

A Simple Quantum Oscillator

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Supplement to “Time Travels with Schrödinger’s Cat”

Einstein for Everyone

[http://www.pitt.edu/~jdnorton/teaching/HPS_0410/chapters/time_travel_cat/
time_travel_cat.html](http://www.pitt.edu/~jdnorton/teaching/HPS_0410/chapters/time_travel_cat/time_travel_cat.html)

1. Setting Up

The spacetime is a Minkowski spacetime rolled up in the timelike direction, called here a “cylinder” universe. Hence, locally, quantum systems behave as they would in an ordinary Minkowski spacetime. The calculations below do not consider the further global constraints required by the rolling up of the spacetime. It is assumed that the mass in the system is so slight as not to cause appreciable deviations from flatness of the spacetime.

The oscillator has two states, $|green\rangle$ and $|red\rangle$ that are the basis vectors of a two dimensional Hilbert space. The general quantum state is

$$|\psi(t)\rangle = r(t) |red\rangle + g(t) |green\rangle$$

Schrödinger’s equation is

$$i\hbar \frac{\partial}{\partial t} |\psi(t)\rangle = H |\psi(t)\rangle$$

where all the energy of the Hamiltonian H is located in the interaction term.

$$H = iE(|red\rangle\langle green| - |green\rangle\langle red|)$$

This simplification reflects the fact that our sole interest is the oscillations between the two basis states over time.

2. Solving Schrödinger’s Equation

The Schrödinger equation is easily solved if we write the state as a column vector and the Hamiltonian as a matrix:

$$i\hbar \frac{\partial}{\partial t} \begin{bmatrix} r(t) \\ g(t) \end{bmatrix} = \begin{bmatrix} 0 & iE \\ -iE & 0 \end{bmatrix} \begin{bmatrix} r(t) \\ g(t) \end{bmatrix}$$

Multiplying out the matrix, we arrive at

$$\frac{dr(t)}{dt} = \frac{E}{\hbar} g \quad \frac{dg(t)}{dt} = -\frac{E}{\hbar} r$$

Taking the second derivative of the first equation and substituting with the second, we have

$$\frac{d^2 r(t)}{dt^2} = -\frac{E}{\hbar} \frac{dg}{dt} = -\left(\frac{E}{\hbar}\right)^2 r(t)$$

Treating $g(t)$ similarly, we arrive at two second order differential equation:

$$\frac{d^2 r(t)}{dt^2} + \left(\frac{E}{\hbar}\right)^2 r(t) = 0, \quad \frac{d^2 g(t)}{dt^2} + \left(\frac{E}{\hbar}\right)^2 g(t) = 0 \quad \text{coupled by} \quad \frac{dr(t)}{dt} = \frac{E}{\hbar} g$$

If we set as the initial conditions that

$$|\psi(t)\rangle = |\text{green}\rangle \quad \text{so that} \quad r(0) = 0 \quad \text{and} \quad g(0) = 1$$

we find the unique solution is

$$|\psi(t)\rangle = \sin\left(\frac{E}{\hbar}t\right) |\text{red}\rangle + \cos\left(\frac{E}{\hbar}t\right) |\text{green}\rangle$$

