First Welfare Theorem

Econ 2100  Fall 2015

Lecture 17, November 2

Outline

1. First Welfare Theorem
2. Preliminaries to Second Welfare Theorem
A feasible allocation \((x, y)\) is **Pareto optimal** if there is no other feasible allocation \((x', y')\) such that
\[ x'_i \succeq_i x_i \quad \text{for all } i \quad \text{and} \quad x'_i \succ_i x_i \quad \text{for some } i. \]

An allocation \((x^*, y^*)\) and a price vector \(p^* \in \mathbb{R}^L\) are a **competitive equilibrium** if
1. for each \(j = 1, \ldots, J\): \(p^* \cdot y_j \leq p^* \cdot y_j^*\) for all \(y_j \in Y_j\);
2. for each \(i = 1, \ldots, I\):
   \[ x_i^* \succeq_i x_i \quad \text{for all } x_i \in B_i (p^*) = \{ x_i \in X_i : p^* \cdot x_i \leq p^* \cdot \omega_i + \sum_j \theta_{ij} (p^* \cdot y_j^*) \} ; \]
   and
3. \( \sum_i x_i^* = \sum_i \omega_i + \sum_j y_j^* \).

**What is the relationship between competitive equilibrium and Pareto efficiency?**

- Is any competitive equilibrium Pareto efficient? **First Welfare Theorem.**
  - This is about excluding something can Pareto dominate the equilibrium allocation.

- Is any Pareto efficient allocation (part of) a competitive equilibrium? **Second Welfare Theorem.**
  - This is about finding prices that make the efficient allocation an equilibrium.
First Welfare Theorem: A Picture
Things seem easy
First Welfare Theorem: Counterexample

An Edgeworth Box Economy

- Consider a two-person, two-good exchange economy.
- Person $a$ has utility function $U_a(x_{1a}, x_{2a}) = 7$ and person $b$ has utility function $U_b(x_{1b}, x_{2b}) = x_{1b}x_{2b}$.
- The initial endowments are $\omega_a = (2, 0)$ and $\omega_b = (0, 2)$.
- CLAIM: $x_a^* = (1, 1)$, $x_b^* = (1, 1)$, and prices $p^* = (1, 1)$ form a competitive equilibrium.
  - $a$’s utility is maximized.
  - $b$’s utility when her income equals 2 is maximized (this is a Cobb-Douglas utility function with equal exponents, so spending half her income on each good is optimal).
  - $x_a^* + x_b^* = (2, 2) = \omega$.
- Is this allocation Pareto optimal? No:
  - $\hat{x}_a = (0, 0)$ and $\hat{x}_b = (2, 2)$ Pareto dominates $x_a^*$, $x_b^*$ since consumer $a$ has the same utility while consumer $b$’s utility is higher.

- How do we rule examples like this out?
- Need consumers preferences that are locally non satiated: there is always something nearby that makes a consumer better off.
Local Non Satiation

**Definition**
A preference ordering \( \succeq_i \) on \( X_i \) is *satiated* at \( y \) if there exists no \( x \) in \( X_i \) such that \( x \succ_i y \).

**Definition**
The preference relation \( \succeq_i \) on \( X_i \) is *locally non-satiated* if for every \( x \) in \( X_i \) and for every \( \varepsilon > 0 \) there exists an \( x' \) in \( X_i \) such that \( \|x' - x\| < \varepsilon \) and \( x' \succ_i x \).

- Remember: \( \|y - z\| = \sqrt{\sum_{l=1}^{L} (y_l - z_l)^2} \) is the Euclidean distance between two points.

**Remark**
- If \( \succeq_i \) is continuous and locally non-satiated there exist a locally non-satiated utility function; then, any closed consumption set must be unbounded (or there would be a global satiation point).
Suppose $\succsim_i$ is locally non-satiated, and let $x_i^*$ be defined as:

$$x_i^* \succsim_i x_i$$

for all $x_i \in \{ x_i \in X_i : p \cdot x_i \leq w_i \}$.

Then

$$x_i \succsim_i x_i^* \quad \text{implies} \quad p \cdot x_i \geq w_i$$

and

$$x_i \succ_i x_i^* \quad \text{implies} \quad p \cdot x_i > w_i$$

In words, if a consumption vector is preferred to a maximal consumption bundle (i.e. an element of the Walrasian demand correspondence) it must cost more.

