Properties of Walrasian Demand

Econ 2100  Fall 2017

Lecture 5, September 12

- Problem Set 2 is due in Kelly’s mailbox by 5pm today

Outline

1. Properties of Walrasian Demand
2. Indirect Utility Function
3. Envelope Theorem
Summary of Constrained Optimization

When $x^*$ is solves $\max f(x)$ subject to $g_i(x) \leq 0$ then $i = 1, \ldots, m$

$$\nabla L(x^*, \lambda) = \nabla f(x^*) - \sum_{i=1}^{m} \lambda_i \nabla g_i(x^*) = 0$$

and $g_i(x^*) \leq 0$, $\lambda_i \geq 0$, and $\lambda_i g_i(x^*) = 0$ for $i = 1, \ldots, m$.

Important details:

- If the better than set or the constraint sets are not convex: big trouble.
- If functions are not differentiable: small trouble.
  - If the geometry still works we can find a more general theorem (see convex analysis).

When does this fail?

- If the constraint qualification condition fails.
- If the objective function is not quasi concave.

This means you must check the second order conditions when in doubt.
Walrasian Demand

**Definition**

Given a utility function $u : \mathbb{R}^n_+ \rightarrow \mathbb{R}$, the Walrasian demand correspondence $x^* : \mathbb{R}^n_+ \times \mathbb{R}_+ \rightarrow \mathbb{R}^n_+$ is defined by

$$x^*(p, w) = \arg \max_{x \in B(p, w)} u(x)$$

where

$$B(p, w) = \{ x \in \mathbb{R}^n_+ : p \cdot x \leq w \}.$$

- When the utility function is quasi-concave and differentiable, the First Order Conditions for utility maximization say:

  $$\nabla(\text{utility function}) - \lambda \nabla(\text{budget constraint}) - \sum \lambda_{\text{non negativity constraints}} \nabla(\text{non negativity constraints}) = 0$$

- So, at a solution $x^* \in x^*(p, w)$:

  $$\frac{\partial L(x^*)}{\partial x_i} = \frac{\partial u(x^*)}{\partial x_i} - \lambda w x_i^* + \lambda_i = 0 \text{ for all } i = 1, \ldots, n.$$
Marginal Rate of Substitution

- Suppose we have an optimal consumption bundle $x^*$ where some goods are consumed in strictly positive amounts.

- Then, the corresponding non-negativity constraints hold and the corresponding multipliers equal 0.

- At such a solution $x^*$, the first order condition is:
  \[
  \frac{\partial u(x^*)}{\partial x_i} = \lambda_w p_i \quad \text{for all } i \text{ such that } x_i^* > 0
  \]

- This expression implies
  \[
  \frac{\partial u(x^*)}{\partial x_j} = \frac{p_j}{p_k} \quad \text{for any } j, k \text{ such that } x_j^*, x_k^* > 0
  \]

- This is the familiar condition about equalizing marginal rates of substitutitons to price ratios.
Marginal Utility Per Dollar Spent

At an optimal consumption bundle \( x^* \) where some goods are consumed in strictly positive amounts, the first order condition is:

\[
\frac{\partial u(x^*)}{\partial x_i} = \lambda_i p_i \quad \text{for all } i \text{ such that } x_i^* > 0
\]

Another way to rearrange it gives

\[
\frac{\partial u(x^*)}{\partial x_j} \frac{p_j}{p_k} = \frac{\partial u(x^*)}{\partial x_k} \quad \text{for any } j, k \text{ such that } x_j^*, x_k^* > 0
\]

the marginal utility per dollar spent must be equal across goods.

