Problem 1. Steady state values for two countries with different savings rates and population growth rates.

To make the problem more general, begin with a common exponent on the production function = alpha or $\alpha$

Production function: $y = k^{\alpha}$ so that the capital/output ratio: $k/y = k / k^{\alpha} = k^{1-\alpha}$

where $y = Y / LE$ and $k = K/LE$

We will refer to the equations derived below in other problems, so be sure you understand them.

Apply the percentage change rule to the definition of $k$ to get:

%. $\Delta k = % \Delta K - % \Delta L - % \Delta E$ or since $% \Delta L = n$ and $% \Delta E = g$ by definition:

Eq. 1 A $% \Delta k = % \Delta K - n - g$

Next, note that $\Delta K = s Y - \delta K$ or dividing through by $K$:

$\Delta K / K = s Y / K - \delta$

Since $\Delta K / K = % \Delta K$, we can rewrite Eq 1 A as:

$% \Delta k = s Y / K - \delta - n - g$

and as $Y/K = Y/LE / K/LE = y/k$ we have:

Eq. 1 B $% \Delta k = s y/k - \delta - n - g$

The definition of the steady state in the extended Solow model is that the **capital stock per efficiency unit of labor** does not change, or that $% \Delta k = 0$. Hence:

$s y/k - \delta - n - g = 0$ or in the steady state the capital/output ratio will be:

**Capital/output ratio**: $k/y = s / (\delta + n + g)$ or $k^{1-\alpha} = s / (\delta + n + g)$

Solving the problem is now a matter of substitution. For the developed country,

$k^{1-\alpha} = s / (\delta + n + g) = .28 / (.01 + .02 + .04) = .28 / .07 = 4$

for the underdeveloped country:

$k^{1-\alpha} = s / (\delta + n + g) = .10 / (.04 + .02 + .04) = .10 / .10 = 1$

If alpha = 0.25 (to make things more interesting than the text), we would find that for the developed country, $k = 4$ to the $1 / 1 - .25$ power or 4 to the 4/3 power = 6.3496 and $y = (pow 6.3496 - 0.25) = 1.5874$.

[for the text problem, with alpha = 1/2, you should find k = 16 and y = 4 in the steady state]

For the underdeveloped country, $k = 1$ and $y = 1$ no matter what the value of alpha.

Note that the underdeveloped country does not seem to have any problems that affect overall efficiency (since the production function is the same), so their choices seem limited to giving incentives for saving, attracting investment from abroad, and population policies. Nothing can be done about technical progress in the Solow model, where it is simply taken as exogenous, and nothing can be done about depreciation in any model.
Problem 2. US savings rate and the Golden Rule

Data: Capital share of GDP = rK/Y = alpha = .30
% Δ y = g = .03 (3 percent per year growth rate; in steady state, % Δ y = g)
k/y = 2.5 (capital-output ratio)
δ = .04 (depreciation rate)
n = .01 (not given, but assumed and approximately true).

a. If the US is in a steady state, as assumed by the question:

COR: \[ \frac{k}{y} = \frac{s}{(n + g + \delta)} \] [See problem 1, the equation labeled "Capital/output ratio"]

so \[ 2.5 = \frac{s}{(.01 + .03 + .04)} = \frac{s}{.08} \]

and \[ s = .08 \times 2.5 = 0.20 \]

So in the steady state, \[ k = (\text{pow } 2.5 \ 1/0.7) = 3.7024 \] \[ y = (\text{pow } 3.7024 \ 0.3) = 1.4810 \]

b. Marginal product of capital = \( \frac{dy}{dk} = \frac{dk}{dk} \alpha / dk = \alpha \frac{y}{k} = .3 / 2.5 \) or MPK = 0.12.

Note the relation between MPK = \( \alpha \frac{y}{k} \) (always) and the COR equation above (for the steady state) yields: \( \text{MPK} = \alpha (n + g + \delta) / s = \alpha / \text{COR in the steady state} \). This is used in part d.

c. The MPK which will maximize consumption satisfies the relation: \( \text{MPK} = (n + g + \delta) = .08 \)
   For the logic behind this, see p. 199-200 and p.208-9.
   Basically, since the steady state consumption is \( c^* = \hat{f}(k^*) - (n + g + \delta) k^* \),
   we maximize consumption by taking \( dc^* / dk^* = \text{MPK} - (n + g + \delta) \) and setting equal to zero.

