Asset Markets in General Equilibrium

Econ 2100 Fall 2015

Lecture 25, December 2

Outline

1. Assets
2. Asset Markets and Equilibrium
3. Linear Pricing
4. Complete Markets
5. Span
The objective is to explicitly model assets (rather than state contingent commodities), in a setup similar to Radner’s. Assets distribute amounts of money; we adopt the convention that good 1 is the unit of accounts.

**Definition**
An asset is a title to receive $r_s$ units of good 1 at date 1 if and only if state $s$ occurs.

An asset is completely characterized by its return vector

$$r = (r_1, \ldots, r_S) \in \mathbb{R}^S$$

$r_s$ is the dividend paid to the holder of a unit of $r$ if and only if state $s$ occurs.

**Definition**
The return matrix $R$ is an $S \times K$ matrix whose $k$th column is the return vector of asset $k$. That is:

$$R = \begin{bmatrix}
  r_{11} & \ldots & r_{k1} & \ldots & r_{K1} \\
  \vdots & \ddots & \vdots & \ddots & \vdots \\
  r_{1s} & \ldots & r_{ks} & \ldots & r_{KS} \\
  \vdots & \ddots & \vdots & \ddots & \vdots \\
  r_{1S} & \ldots & r_{kS} & \ldots & r_{KS}
\end{bmatrix}$$

A row indicates the returns of all assets in one particular state.
Assets: Examples

Example
An asset that delivers one unit of good 1 in all states:

\[ r = (1, \ldots, 1) \]

If there is only one good, \( L = 1 \), this the risk-free (or safe) asset.

- Why is this not safe with many goods?
  - Because the price of good 1 can change from state to state.
    - An asset matters insofar as it can be transformed into consumption goods; the rate at which one can do this depends on relative prices.

Example
An asset that delivers one unit of good 1 in one state, and zero otherwise:

\[ r = (0, \ldots, 1, \ldots, 0) \]

This is an Arrow security.
Derivative Assets: Example

- Assets whose returns are defined in terms of other assets are called derivatives.
- These are very common in financial markets.

**Example**

A European Call Option on asset $r$ at strike price $c$ gives its holder the right to buy, after the state is revealed but before the returns on the asset are paid, one unit of asset $r$ at price $c$.

- What is the return vector of this European call option?
- The option will be exercised only if $r_s > c$ since in the opposite case one loses money (equality does not matter);
- hence
  
  $r(c) = (\max \{0, r_1 - c\}, \ldots, \max \{0, r_s - c\})$

- If $r = (1, 2, 3, 4)$, then
  
  $r(1.5) = (0, 0.5, 1.5, 2.5)$

  $r(2) = (0, 0, 1, 2)$

  $r(3) = (0, 0, 0, 1)$
Budget Constraints with Asset Markets

Given an asset matrix $R$, one can define prices and holdings of each asset.

- $q = (q_1, \ldots, q_K) \in \mathbb{R}^K$ are the asset prices, where $q_k$ is the price of asset $k$.
- $z_i = (z_{1i}, \ldots, z_{Ki}) \in \mathbb{R}^K$ are consumer $i$’s holdings of each asset.
  - This is called a **portfolio**: it shows how many units of each asset $i$ owns.

Assets are traded at time 0, while returns are realized at time 1. At that time, agents decide how much to consume and they trade on spot markets.

- As in Radner, $i$’s budget constraints are

  $$\sum_{k=1}^{K} q_k z_{ki} \leq 0 \quad \text{and} \quad p_s \cdot x_{si} \leq p_s \cdot \omega_{si} + \sum_{k=1}^{K} p_{1s} z_{ki} r_{sk} \quad \text{for each } s$$

- Income at time 1 is given by the value of endowment plus the income one obtains by selling the returns of the assets one owns.
  - As usual, one can normalize the spot price of good one to be 1.
Question

What is the difference between these budget constraints

\[
\sum_{k=1}^{K} q_k z_{ki} \leq 0 \quad \text{and} \quad p_s \cdot x_{si} \leq p_s \cdot \omega_{si} + \sum_{k=1}^{K} p_{1s} z_{ki} r_{sk} \quad \text{for each } s
\]

and the ones in a Radner equilibrium?