Something strictly preferred to a maximal bundle must not be affordable (or the consumer would have chosen it).
**Theorem (First Fundamental Theorem of Welfare Economics)**

Suppose each consumer’s preferences are locally non-satiated. Then, any allocation \( x^*, y^* \) that with prices \( p^* \) forms a competitive equilibrium is Pareto optimal.

- There are almost no assumptions...
- Local non-satiation has bite: there is always a more desirable commodity bundle nearby.
- Among the economic assumptions implicit in our definition of an economy one is important for this result: no externalities.
  - Externalities are present if one person’s consumption influences another person’s preferences
    - For example: in a two-person, two-good exchange economy person \( a \) has utility function \( U_a(x_{a1}, x_{a2}) = x_{a1}x_{a2} - x_{b1} \).
Proof of the First Welfare Theorem (by contradiction)

Suppose not: there exists a feasible allocation \( x, y \) such that \( x_i \preceq_i x_i^* \) for all \( i \), and \( x_i \succ_i x_i^* \) for some \( i \).

- By local non satiation,
  - \( x_i \preceq_i x_i^* \) implies \( p^* \cdot x_i \geq p^* \cdot \omega_i + \sum_j \theta_{ij}(p^* \cdot y_j^*) \)
  - \( x_i \succ_i x_i^* \) implies \( p^* \cdot x_i > p^* \cdot \omega_i + \sum_j \theta_{ij}(p^* \cdot y_j^*) \)

Therefore, summing over consumers

\[
\sum_{i=1}^I p^* \cdot x_i > \sum_{i=1}^I p^* \cdot \omega_i + \sum_{i=1}^I \sum_{j=1}^J \theta_{ij}(p^* \cdot y_j^*) \quad \text{accounting} \quad \sum_{j=1}^J p^* \cdot y_j^* = p^* \cdot \omega + \sum_{j=1}^J p^* \cdot y_j^*
\]

- Since each \( y_j^* \) maximizes profits at prices \( p^* \), we also have

\[
\sum_{j=1}^J p^* \cdot y_j^* \geq \sum_{j=1}^J p^* \cdot y_j.
\]

Substituting this into the previous inequality:

\[
\sum_{i=1}^I p^* \cdot x_i > p^* \cdot \omega + p^* \cdot \sum_{j=1}^J p^* \cdot y_j^* \geq p^* \cdot \omega + \sum_{j=1}^J p^* \cdot y_j
\]

- But then \( x, y \) cannot be feasible since if it were we would have

\[
\sum_{i=1}^I x_i = \omega + \sum_{j=1}^J y_j \quad \Rightarrow \quad \sum_{i=1}^I p \cdot x_i = p \cdot \omega + \sum_{j=1}^J p \cdot y_j
\]

which contradicts the expression above.
Any competitive equilibrium is in the core.

Proof.

Homework. This is very very similar to the proof of the First Welfare Theorem.

- A ‘converse’ can be established in cases in which the economy is “large”, that is, it contains many individuals.
  - That is called the core convergence theorem and I do not think we will have time for it.
Second Welfare Theorem: Preliminaries

- This is a converse to the First Welfare Theorem.
- The statement goes: under some conditions, any Pareto optimal allocation is part of a competitive equilibrium.
  - Next, we try to understand what these conditions must be. We state and prove the theorem next class.
- In order to prove a Pareto optimal allocation is part of an equilibrium one needs to find the price vector that ‘works’ for that allocation, since an equilibrium must specify an allocation and prices.
- First, we see a simple sense in which this cannot work: Pareto optimality disregards the budget constraints.
  - This is fixed by adjusting the definition of equilibrium.
- Then we see two counterexamples that stress the need for convexities.
  - These are fixed by assuming production sets and better-than sets are convex.
- Finally, we see an example showing that an equilibrium may not exist, and therefore there is no way the theorem holds.
  - This is fixed by, again, adjusting the definition of equilibrium.
The way to make the Pareto optimal allocation an equilibrium is to stick a price vector tangent to both indifference curves. This is not enough, however, since the Pareto optimal allocation violates the budget constraint of consumer 2. From the Edgeworth box, you can see that there is no hope of going from any Pareto optimal allocation to prices compatible with maximization and initial endowment.

There is hope, however, to keep the slope of the budget constraint fixed and adjust income so that things work out.

One needs to adjust income appropriately.