   Since MPK with the initial values is greater than .08, we have too little capital (note that more capital will drive the MPK down from the initial value of 0.12)

d. The appropriate value of the savings rate will satisfy: \( \text{MPK} = .08 = \frac{.3}{\text{COR}*} \) so that the appropriate \( \text{COR}* \) will be \( .3 / .08 = 3.75 \)

e. Since \( \text{COR}* = 3.75 = s / (n + g + \delta) \) from the COR equation above,
   we have: \[ 3.75 = s / .08 \text{ or } s = .08 \times 3.75 = 0.3 \text{ will get us to Golden Rule Savings.} \]

This was a rather indirect route to find that the optimal savings rate is equal to alpha.
Problem 3. Prove the following propositions for the Solow model with population growth at rate \( n \), and technical progress at rate \( g \).

a. The capital-output ratio is constant. Note that this means \( K/Y \) is constant, not just \( k/y \).
We showed that \( k/y = s / (n + g + \delta) \) in the last problem (equation COR) and all of \( s, n, g \) and \( \delta \) were constant.
But does the constancy of \( k/y \) imply the constancy of \( K/Y \) ?

This is not obvious, but \( k \times LE = K \) and \( y \times LE = Y \), so if we take the constant \( k/y \) and multiply by \( LE / LE \) (obviously equal to the constant value of 1), we get \( K/Y \) which must also be a constant.

b. Capital and labor each earn a constant share of the economy's income.
Note that strictly the capital share of income is defined as \( rK / Y \), not \( rk / y \). The point is important in the next part of this question.

Since in a competitive market the return to capital is its marginal product, we have:

\[
\begin{align*}
  r &= MPK = ak^{\alpha-1} = \frac{ak^\alpha}{k} \\
  rk &= \alpha k \\
  rk &= \alpha y \quad \text{or, multiplying both sides by LE, } rK = \alpha Y \\
  rk / y &= \alpha \quad \text{But } rk / y = rK / Y \text{ is the capital share of income; since } \alpha \text{ is constant, the capital share income is constant.}
\end{align*}
\]

If the capital share of income is constant, the labor share of income must also be constant in a model with only two factors of production. \( wL = (1 - \alpha) Y \) follows from \( rK + wL = Y \)

\[
\begin{align*}
  % \Delta rK &= % \Delta \alpha + % \Delta Y \\
  % \Delta rK &= % \Delta Y = n + g
\end{align*}
\]

Also, clearly, \( % \Delta wL = % \Delta (1 - \alpha) + % \Delta Y = % \Delta Y = n + g, \) since \( (1 - \alpha) \) is constant.

c. Total capital and labor income both grow at the rate \( n + g \).
Note that while \( rK / Y \) is the capital share of income, \( rK = \alpha Y \) is the total capital income.
We can apply the percentage change relationship to find that

\[
\begin{align*}
  % \Delta rK &= % \Delta \alpha + % \Delta Y \\
  % \Delta rK &= % \Delta Y = n + g
\end{align*}
\]

Also, clearly, \( % \Delta wL = % \Delta (1 - \alpha) + % \Delta Y = % \Delta Y = n + g, \) since \( (1 - \alpha) \) is constant.

d. Part c might lead you to think that both \( r \) and \( w \) will also be growing at the rate \( n + g \).
This does not follow; in fact, from part (b) we note that \( rk = \alpha y \) so \( r^* = \alpha y^* / k^* \) (the stars referring to steady-state variables)
and in percentage change form, since alpha, \( y^* \) and \( k^* \) are all constant:

\[
% \Delta r^* = % \Delta \alpha + % \Delta y^* - % \Delta k^* = 0
\]

The steady state rate of return to capital is constant.

For labor, since \( wL = (1 - \alpha) Y \) we must also have \( % \Delta w + % \Delta L = % \Delta (1 - \alpha) + % \Delta Y, \) so

\[
% \Delta w + n = n + g \quad \text{or} \quad % \Delta w = g
\]

The growth rate of real wages does keep pace with the rate of technical progress.

(Note: this does not mean that capitalism is doomed to disappear: although the rate of profit per unit of capital does not increase, the number of units of capital increases along with output, and we found above that capitalists as a whole will keep the same share of the economy's income in Solow's model).
Problem 4. Richland and Poorland and relative efficiency.

