0,2)} |\nabla\eta|^2\,|(\tilde{v}-\mu)_+|^2.$$
and if $p \leq 2$
$$\int_{B(0,2)} |\nabla(\tilde{v}-\mu)_+|^2\eta^2 \overset{|\nabla u|^{p-2}\geq 1}{\leq} \int_{B(0,2)} |\nabla u|^{p-2}|\nabla(\tilde{v}-\mu)_+|^2\eta^2 \overset{(13.12)}{\leq} \left(\frac{1}{2}(1+\mu)\right)^{p-2} \int_{B(0,2)} |\nabla\eta|^2\,|(\tilde{v}-\mu)_+|^2.$$
- Similarly, we have for w the same inequality as in Lemma 10.18: namely for any $\mu > 0$, $k \geq 1$, and $\eta \in C_c^\infty(B(0,2))$. Then we have for
$$\int_{B(0,2)} |\nabla(\eta(w_k-\mu)_+)|^2 \leq C\,\|\nabla\eta\|_{L^\infty}^2 \int_{\operatorname{supp}\eta} |(w_k-\mu)_+|^2$$

From Proposition 10.19 (see also Exercise 10.20), which just relies on the above Cacciopoli estimate and otherwise does not need w_k to solve any PDE, we conclude that
$$\|w_k\|_{L^\infty(B(0,1/2))} = \sup_{B(0,1/2)} (w_k)_+ \precsim \|w_k\|_{L^2(B(0,1)} \quad \forall k \geq 1. \tag{13.16}$$

So we will now work on showing that for sufficiently large k, $\|w_k\|_{L^2(B(0,1))} \ll 1$.

- In particular from the above any $k \geq 0$,

$$\int_{B(0,1)} |\nabla w_k|^2 \leq C(n) \int_{B(0,2)} |w_k|^2 \leq C(n)|B(0,2)|. \tag{13.17}$$

- From De Giorgi's isoperimetric inequality, Lemma 10.29, we obtain

$$(\mathcal{L}^n(\{w_k \leq 0\}))^2 \left(\mathcal{L}^n(\{w_k \geq \frac{1}{2}\})\right)^2 \leq C(n)\, \mathcal{L}^n(\{0 < w_k < \frac{1}{2}\})\, \|\nabla w_k\|^2_{L^2(B(0,1))}.$$

In view of (13.17), (13.14), (13.13) we conclude

$$\begin{aligned}C(n)\, \mathcal{L}^n(\{0 < w_k < \frac{1}{2}\} \cap B(0,1)) \geq& (\mathcal{L}^n(\{w_k \leq 0\}))^2 \left(\mathcal{L}^n(\{w_k \geq \frac{1}{2}\}) \cap B(0,1)\right)^2\\ \geq& \mu\ (\mathcal{L}^n(\{w_{k+1} \geq 0\} \cap B(0,1))^2\end{aligned}$$

Since $0 \leq w_{k+1} \leq 1$ we have

$$\mathcal{L}^n(\{w_{k+1} \geq 0\} \cap B(0,1)) \geq \int_{B(0,1)} (w_{k+1})^2.$$

so that we have

$$\int_{B(0,1)} (w_{k+1})^2 \leq \frac{C(n)}{\mu}\, \mathcal{L}^n(\{0 < w_k < \frac{1}{2}\} \cap B(0,1)). \tag{13.18}$$

Fix now some $\delta > 0$ (to be specified later). We claim that there exists a number k_0 (depending on δ, n, Λ, λ, but otherwise independent) so that for some $\bar{k} \in \{1,\ldots,k_0\}$ we have

$$\int_{B(0,1)} (w_{\bar{k}})^2 < \delta^2.$$

Indeed if we have

$$\int_{B(0,1)} (w_k)^2 \geq \delta^2 \quad \text{for all } k = 1,\ldots,k_0$$

we conclude from (13.18) that

$$\frac{\mu}{C(n)}\, \delta^2 \leq \mathcal{L}^n(\{0 < w_k < \frac{1}{2}\} \cap B(0,1)) \quad \text{for all } k = 1,\ldots,k_0$$