We achieve this by modifying the definition of equilibrium.
Equilibrium With Transfers

**Definition**

Given an economy \( \left( \{X_i, \sim_i\}_{i=1}^{I}, \{Y_j\}_{j=1}^{J}, \omega \right) \), an allocation \( x^*, y^* \) and a price vector \( p^* \) constitute a **price equilibrium with transfers** if there exists a vector of wealth levels

\[
w = (w_1, w_2, ..., w_I)
\]

with

\[
\sum_{i=1}^{I} w_i = p^* \cdot \omega + \sum_{j=1}^{J} p^* \cdot y_j^*
\]

such that:

1. For each \( j = 1, ..., J \):
   \[
p^* \cdot y_j \leq p^* \cdot y_j^* \quad \text{for all } y_j \in Y_j.
   \]

2. For each \( i = 1, ..., I \):
   \[
x_i^* \sim_i x_i \quad \text{for all } x_i \in \{x_i \in X_i : p^* \cdot x_i \leq w_i\}
   \]

3. \[
\sum_{i=1}^{I} x_i^* = \sum_{i=1}^{I} \omega_i + \sum_{j=1}^{J} y_j^*
\]

- Aggregate wealth is divided among consumers so that the budget constraints are satisfied.
- How is the wealth of each consumer effected? They get a positive or negative transfer relative to the value of their endowment at the equilibrium prices.
- A competitive equilibrium satisfies this: set \( w_i = p^* \cdot \omega_i + \sum_j \theta_{ij}(p^* \cdot y_j^*) \).
Equilibrium With Transfers

Remark

The income transfers (across consumers) that achieve the budget levels in the previous definition are:

\[ T_i = w_i - \left[ p^* \cdot w_i + \sum_{j=1}^{J} \theta_{ij} \left( p^* \cdot y_j^* \right) \right] \]

- Summing over consumers, we get

\[
\sum_i T_i = \sum_{i=1}^{l} w_i - \left[ \sum_{i=1}^{l} p^* \cdot w_i + \sum_{i=1}^{l} \sum_{j=1}^{J} \theta_{ij} \left( p^* \cdot y_j^* \right) \right] \\
= \sum_{i=1}^{l} w_i - \left[ p^* \cdot \omega + \sum_{j=1}^{J} p^* \cdot y_j^* \right] \\
= 0
\]

- Transfers redistribute income so that the ‘aggregate budget’ balances.
- This is important: in a general equilibrium model nothing should be ‘outside’ the economy.
Counterexample 1 to Second Welfare Theorem

Need convex preferences for the Second Welfare Theorem

$x^*$ is Pareto optimal, but one can see it is not an equilibrium at prices $p$
Counterexample II to Second Welfare Theorem

One Producer One Consumer Economy (Robinson Crusoe)

\( x \) is the preferences maximizing bundle at the given prices

\( x \) does not maximizes profits in production set \( Y \) at the given prices (not even locally)

Need convex production sets for Second Welfare Theorem
Convexity

**Definition**

A preference relation $\succeq$ on $X$ is convex if the set $\{y \in X \mid y \succeq x\}$ is convex for every $x$.

- If $x'$ and $x''$ are weakly preferred to $x$ so is any convex combination.

- Convexity implies existence of an hyperplane that ‘supports’ a consumer’s better than set.

**Definition**

In an exchange economy, an allocation $x$ is **supported** by a non-zero price vector $p$ if: for each $i = 1, \ldots, I$

$$x' \succeq_i x_i \implies p \cdot x' \geq p \cdot x_i$$

- Convexity also yields an hyperplane that ‘supports’ all producers’ better than set at the same time.

- This hyperplane is the price vector that makes a Pareto optimal allocation an equilibirum.
Counterexample III to Second Welfare Theorem

- $\omega$ is the initial endowment
- Consumer 1 owns all of good 2
- Consumer 2 owns all of good 1

For any $p \geq 0$,
- 2 demands $\omega_2$ when her wealth is $w_2$;
- but $\omega_1$ is never optimal for 1 if $p \geq 0$... she would demand an infinite quantity of good 1

Consumer 2 only cares about good 1
Consumer 1 preferences have infinite slope at $\omega_1$

For any $p \geq 0$, $2$ demands $\omega_2$ when her wealth is $w_2$; but $\omega_1$ is never optimal for 1 if $p \geq 0$... she would demand an infinite quantity of good 1.

There is no Walrasian equilibrium in this case.