In this problem, Mankiw takes the "technical progress" parameter as total factor productivity rather than as purely labor-enhancing, and writes the production function as:

\[ Y = \alpha K^\alpha L^{1-\alpha} \]

The production function can be rewritten, if we define \( E = A \) to the \( 1/(1-a) \) power, as:

\[ Y = K^\alpha (EL)^{1-\alpha} \]

and it will be more convenient to use this form in what follows.

The production function per effective unit of labor is:

\[ y = k^\alpha \]

as usual, and we can also use the result from problem 1 that \( k/y = s / (n + g + d) \)

In Richland, we have:

\[ \frac{y}{k} = \frac{.32}{(.01 + .02 + .05)} = \frac{.32}{.08} = 4 \]

and since \( y = k^\alpha \), we have \( k/y = k \) to the \( 1-a \) power, and hence (using \( k^* \) to mean steady-state capital per effective unit of labor)

\[ k^* = 4 \] to the \( 1/(1-a) \) power,

and finally:

\[ y^* = 4 \] to the \( a/(1-a) \) power. If \( a = .3 \), this means that \( a / (1-a) = 3/7 \),

and hence \( y^* = 1.814 \)

In Poorland, we have \( k/y = .10 / (.03 + .02 + .05) = .10/.10 = 1 \), and so:

\[ k^* = 1 \] to the \( 1/a \) power and hence \( k^* = 1 \) whatever the value of \( a \).

Likewise,

\[ y^* = k^* \] to the \( a/(1-a) \) power, and hence \( y^* = 1 \) whatever the value of \( a \).

If this seems arbitrary, note that we can index all values to Richland to the values of Poorland:

if I index all grades on the next exam to 50 points, and student A gets 50, his score will be 1.00 and student B who received 30 points will get a grade of 30/50 = 0.60.

So we expect Richland to be nearly twice as rich as Poorland with \( a = .3 \) in Richland.

Problem: Check the PWT 6.2 variable \( Y = GDP \) relative to the US.

(stats \( Y \)) will tell you that 25 percent of the values of \( Y \) are below 7.26; or the US is more than 14 times as rich as the bottom quarter of all observations.

\( y.chn, y.bgd, y.hnd, y.phl, y.zar, y.cmr \) will show the time track for India, China, Bangladesh, Honduras, Philippines, Zaire (Congo), and Cameroon.

Can the alpha for the US production function be high enough to give an answer of \( y^* = 14 \) for the US? We need to solve the equation \( 4 \) to the \( a / (1-a) \) power = 14. We can solve this sort of equation by taking logarithms: \( (a / (1-a)) \ln(4) = \ln(14) \) or \( a / (1-a) = \ln(14) / \ln(4) = 2.6391 / 1.3863 = 1.9037 \)

Since \( a/(1-a) = 1.9 \), we have \( a = 1.9 - 1.9a \) or \( 2.9a = 1.9 \) or \( a = 1.9 / 2.9 = 0.6552 \).

There is no logical objection to alpha taking on the value of 0.655, but remember that this should be the capital share of income -- and in the US, this is more like 1/3 than .655. So unless there is something badly wrong with the basics of the Solow theory, this is not likely. We do therefore need to allow for differences in the efficiency with which each country uses available technology. The difference may be due to the quality of institutions, to policies such as trade-restricting policies, or to quality of education.
Problem 6. Endogenous Growth.  [See text, pages 235-238]

The key feature is a distinction between knowledge-intensive industries which do NOT face diminishing returns. Knowledge is not a "rival" good, but usable by many at the same time; and more knowledge can lead to sharply increasing returns by raising the likelihood of scientific inventions or practical innovations applying new technology.

The manufacturing sector remains traditional:
$$Y = K^α (E (1 - u) L)^{1-α} = K^α (E W)^{1-α}$$

where $(1 - u)$ is the share of the labor force actually working in traditional manufacturing.

The "u" in this model does NOT represent "unemployment" but rather can be thought of as "university" -- the share of the work force in knowledge intensive industries.

"University" is in quotation marks because industrial research labs count as well; Google and Microsoft are certainly in the knowledge intensive sector.

The "product" of the "university" sector is an increase in knowledge or $Δ E$
(either through research or teaching future workers).

The production function for the "university" sector is:
$$Δ E = g(uL) E$$

so knowledge will grow at the rate $g(uL) = Δ E / E$.