Thus the oscillator completes one cycle in the time T given by

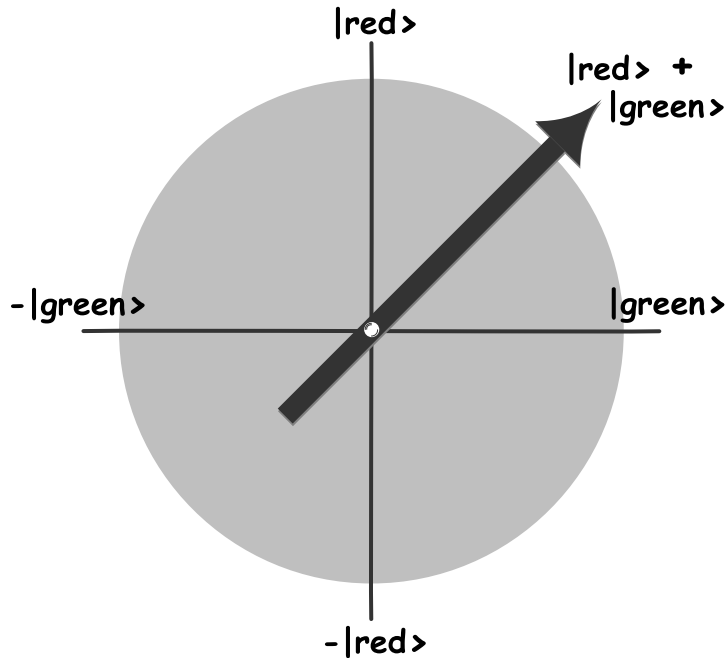
$$\left(\frac{E}{\hbar}\right)T = 2\pi \quad \text{which is equivalent to} \quad T = 2\pi\left(\frac{\hbar}{E}\right) = \frac{h}{E}.$$

3. Time Evolution between States

From this solution of the Schrödinger equation, we can read off the times needed for the transitions that interest us. The system evolves from $|\text{green}\rangle$ to $|\text{red}\rangle$ in passing from $t=0$ to $t=T/4$:

$$|\text{green}\rangle \rightarrow |\text{red}\rangle \quad \text{in time} \quad T/4$$

One might think that it will take the same time to evolve back from $|\text{red}\rangle$ to $|\text{green}\rangle$. However the solution shows that this transition occurs from $t=T/4$ to $t=T$. That is, it requires time $3T/4$. The reason is that, in elapsed time $T/4$, the state $|\text{red}\rangle$ does not evolve into $|\text{green}\rangle$, but into $-|\text{green}\rangle$. That is, in words “minus $|\text{green}\rangle$.” That this but happen is made clear by a geometric picture of the state space:



The time evolution of the system consists simply in the rotation of the state, represented by the arrow, over the four quadrants of the circle. As t increases, the evolution is

$$|\text{green}\rangle \rightarrow |\text{red}\rangle \rightarrow -|\text{green}\rangle \rightarrow -|\text{red}\rangle \rightarrow |\text{green}\rangle$$

The figure also makes clear something that is also recoverable from the solution of the Schrödinger equation: the superposition $|\text{red}\rangle + |\text{green}\rangle$ returns to its state $|\text{red}\rangle + |\text{green}\rangle$ after time T . Collecting all these results we have

$$|\text{green}\rangle \rightarrow |\text{red}\rangle \text{ in time } T/4$$

$$|\text{red}\rangle \rightarrow |\text{green}\rangle \text{ in time } 3T/4$$

$$|\text{red}\rangle + |\text{green}\rangle \rightarrow |\text{red}\rangle + |\text{green}\rangle \text{ in time } T$$

4. Energy Eigenstates

While this completes the calculations needed to support the main text, for completeness, the energy eigenstates are given here. They are found by solving the matrix equation

$$\begin{bmatrix} 0 & iE \\ -iE & 0 \end{bmatrix} \begin{bmatrix} r \\ g \end{bmatrix} = \lambda \begin{bmatrix} r \\ g \end{bmatrix}$$

for an eigenvalue λ . With a little algebra it is easy to show that there are just two eigenvalues:

$$\lambda = E, \text{ whenever } g = -ir; \text{ and}$$

$$\lambda = -E, \text{ whenever } g = ir.$$

Hence the two energy eigenvectors are, up to a constant of unit norm:

$$\psi_E(t) = \frac{1}{\sqrt{2}} \exp\left(-\frac{iEt}{\hbar}\right) (|red\rangle + i|green\rangle)$$

$$\psi_{-E}(t) = \frac{1}{\sqrt{2}} \exp\left(+\frac{iEt}{\hbar}\right) (|red\rangle - i|green\rangle)$$

Their time dependence has been indicated by adding a time dependent phase factor of unit norm.

The factor of $1/\sqrt{2}$ has been included to normalize the vectors to unity.