- If not, there are \( j \) and \( k \) for which \( \frac{\partial u(x^*)}{\partial x_j} \frac{p_j}{p_k} < \frac{\partial u(x^*)}{\partial x_k} \frac{p_k}{p_k} \)

- DM can buy \( \frac{\varepsilon}{p_j} \) less of \( j \), and \( \frac{\varepsilon}{p_k} \) more of \( k \), so the budget constraint still holds, and

- by Taylor’s theorem, the utility at the new choice is

\[
u(x^*) + \frac{\partial u(x^*)}{\partial x_j} \left( -\frac{\varepsilon}{p_j} \right) + \frac{\partial u(x^*)}{\partial x_k} \left( \frac{\varepsilon}{p_k} \right) + o(\varepsilon) = u(x^*) + \varepsilon \left( \frac{\partial u(x^*)}{\partial x_k} \frac{p_k}{p_k} - \frac{\partial u(x^*)}{\partial x_j} \frac{p_j}{p_j} \right) + o(\varepsilon)
\]

which implies that \( x^* \) is not an optimum.

- Think about the case in which some goods are consumed in zero amount.
Demand and Indirect Utility Function

The Walrasian demand correspondence \( x^*: \mathbb{R}^{n+} \times \mathbb{R}_+ \to \mathbb{R}^n \) is defined by

\[
x^*(p, w) = \arg \max_{x \in B(p, w)} u(x) \quad \text{where} \quad B(p, w) = \{x \in \mathbb{R}^n_+ : p \cdot x \leq w\}.
\]

Definition

Given a continuous utility function \( u: \mathbb{R}^n_+ \to \mathbb{R} \), the indirect utility function \( v: \mathbb{R}^{n+} \times \mathbb{R}_+ \to \mathbb{R} \) is defined by

\[
v(p, w) = u(x^*) \quad \text{where} \quad x^* \in x^*(p, w).
\]

- The indirect utility function measures changes in the ‘optimized’ value of the objective function as the parameters (prices and wages) change and the consumer adjusts her optimal consumption accordingly.

Results

- The Walrasian demand correspondence is upper hemi continuous
  - To prove this we need properties that characterize continuity for correspondences.

- The indirect utility function is continuous.
  - To prove this we need properties that characterize continuity for correspondences.
Berge’s Theorem of the Maximum

- The theorem of the maximum lets us establish the previous two results.

**Theorem (Theorem of the Maximum)**

If \( f : X \rightarrow \mathbb{R} \) is a continuous function and \( \varphi : Q \rightarrow X \) is a continuous correspondence with nonempty and compact values, then

- the mapping \( x^* : Q \rightarrow X \) defined by
  
  \[
  x^*(q) = \arg \max_{x \in \varphi(q)} f(x)
  \]

  is an upper hemicontinuous correspondence and

- the mapping \( v : Q \rightarrow \mathbb{R} \) defined by
  
  \[
  v(q) = \max_{x \in \varphi(q)} f(x)
  \]

  is a continuous function.

Berge’s Theorem is useful when exogenous parameters enter the optimization problem only through the constraints, and do not directly enter the objective function.

This happens in the consumer’s problem: prices and income do not enter the utility function, they only affect the budget set.
Properties of Walrasian Demand

**Proposition**

If \( u(x) \) is continuous, then \( x^*(p, w) \) is upper hemicontinuous and \( v(p, w) \) is continuous.

**Proof.**

Apply Berge’s Theorem:
If \( u : \mathbb{R}^n_+ \to \mathbb{R} \) is a continuous function and \( B(p, w) : \mathbb{R}^n_+ \times \mathbb{R}_+ \to \mathbb{R}^n_+ \) is a continuous correspondence with nonempty and compact values. Then:

(i): \( x^* : \mathbb{R}^n_+ \times \mathbb{R}_+ \to \mathbb{R}^n_+ \) defined by \( x^*(p, w) = \text{arg max}_{x \in B(p, w)} u(x) \) is an upper hemicontinuous correspondence and

(ii): \( v : \mathbb{R}^n_+ \times \mathbb{R}_+ \to \mathbb{R} \) defined by \( v(p, w) = \max_{x \in B(p, w)} u(x) \) is a continuous function.

- We need continuity of the correspondence from price-wage pairs to budget sets.
- We must show that \( B : \mathbb{R}^n_+ \times \mathbb{R}_+ \to \mathbb{R}^n_+ \) defined by
  \[
  B(p, w) = \{ x \in \mathbb{R}^n_+ : p \cdot x \leq w \}
  \]
  is continuous and we are done.
Reminder from math camp.