- Here, the dividends of the assets are given; one focuses only on the portfolio choice.
- In Radner, the dividends are constructed by the consumers’ choice of trades in the state-contingent commodity.
- Formally, in Radner one implicitly assumes \( S \) different assets, each with returns \( r_s = 1 \) in state \( s \) and zero otherwise.
- If \( S = K \), we can write Radner using the \( z \) and \( r \) above:

\[
Z_s^{Radner} = \sum_{s=1}^{S} Z_s r_s.
\]
What do budget sets look like?

- Using the asset matrix, rewrite $i$’s budget set as:

$$B(p, q, R) = \left\{ x_i \in \mathbb{R}^{L_S}_+ : \begin{array}{c} \text{there is } z_i \in \mathbb{R}^K \
\text{such that} \\
q \cdot z_i \leq 0 \\
\text{and} \\
p \cdot (x_i - \omega_i) \leq R z_i \end{array} \right\}$$

with $p \in \mathbb{R}^{LS}$, and where we normalized $p_{1s} = 1$.

- The second part of the budget set defines $S$ inequalities

$$\begin{pmatrix}
p_1 \cdot (x_1 - \omega_1) \\
\vdots \\
p_s \cdot (x_s - \omega_s) \\
\vdots \\
p_s \cdot (x_{Si} - \omega_{Si})
\end{pmatrix} \leq \begin{pmatrix}
z_1 r_{11} + \ldots + z_{ki} r_{k1} + \ldots + z_{Ki} r_{K1} \\
\vdots \\
z_1 r_{1s} + \ldots + z_{ki} r_{ks} + \ldots + z_{Ki} r_{Ks} \\
\vdots \\
z_1 r_{1S} + \ldots + z_{ki} r_{kS} + \ldots + z_{Ki} r_{KS}
\end{pmatrix}$$

where each $p_s \in \mathbb{R}^L$

- $R z_i$ represents the ‘financial income’ attainable by a consumer who chooses portfolio $z_i$.

- For a fixed $R$, the set of all possible values for $R z_i$ represents the set of all possible incomes available to the consumer. As $R$ changes so does this set.
A Radner equilibrium with asset markets is given by assets prices $q^* \in \mathbb{R}^K$, spot prices $p^*_s \in \mathbb{R}^L$ in each state $s$, portfolios $z^*_i \in \mathbb{R}^K$, and spot market plans $x^*_s \in \mathbb{R}^L$ for each $s$ such that:

1. for each $i$, $z^*_i$ and $x^*_i$ solve

$$\max_{z_i \in \mathbb{R}^K, x_{si} \in \mathbb{R}_+^L} U_i (x_{1i}, \ldots, x_{si})$$

subject to

$$\sum_{k=1}^{K} q^*_k z_{ki} \leq 0 \quad \text{and} \quad p^*_s \cdot x_{si} \leq p^*_s \cdot \omega_{si} + \sum_{k=1}^{K} p^*_1 z_{ki} r_{sk}$$

2. all markets clear; that is:

$$\sum_{i=1}^{I} z^*_k \leq 0 \quad \text{and} \quad \sum_{i=1}^{I} x^*_s \leq \sum_{i=1}^{I} \omega_{si} \quad \text{for all } s \text{ and } k$$

- The definition is familiar, but there are two new objects: the optimal portfolio and the equilibrium asset prices.
- What can one say about them?
Linear Pricing

Main Question
We do we know about the equilibrium asset prices $q^*$?