But by (13.15) we have disjointness, so

$$k_0 \frac{\mu}{C(n)}\, \delta^2 \leq \sum_{k=1}^{k_0} \mathcal{L}^n(\{0 < w_k < \frac{1}{2}\} \cap B(0,1)) \overset{(13.15)}{\leq} \mathcal{L}^n(B(0,1)).$$

This leads to a contradiction if

$$k_0 := \mathcal{L}^n(B(0,1)) \frac{C(n)}{\mu\, \delta^2}$$

So there must be some $\bar{k} \in \{1,\ldots,k_0\}$ such that

$$\int_{B(0,1)} (w_{\bar{k}})^2 < \delta^2.$$

Combining this with (13.16) we have

$$\|w_{\bar{k}}\|_{L^\infty(B(0,1/2))} \leq C(n,\Lambda,\lambda)\, \|w_{\bar{k}}\|_{L^2(B(0,1/2))} \lesssim \delta.$$

So if we choose δ small enough, we can ensure that there exists some $\bar{k} \in \{1,\ldots,k_0\}$ (where k_0 is a constant depending only on permissible data), such that

$$\|w_{\bar{k}}\|_{L^\infty(B(0,1/2))} \leq \frac{1}{2}$$

But then for $x \in B(0,1/2)$

$$\left(\tilde{v}(x) - 1 + 2^{-\bar{k}}\right)_+ = 2^{-\bar{k}} w_{\bar{k}} < 2^{-1-\bar{k}}.$$

and thus

$$\tilde{v}_+(x) \leq 2^{-1-\bar{k}} + 1 - 2^{-\bar{k}} = 1 - \left(2^{-\bar{k}} - 2^{-1-\bar{k}}\right).$$

Setting

$$\gamma := \min_{k=1,\ldots,k_0} \left(2^{-k} - 2^{-1-k}\right) > 0$$

(and observe once more that k_0 only depends on the data) we conclude

$$2\left(\partial_{x_i} u - \frac{1}{2}\right) \leq \tilde{v}_+ \leq 1-\gamma$$

that is

$$\left(\partial_{x_i} u - \frac{1}{2}\right) \leq \frac{2-\gamma}{2} =: \theta$$

We can conclude. □

13.4. **Comments on extensions and miscellaneous results.**

- A parabolic version of the above $C^{1,\alpha}$-argument has been worked out in [Adimurthi and Banerjee, 20 More precisely, the presentation above is the elliptic version of their parabolic argument, as explained to me by Adimurthi.
- For suitably nice f, one can also obtain results for the inhomogeneous version

  $$\operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) = f.$$

  For the current state of the art we refer to e.g. in [Kuusi and Mingione, 2018] and references therein.
- If $\nabla u(x_0) \neq 0$ then the p-Laplace equation becomes uniformly elliptic, and u is even analytic around x_0. So $u \notin C^{1,\alpha}$ implies that $\nabla u(x_0) = 0$. Lewis [Lewis, 1980] showed that *in 2 space dimensions* if $u \in C^k$ and $\nabla u(x_0) = 0$ then $\nabla^\ell u(x_0) = 0$ for all $k \in \{1,\ldots,k\}$. In particular if u is analytic around x_0 then $u \equiv 0$. The case of arbitrary dimension is an interesting open problem. On the other hand, there are examples of nonconstant C^k-solutions of the p-Laplace equation that have a vanishing gradient at a certain point x_0.

- In Weyl's lemma Theorem 2.40 we showed that even if $u \in L^1_{loc}$ and in distributional sense solves $\Delta u = 0$ then $u \in C^\infty$.

  If $p \neq 2$ then $\operatorname{div}(|\nabla u|^{p-2} \nabla u) = 0$ makes distributionally sense for $u \in W^{1,p-1}$. However a p-Laplace version of Weyl's lemma fails, as was recently shown by Colombo-Tione [Colombo and Tione, 2022] disproving the so-called Iwaniec conjecture.

Email address: `armin@pitt.edu`