**Definition**

A correspondence \( \varphi : X \to Y \) is

- **upper hemicontinuous** at \( x \in X \) if for any neighborhood \( V \subseteq Y \) containing \( \varphi(x) \), there exists a neighborhood \( U \subseteq X \) of \( x \) such that \( \varphi(x') \subseteq V \) for all \( x' \in U \).

- **lower hemicontinuous** at \( x \in X \) if for any neighborhood \( V \subseteq Y \) such that \( \varphi(x) \cap V \neq \emptyset \), there exists a neighborhood \( U \subseteq X \) of \( x \) such that \( \varphi(x') \cap V \neq \emptyset \) for all \( x' \in U \).

A correspondence is upper (lower) hemicontinuous if it is upper (lower) hemicontinuous for all \( x \in X \).

A correspondence is continuous if it is both upper and lower hemicontinuous.
The following sufficient conditions are sometimes easier to use.

**Proposition (A)**

Suppose $X \subseteq \mathbb{R}^m$ and $Y \subseteq \mathbb{R}^n$. A compact-valued correspondence $\varphi : X \to Y$ is upper hemicontinuous if, and only if, for any domain sequence $x_j \to x$ and corresponding range sequence $y_j$ such that $y_j \in \varphi(x_j)$, there exists a convergent subsequence $\{y_{j_k}\}$ such that $\lim y_{j_k} \in \varphi(x)$.

- Note that compactness of the image is required.

**Proposition (B)**

Suppose $A \subseteq \mathbb{R}^m$, $B \subseteq \mathbb{R}^n$, and $\varphi : A \to B$. Then $\varphi$ is lower hemicontinuous if, and only if, for all $\{x_m\} \in A$ such that $x_m \to x \in A$ and $y \in \varphi(x)$, there exist $y_m \in \varphi(x_m)$ such that $y_m \to y$. 

Continuity for Correspondences: Examples

Exercise
Suppose \( \varphi : \mathbb{R} \rightarrow \mathbb{R} \) is defined by:

\[
\varphi(x) = \begin{cases} 
\{1\} & \text{if } x < 1 \\
[0, 2] & \text{if } x \geq 1
\end{cases}
\]

Prove that \( \varphi \) is upper hemicontinuous, but not lower hemicontinuous.

Exercise
Suppose \( \varphi : \mathbb{R} \rightarrow \mathbb{R} \) is defined by:

\[
\varphi(x) = \begin{cases} 
\{1\} & \text{if } x \leq 1 \\
[0, 2] & \text{if } x > 1
\end{cases}
\]

Prove that \( \varphi \) is lower hemicontinuous, but not upper hemicontinuous.
Continuity of the Budget Set Correspondence

Question 1, Problem Set 3; due next Tuesday.

Show that the correspondence from price-wage pairs to budget sets, 
\( B : \mathbb{R}^n_+ \times \mathbb{R}_+ \to \mathbb{R}_+ \) defined by 
\[
B(p, w) = \{ x \in \mathbb{R}^n_+ : p \cdot x \leq w \}
\]
is continuous.

- First show that \( B(p, w) \) is upper hemi continuous.
- Then show that \( B(p, w) \) is lower hemi continuous.
- In both cases, use the propositions in the previous slide (and Bolzano-Weierstrass).
- There was a very very similar problem in math camp.
Properties of Walrasian Demand

**Definitions**

Given a utility function \( u : \mathbb{R}_+^n \rightarrow \mathbb{R} \), the **Walrasian demand correspondence** \( x^* : \mathbb{R}^n_{++} \times \mathbb{R}_+ \rightarrow \mathbb{R}^n_+ \) is defined by

\[
x^*(p, w) = \arg \max_{x \in B_{p,w}} u(x)
\]

where

\[
B_{p,w} = \{ x \in \mathbb{R}^n_+ : p \cdot x \leq w \}.
\]

and the **indirect utility function** \( v : \mathbb{R}^n_{++} \times \mathbb{R}_+ \rightarrow \mathbb{R} \) is defined by

\[
v(p, w) = u(x^*(p, w))
\]

where

\[
x^*(p, w) \in \arg \max_{x \in B_{p,w}} u(x).
\]