Proposition
Assume every return vector $r_k$ is nonnegative and non zero ($r_k \geq 0$ and $r_k \neq 0$ for all $k$) and preferences are strictly monotone. Then, for every asset prices $q^* \in \mathbb{R}^K$ which are part of a Radner equilibrium, we can find $S$ non-negative numbers $\mu_s \geq 0$ which satisfy

$$q^*_k = \sum_{s=1}^{S} \mu_s r_{sk} \quad \text{for all } k$$

- In words, equilibrium asset prices are linear in their returns.
- The equilibrium asset prices $q^*$ depend linearly on the asset returns.
- The linear ‘factors’ are the same for all assets: $\mu_s$ does not depend on $k$.
- Two ways to prove this:
  - next, a proof for the differentiable case;
  - a more general proof as an implication of absence of arbitrage (next class).
We need to show that

\[ q_k^* = \sum_{s=1}^{S} \mu_s r_{sk} \quad \text{for all } k \]

**Proof.**

- Let \( z_i^* \in \mathbb{R}^K \) be an optimal portfolio; define 
  \[ w_{si}^* = p_s^* \cdot \omega_{si} + \sum_{k=1}^{K} p_{1s}^* z_{ki}^* r_{sk}. \]

- A utility maximizing portfolio solves
  \[
  \max_{z_1, \ldots, z_K} \sum_{s=1}^{S} \pi_{si} v_{si} \left( p_s^*, p_s^* \cdot \omega_{si} + \sum_{k=1}^{K} p_{1s}^* z_{ki}^* r_{sk} \right) \quad \text{subject to} \quad \sum_{k=1}^{K} q_{k}^* z_{ki} \leq 0
  \]
  where \( v_{si}(\cdot) \) is \( i \)'s indirect utility function in state \( s \) (how does one get this?).

- Since unlimited short sales are possible, an optimal portfolio must be interior and thus solve the following first order conditions (with \( \alpha_i > 0 \)):
  \[
  \sum_{s=1}^{S} \pi_{si} \frac{\partial v_{si} \left( p_s^*, w_{si}^* \right)}{\partial w_{si}} r_{sk} = \alpha_i q_k^*
  \]

- Define
  \[
  \mu_{si} = \frac{\pi_{si}}{\alpha_i} \frac{\partial v_{si} \left( p_s^*, w_{si}^* \right)}{\partial w_{si}}
  \]
  and rearrange so that
  \[
  q_k^* = \sum_{s=1}^{S} \mu_{si} r_{sk}
  \]

- Thus we define the \( \mu_s \) by picking some consumer \( i \) and letting \( \mu_s = \mu_{si} \).
Linear Pricing At Work

- Linear pricing can have stark implications for the prices of derivative assets.

**Pricing Linear Combinations of Assets**

Assume the return vectors of all assets are non-negative and non-zero, and assume preferences are strictly monotone.

- Suppose asset 3 satisfies
  \[ r_3 = \gamma_1 r_1 + \gamma_2 r_2. \]

- Then, one can use the linear pricing formula as follows:
  \[
  q_3^* = \sum_{s=1}^{S} \mu_s r_{s3} = \sum_{s=1}^{S} \mu_s (\gamma_1 r_{1s} + \gamma_2 r_{2s}) = \gamma_1 \sum_{s=1}^{S} \mu_s r_{1s} + \gamma_2 \sum_{s=1}^{S} \mu_s r_{2s} = \gamma_1 q_1^* + \gamma_2 q_2^*
  \]

  - where the last equality follows from the previous proposition.

- Therefore equilibrium prices must satisfy
  \[ q_3^* = \gamma_1 q_1^* + \gamma_2 q_2^*. \]

- If assets are linearly dependent, their prices are linearly dependent too.
Linear Pricing

In proving the linear pricing proposition we have shown that

\[ q_k^* = \sum_{s=1}^{S} \mu_{si} r_{sk} = \sum_{s=1}^{S} \pi_{si} \frac{\partial v_{si}(p_s^*, w_{si}^*)}{\partial w_{si}} \frac{\partial w_{si}}{\partial \alpha_i} r_{sk} \]

for all \( k \)

- What are the \( \mu_s \)?
- They are given by the ratio of two numbers.
  - \( \pi_{si} \frac{\partial v_{si}(p_s, w_{si}^*)}{\partial w_{si}} \) measures the expected utility (at time 0) of an extra unit of wealth in state \( s \) (at time 1),
  - \( \alpha_i \) is the Lagrange multiplier that measures the utility of an extra unit of wealth at time 0.
- Since \( \mu_{si} \) differs across consumers, the \( \mu \)'s are in general not uniquely defined.
- This implies equilibrium asset prices are also not uniquely defined in general.
- Next, a restriction that eliminates this multiplicity and that has all sorts of important implications.
Complete Markets

**Definition**
An asset structure $R$ is **complete** if $\text{rank } R = S$.

- Markets are complete if there are at least $S$ assets with linearly independent returns.