- $\omega$ is the unique Pareto optimal allocation, hence the only candidate for an equilibrium.
- The unique price vector that can support it as equilibrium (normalizing the price of the first good to 1) is $p = (1, 0)$.
- The corresponding wealth is $w_1 = (1, 0) \cdot (0, \omega_2) = 0$.
- However, $\omega_1$ is not maximal for consumer 1 at $p$ since she would like more of good 2 (it has zero price, hence she can afford it).
- There is no competitive equilibrium in this example.
Quasi-Equilibrium

To fix the ‘existence at the boundary’ problem, with make a small change to the definition of equilibrium.

Definition

Given an economy \( \{X_i, \succeq_i\}_{i=1}^I \), \( \{Y_j\}_{j=1}^J \), \( \omega \), an allocation \( x^*, y^* \) and a price vector \( p^* \) constitute a quasi-equilibrium with transfers if there exists a vector of wealth levels

\[
w = (w_1, w_2, ..., w_I)
\]

such that:

1. For each \( j = 1, ..., J \):

\[p^* \cdot y_j \leq p^* \cdot y_j^* \quad \text{for all } y_j \in Y_j.\]

2. For every \( i = 1, ..., I \):

\[\text{if } x \succeq_i x_i^* \quad \text{then } p^* \cdot x \geq w_i\]

3. \[
\sum_{i=1}^I x_i^* = \sum_{i=1}^I \omega_i + \sum_{j=1}^J y_j^*
\]

Make sure you see why this deals with the problem in the previous slide.

Any equilibrium with transfers is a quasi-equilibrium (make sure you check this).
Consider a competitive model with one input, two outputs, and two firms with production functions

\[ y_1 = f_1(l_1) = \sqrt{2l_1} \quad \text{and} \quad y_2 = f_2(l_2) = \sqrt{2l_2} \]

(where \( l_j \) denotes the amount of the input \( l \) used by firm \( j \), and firm \( j \) produces only good \( j \)). Consumer \( a \) is endowed with 25 units of the input \( l \) and owns no shares in the firms, and has utility function \( U_a(x_{a1}, x_{a2}) = x_{a1}x_{a2} \). Consumer \( b \) owns both firms, but has zero endowment, and has utility function \( U_b(x_{b1}, x_{b2}) = x_{b1} + x_{b2} \). Find a competitive equilibrium in this model. Is it unique?

Prove that any competitive equilibrium is in the core (for an exchange economy).

Consider a two-person, two-good exchange economy in which person \( a \) has utility function \( U_a(x_{a1}, x_{a2}) = x_{a1}x_{a2} - x_{b1} \) and person \( b \) has utility function \( U_b(x_{b1}, x_{b2}) = x_{b1} + x_{b2} \). The initial endowments are \( \omega_a = (2, 0) \) and \( \omega_b = (0, 2) \).

1. Show that \( \rho = (1, 1) \) and \( x_a = (1, 1) \), \( x_b = (1, 1) \) is a competitive equilibrium.
2. Prove or provide a counterexample to the following statement: in this economy any competitive equilibrium is Pareto optimal.

Consider a two-person, two-good exchange economy in which person \( a \) has utility function \( U_a(x_{a1}, x_{a2}) = 1 \) if \( x_{a1} + x_{a2} < 1 \) and \( U_a(x_{a1}, x_{a2}) = x_{a1} + x_{a2} \) if \( x_{a1} + x_{a2} \geq 1 \). Person \( b \) has utility function \( U_b(x_{b1}, x_{b2}) = x_{b1}x_{b2} \). The initial endowments are \( \omega_a = (1, 0) \) and \( \omega_b = (0, 1) \).

1. Show that \( \rho = (1, 1) \) and \( x_a = (\frac{1}{2}, \frac{1}{2}) \), \( x_b = (\frac{1}{2}, \frac{1}{2}) \) is a competitive equilibrium.
2. Is this allocation in the core? Explain your answer.
3. Does the first welfare theorem hold for this economy? Explain your answer.

Consider a two-person, two-good exchange economy where the agents’ utility functions are \( U_a(x_{a1}, x_{a2}) = x_{a1}x_{a2} \) and \( U_b(x_{b1}, x_{b2}) = x_{b1}x_{b2} \), and the initial endowments are \( \omega_a = (1, 5) \) and \( \omega_b = (5, 1) \).

1. Find the Pareto optimal allocations and the core. Draw the Edgeworth box for this economy.
2. Find the individual and market excess demand functions. Find the equilibrium prices and allocations.
3. Show directly that every interior Pareto optimal allocation in this economy is a price equilibrium with transfers by finding the associated prices and transfers.