Note that the main input in increasing knowledge is knowledge: $E$ produces $Δ E$.

Of course, we also must have researchers and teachers, the "u" variable, whose productivity depends on the function $g(u)$ -- which might be a simple numerical value as the productivity coefficient in the Ricardian model, in which case we could write
$$Δ E = g(uL) = 5 uL * E$$
or might itself show diminishing returns, so the production function would be:
$$Δ E = (pow uL 0.5) * E$$

But even if so, the production function would still show no diminishing returns to knowledge itself.

The important point for this problem is that a higher proportion of workers in the "university" sector means a higher multiplier for knowledge.

The model looks exactly like the Solow model on the surface, once we define:

$$k = K / [(1-u) LE] \text{ and } y = Y / [(1-u) LE]$$

The equilibrium condition will be:

$$% Δ k = % Δ K - % Δ (1 - u) - % Δ L - % Δ E$$

or:

$$% Δ k = s y/k - δ - n - g (u)$$

since $% Δ K = sY - δ K$ and division by $K$ gets us to the result desired.

Note that $% Δ (1 - u) = 0$ since $u$ will be treated as an exogenous variable here.

It would of course be better to "endogenize" $u$, and much of the recent work in growth theory (see Jones in particular) tries to do this.

We are left with $k^{1- u} = s / (δ + n + g (u))$. An increase in $u$ will clearly directly reduces production in the manufacturing sector, and so would decrease living standards in the short run, but will mean a faster growth rate in the long run. The tradeoff is much like that involved in the simple model when the savings rate increases.
Study Guide for Mankiw, *Macroeconomics*, Chapters 7 and 8
Intermediate Macroeconomics -- Dr. McGahagan
Spring 2008

**Basic objective of these chapters:**
-- Master the basic Solow growth model and its extensions to population growth, productivity growth, and endogenous growth.
-- Be aware of the empirical data which can be brought to bear on the model, how well the Solow model fits the data, and the revisions that have been suggested for a better fit.

**Model details:**

**Cobb-Douglas production function.** How does it show diminishing returns to a factor, constant returns to scale. How do you show it into a per capita production function? How do you represent total factor productivity? labor-enhancing productivity?

**Equilibrium condition:** percent change in \(k = 0\).

How does this apply in the basic model and in the extended models (pop. growth, productivity growth, endogenous growth)? Be able to start from this and derive the relation between the capital output ratio \(k/y\) and \(s, d, n, g\) expressed in the formula \(y/k = s / (n+g+d)\)

See text problem 1 of chapter 8 for an example.

Be able to show the model graphically, and explain the meaning of intersections and slopes.

What are the **dynamics of out-of-equilibrium growth**? What happens if the capital stock begins at half the steady state value? What happens if the savings rate suddenly increases? Be able to present a time-series graph of out-of-equilibrium behavior similar to those on pages 204-5 (or from the computer program).

What is the **Golden Rule**? How can we derive the condition \(MPK = d + g + n\)?

**Empirics:**

What does it mean to say that the Solow model predicts "balanced growth"? How does this compare to Marx's predictions of the effects of economic growth?

What is **convergence** and why should we accept it on the basis of the Solow model? What is the actual evidence on convergence from the Penn World Tables? Does it force us to extend the model beyond capital and labor as the only factors of production?

Be able to discuss the evidence presented by Mankiw-Romer-Weil and Jeffrey Sachs.

Is there evidence for efficiency differences? (see chapter 8, problem 4)

What is the role of human capital, international trade, institutions?

**Names:** Identify the following:
- Robert Solow
- Daron Acemoglu
- Joseph Schumpeter
- Thomas Malthus
- Dani Rodrik
- Paul Romer
- Michael Kremer
- Andrei Shleifer
- Robert Lucas
- Jeffrey Sachs
- David Warsh

**Problems:** Chapter 7: Problems 1,3,5,8
Chapter 8: Problems 1,3,4,6  Don't forget to look over the Questions for Review

**Endogenous Growth:**

What are the special characteristics of knowledge as a good? Mention "rivalry", "excludability" and "externalities" in your discussion.

What is the AK model and what are its implications? Is there a "steady-state"?

What is the two-sector model of endogenous growth, and what are the differences and similarities with the Solow model?

What is the distinction between innovation and invention? What is "creative destruction"?

**Growth accounting:** Don't forget the appendix to chapter 8, and the problems in the appendix.