**Question 4, Problem Set 13**
Prove that if markets are complete, then the $\mu$’s in the previous proposition are uniquely defined.
Complete Markets: Examples

Example
There are $S$ different Arrow securities; then

$$R = \begin{bmatrix} 1 & 0 & \ldots & 0 \\ 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \ldots & \ldots & 0 & 1 \end{bmatrix}$$

which is the identity matrix and therefore has rank $S$.

Example
There are three states, so $S = 3$, and $R = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$ which has full rank.
Consider the European Call Option on asset $r = (r_1, r_2, r_3, r_4)$ at strike price $c$.

- The return vector of this call option is
  \[
  r(c) = (\max\{0, r_1 - c\}, \ldots, \max\{0, r_4 - c\})
  \]
- Let $r = (4, 3, 2, 1)$, consider three options given by $r(3.5)$, $r(2.5)$, and $r(1.5)$.
- The return matrix given by these four assets
  \[
  R = \begin{bmatrix}
  4 & 0.5 & 1.5 & 2.5 \\
  3 & 0 & 0.5 & 1.5 \\
  2 & 0 & 0 & 0.5 \\
  1 & 0 & 0 & 0 \\
  \end{bmatrix}
  \]
  which has full rank.
- So even if there is only one asset and four states (so that $K < S$ which obviously would imply $\text{rank } R < S$), one can have complete markets by allowing for options to exist.
Remark

The crucial feature of an asset structure is the wealth it can generate.

- This is the set of all incomes that can be achieved constructing some portfolio:
  \[ \text{Range } R = \{ w \in \mathbb{R}^S : w = Rz \text{ for some } z \in \mathbb{R}^K \} \]
- When two return matrices have the same range (i.e. they ‘span’ the same wealth levels) they yield the same equilibrium allocation.

Proposition

Suppose prices \( q^* \in \mathbb{R}^K \) and \( p^* \in \mathbb{R}^{LS} \) and plans \( x^* \in \mathbb{R}^{LSI} \) and \( z^* \in \mathbb{R}^{KI} \) constitute a Radner equilibrium for an asset structure with \( S \times K \) return matrix \( R \). Let \( \hat{R} \) be an \( S \times \hat{K} \) second asset returns structure. If Range \( R = \text{Range } \hat{R} \), then \( x^* \) is also an equilibrium allocation for the economy that has \( \hat{R} \) as its asset matrix.

- The proof has two steps.
  - First, show that the two budget sets corresponding to \( R \) and \( \hat{R} \) are the same.
  - Second, find portfolios \( z \) such that
    \[
    \sum_i z_i = 0 \text{ and } m_i = [p_1^* \cdot (x_{1i}^* - \omega_{1i}), \ldots, p_S^* \cdot (x_{Si}^* - \omega_{Si})]^T = \hat{R}z_i
    \]

- This is question 5 in Problem Set 13
Individual $i$’s budget constraints are:

\[
\sum_{k=1}^{K} q_k z_{ki} \leq 0 \quad \text{and} \quad \sum_{l=1}^{L} p_{ls} x_{lsi} \leq \sum_{l=1}^{L} p_{ls} \omega_{lsi} + p_{1s} \sum_{k=1}^{K} z_{ki} r_{sk} \quad \text{for each } s
\]

Rewrite $i$’s budget set as:

\[
B(p, q, R) = \left\{ x_i \in \mathbb{R}^{Ls} : \right. \begin{array}{l}
x_i \in \mathbb{R}_+^{Ls} : \text{there is } z_i \in \mathbb{R}^K \\
such that \quad q \cdot z_i \leq 0 \quad \text{and} \quad p \cdot (x_i - \omega_i) \leq Rz_i
\end{array} \right\}
\]

where we normalized $p_{1s} = 1$.

