Review Problems (II)

1. Let

\[ S = \left\{ \frac{n - m}{n + m} : n, m \in \mathbb{N} \right\}. \]

Find sup\( S \) and justify your answer.

2. Let \( x, y, t \in \mathbb{R} \). Suppose that \( x < y \) and \( t > 0 \). Show that there exists an irrational number \( z \) such that

\[ x < \frac{z}{t} < y. \]

3. (i) Use the definition of the limit of a sequence to prove that

\[ \lim_{n \to \infty} \left( \frac{n + 1}{2n + 1} \right) = \frac{1}{2}. \]

(ii) Prove that \( \left( \frac{2 + (-1)^n n}{2 + n} \right) \) is divergent.

4. Let \( x, y \) be two positive real numbers. Suppose that \( x > 1 \). Prove that there exists an \( m \in \mathbb{N} \) such that \( x^m > y \).

5. Let \( S \) be a nonempty subset of \( \mathbb{R} \). Determine whether the given statement is true or false.

   (i) If sup\( S \) exists and \( u = \sup S \), then \( u \in S \).

   (ii) If inf\( S \) exists and \( v = \inf S \), then for any \( n \in \mathbb{N} \) there exists an \( s_n \in S \) such that

   \[ v < s_n + \frac{1}{n^2}. \]

   (iii) If both sup\( S \) and inf\( S \) exist and sup\( S = \inf S \), then \( S \) contains exactly one element.

   (iv) If \( x \in \mathbb{R} \) and \( n \in \mathbb{N} \), then there exists an irrational number \( \xi \) such that \( |x - \xi| < \frac{1}{n^2} \).

   (v) Let \( I_n = [a_n, b_n], n \in \mathbb{N} \), be a nested sequence of closed bounded intervals. Suppose that \( b_n - a_n > 0 \) for each \( n \in \mathbb{N} \). Then \( \bigcap_{n=1}^{\infty} I_n \) contains more than one element.