**Properties of Walrasian demand:**

- if \( u \) is continuous, then \( x^*(p, w) \) is nonempty and compact.
- \( x^*(p, w) \) is homogeneous of degree zero: for any \( \alpha > 0 \), \( x^*(\alpha p, \alpha w) = x^*(p, w) \)
- if \( u \) represents a locally nonsatiable \( \succsim \), then \( p \cdot x = w \) for any \( x \in x^*(p, w) \).
- if \( u \) is quasiconcave, then \( x^*(p, w) \) is convex.
- if \( u \) is strictly quasiconcave, then \( x^*(p, w) \) is unique.
- If \( u(x) \) is continuous, then \( x^*(p, w) \) is upper hemicontinuous and \( v(p, w) \) is continuous.

**Next, comparative statics:** How does Walrasian Demand change as income and/or (some) prices change?
Implicit Function Theorem

This is the main tool to perform comparative static analysis (also with constraints)

- Same as math camp.

**Theorem (Implicit Function Theorem)**

Suppose $A$ is an open set in $\mathbb{R}^{n+m}$ and $f : A \to \mathbb{R}^n$ is continuously differentiable. Let $D_x f$ be the $n \times n$ derivative matrix of $f$ with respect to its first $n$ arguments. If $f(\bar{x}, \bar{q}) = 0_n$ and $D_x f(\bar{x}, \bar{q})$ is nonsingular, then there exists a neighborhood $B$ of $\bar{q}$ in $\mathbb{R}^m$ and a unique continuously differentiable $g : B \to \mathbb{R}^n$ such that

$$g(\bar{q}) = \bar{x} \quad \text{and} \quad f(g(\bar{q}), \bar{q}) = 0_n \text{ for all } \bar{q} \in B$$

Moreover,

$$D_q g(\bar{q}) = -\left[D_x f(\bar{x}, \bar{q})\right]^{-1} D_q f(\bar{x}, \bar{q}).$$

- Notation: $(D_x f)_{ij} = \frac{\partial f_i}{\partial x_j}$

IFT gives a way to write, locally, the $x$s as dependent on $q$s via a smooth (differentiable) function.

- We use this in the consumer’s utility maximization problem by setting $f(\cdot)$ to be the FOC for utility maximization;
  - then $g(\cdot)$ is the Walrasian demand, and $\bar{q}$ is a price vector and income.
Bordered Hessian

For comparative statics he following object is useful.

**Definition**

Let \( m = \{ i : x_i^*(\bar{p}, \bar{w}) > 0 \} \) and reindex \( \mathbb{R}^n \) so these \( m \) dimensions come first. The bordered Hessian \( H \) of \( u \) with respect to its first \( m \) dimensions is

\[
H = \begin{bmatrix}
0 & u_1 & u_2 & \cdots & u_m \\
u_1 & u_{11} & u_{21} & \cdots & u_{m1} \\
u_2 & u_{12} & u_{22} & \cdots & u_{m2} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
u_m & u_{1m} & u_{2m} & \cdots & u_{mm}
\end{bmatrix}
\]

where \( u_i = \frac{\partial u}{\partial x_i} \) and \( u_{ij} = \frac{\partial^2 u}{\partial x_i \partial x_j} \).

This only considers demand functions which are not zero and then takes first and second derivatives of the utility function with respect to those goods.
**Proposition**

Suppose $u$ is twice continuously differentiable, locally nonsatiated, strictly quasiconcave, and that there exists $\varepsilon > 0$ such that $x^*_i(p, \bar{w}) > 0$ if and only if $x^*_i(p, w) > 0$, for all $(p, w)$ such that $\|(\bar{p}, \bar{w}) - (p, w)\| < \varepsilon$.\(^a\) If $H$ is nonsingular at $(\bar{p}, \bar{w})$, then $x^*(p, w)$ is continuously differentiable at $(\bar{p}, \bar{w})$.