The set of income levels that can be achieved using some portfolio is:

\[
\text{Range } R = \left\{ w \in \mathbb{R}^S : w = Rz \text{ for some } z \in \mathbb{R}^K \right\}
\]

This is a linear space.

By the previous proposition, if two return matrices have the same range they ‘span’ the same wealth levels and yield the same equilibrium allocation.
Proposition

Assume the asset markets are complete. Then

1. Suppose consumption plans \( x^* \in \mathbb{R}^{LSI} \) and prices \( p^* \in \mathbb{R}^{LS}_{++} \) constitute an Arrow-Debreu equilibrium. Then, there are asset prices \( q^* \in \mathbb{R}^{K}_{++} \) and portfolio choices \( z^* = (z^*_1, \ldots, z^*_I) \in \mathbb{R}^{KI} \) such that: \( z^* \), \( q^* \), \( x^* \), and spot prices \( p^*_s \in \mathbb{R}^{S}_{++} \) for each \( s \) form a Radner equilibrium.

2. Suppose consumption plans \( x^* \in \mathbb{R}^{LSI} \), portfolio plans \( z^* \in \mathbb{R}^{KI} \) and prices \( q^* \in \mathbb{R}^{S}_{++} \) and spot prices \( p^*_s \in \mathbb{R}^{S}_{++} \) for each \( s \) constitute a Radner equilibrium. Then, there are \( S \) strictly positive numbers \( (\mu_1, \ldots, \mu_S) \in \mathbb{R}^{S}_{++} \) such that the allocation \( x^* \) and the state-contingent commodities price vector \( (\mu_1 p^*_1, \ldots, \mu_S p^*_S) \in \mathbb{R}^{LS} \) form an Arrow Debreu Equilibrium.

When asset markets are complete, agents are effectively unrestricted in the amounts of trades they can make across states. Then, the Radner equilibrium with assets is equivalent to an Arrow-Debreu equilibrium.

Because we can always set \( p_{1s} = 1 \), each multiplier is interpreted as the value at time 0 of a dollar at time 1 in state \( s \).

Prove this as Question 6 in Problem Set 13.
Problem Set 13 (the last one!)

Due 7 December, Monday, at the beginning of class

1. Consider the following three consumers (no production), one good economy with two periods (0 and 1) and two states (s and t). Each consumer’s utility is the sum of utility in period 0 and expected utility in period 1; the utility function is \( \ln(x) \), where \( x \) is the amount consumed of the good, in all periods and states. Consumers all agree that both future states are equally likely. The initial endowments of the three consumers are

\[
\omega_1 = (0, 4, 0) \quad \omega_2 = (0, 0, 4) \quad \omega_3 = (4, 0, 0)
\]

where \( \omega_i = (\omega_{0i}, \omega_{1si}, \omega_{1ti}) \). Define an Arrow-Debreu equilibrium and a Radner equilibrium for this two period economy, and find both normalizing the price of the good in period 0 to be 1.

2. Consider the following pure exchange economy with only one commodity and two consumers, Alice and Bob. There are three dates, \( t = 0, 1, 2 \), and there are four states of nature, \( S = \{ s_1, s_2, s_3, s_4 \} \). Time evolves along a tree so that at date 1 consumers know that either the possible states are \( s_1 \) and \( s_2 \), or the possible states are \( s_3 \) and \( s_4 \). Both consumers are expected utility maximizers who think the 4 states are equally likely, and their von Neumann-Morgenstern utility function is

\[
U(x_0, x_1, x_2) = \log(x_0) + \log(x_1) + \log(x_2),
\]

where \( x_t \) denotes the amount consumed of the one good at date \( t \). Both consumers have endowments of 1 unit of the good at date 0, and 9 units of the good at date 2; but their endowments are different at date 1: A is endowed with 2 units of the good in when states \( s_1 \) and \( s_2 \) are possible and 6 units when the possible states are \( s_3 \) and \( s_4 \); B is endowed with 6 units of the good when states \( s_1 \) and \( s_2 \) are possible and 2 units when the possible states are \( s_3 \) and \( s_4 \).

(a) Define an Arrow-Debreu equilibrium for this economy; make sure you carefully write down the information structure, the corresponding date-events, as well as the measurability conditions necessary for your definition to work.

(b) Find this equilibrium, and argue that it is the only one.

Now consider an economy with 2 financial securities and dynamic trading; each security trades at dates 0 and 1; each pays dividends in each of the four states of the world at date 2. The first pays a dividend of 2 if the state of nature is \( s_1 \), 7 if \( s_2 \), 3 if \( s_3 \), and 5 if \( s_4 \). The second pays, in the same order, 12, 2, 4, and 4.

(c) Define the Radner equilibrium corresponding to this economy with dynamic trading.

3. Consider an economy with two time periods, \( t = 0 \) and \( t = 1 \), where consumer \( i \)'s preference are given by

\[
U_i(x_{0i}, x_{1i}) = u_{0i}(x_{0i}) + \sum_{s=1}^{S} \pi_s u_{1i}(x_{0i})
\]

where \( x_{0i}, x_{1i} \in \mathbb{R}^L \) and each \( u_{ti}(\cdot) \) is differentiable.
(a) Prove that the linear pricing proposition still holds.

(b) Consider the case of only one good \((L = 1)\) and express the \(\mu_s\) in terms of the marginal utilities of consumption.

4. In class, we have shown that if every return vector \(r_k \in \mathbb{R}^S\) is nonnegative and non zero \((r_k \geq 0\) and \(r_k \neq 0\) for all \(k\)) and preferences are strictly monotone. Then, for every asset prices \(q^* \in \mathbb{R}^K\) which are part of a Radner equilibrium, we can find \(S\) non-negative numbers \(\mu_s \geq 0\) which satisfy 
\[
q^*_k = \sum_{s=1}^{S} \mu_s r_{sk}
\]
for all \(k\).

(a) Prove that if the asset markets are complete, then the \(\mu_s\) are uniquely defined.

(b) Given an interpretation of these \(\mu_s\) in terms of Arrow securities.

5. Suppose prices \(q^* \in \mathbb{R}^K\) and \(p^* \in \mathbb{R}^{LS}\) and plans \(x^* \in \mathbb{R}^{LSI}\) and \(z^* \in \mathbb{R}^{KI}\) constitute a Radner equilibrium for an asset structure with \(S \times K\) return matrix \(R\). Let \(R'\) be an \(S \times K'\) second asset returns structure. If \(\text{Range } R = \text{Range } R'\) then \(x^*\) is also an equilibrium allocation for the economy that has \(R'\) as its asset matrix.

6. Suppose the asset markets are complete. Then

(a) If consumption plans \(x^* \in \mathbb{R}^{LSI}\) and the prices \(p^* \in \mathbb{R}^{LS}_{++}\) constitute an Arrow-Debreu equilibrium. Then, there are asset prices \(q^* \in \mathbb{R}^{K+}_{++}\) and portfolio choices \(z^* = (z^*_1, ..., z^*_s) \in \mathbb{R}^{K+}\) such that: \(z^*, q^*, x^*, \) and spot prices \(p_s^*\) for each \(s\) form a Radner equilibrium.

(b) Suppose consumption plans \(x^* \in \mathbb{R}^{LSI}\), portfolio plans \(z^* \in \mathbb{R}^{KI}\) and prices \(q^* \in \mathbb{R}^{S+}_{++}\) and \(p^* \in \mathbb{R}^{LS}_{++}\) constitute a Radner equilibrium. Then, there are \(S\) strictly positive multipliers \((\mu_1, ..., \mu_S) \in \mathbb{R}^{S+}_{++}\) such that the allocation \(x^*\) and the state-contingent commodities price vector \((\mu_1 p_1^*, ..., \mu_S p_S^*) \in \mathbb{R}^{LS}\) form an Arrow Debreu Equilibrium.