\(^a\)This condition is automatically satisfied if $x^*_i > 0$ for all $i$, by continuity of $x^*$.

**Proof.**

Question 2, Problem Set 3.

- These are sufficient conditions for differentiability of $x^*(p, w)$, yet we do not have axioms on $\succsim$ that deliver a continuously differentiable utility.
- We also assume demand is strictly positive (locally), but this is the very object we want to characterize.
- This is “bad” math.
Proof.

We want to use the Implicit Function Theorem. Things we know right away:

- $u$ is strictly quasiconcave $\Rightarrow x^*$ is a function.
- $u$ is continuously differentiable $\Rightarrow$ the first order conditions are well defined.
- $u$ is locally nonsatiated $\Rightarrow$ the budget constraint binds.
- $x^*$ is strictly positive in the first $m$ commodities $\Rightarrow$ ignore corresponding nonnegativity constraints.
- It now suffices to show that $x^*$ is differentiable in the first $m$ prices, and $w$, and ignore the last $n - m$ commodities.$^a$
- The remainder of the proof uses IFT to show that $x^*$ is differentiable as desired;
- Fill in the details.

$^a$There is a neighborhood of $(\bar{p}, \bar{w})$ where consumption is zero for the last $n - m$ commodities, and these constant dimensions will have no effect on the differentiability of $x^*$. 
Comparative Statics
Without constraints this is the same as math camp

Problem

Let $\phi : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}$, written $\phi(x; q)$; DM chooses $x$ to maximize $\phi$, while $q$ are parameters she does not control.

Comparative Statics

We want to know how DM adjusts her optimal choice $x^*(q)$ when the parameters $q$ change. What is the derivative of $x^*(q)$?

- If $\phi$ is strictly concave and differentiable, and smooth, the Implicit Function Theorem gives the answer.
**Envelope Theorem Without Constraints**

Define: \( x^*(q) = \arg \max \phi(x; q) \). The maximizer solves the first order condition:
\[
f(x; q) = D_x \phi(x; q) = 0^n \quad .
\]
The Jacobian of \( f \) is \( D_x f(x; q) = D_{xx} \phi(x; q) \), and is nonsingular (why?).

**Reminder from math camp**

By IFT: \( x^*(q) \) is continuously differentiable close to a solution \( x^*(\bar{q}), \bar{q} \), and:
\[
D_q x^*(\bar{q}) = - [D_x f(x^*(\bar{q}); \bar{q})]^{-1} D_q f(x^*(\bar{q}); \bar{q}) = - \left[ D_{xx} \phi(x^*(\bar{q}); \bar{q}) \right]^{-1} D_{qx} \phi(x^*(\bar{q}); \bar{q})
\]

- Using the Chain Rule, the change of \( \phi(x^*(q); q) \) is:
\[
D_q \phi(x^*(\bar{q}); \bar{q}) = D_q \phi(x, q)\big|_{q=\bar{q}} + D_x \phi(x, q)\big|_{q=\bar{q}} D_q x^*(q) = D_q \phi(x, q)\big|_{q=\bar{q}} \quad x=x^*(q)
\]

- the “second order effect” of how the maximizer \( x^* \) responds to \( q \) is irrelevant; only the “first order effect” of how \( q \) changes the objective function evaluated at the fixed maximizer \( x^*(q) \) matters. This observation is sometimes called the Envelope Theorem.

- If \( x \) and \( q \) are scalars, \( D_q x^*(q) \) becomes:
\[
\frac{\partial x^*}{\partial q} = \frac{\partial^2 \phi}{\partial q \partial x^2}
\]
Comparative Statics With Constraints

There are $k$ equality constraints, $F_i(x; q) = 0$ with each $F_i$ smooth.

The maximization problem now becomes:

$$x^*(q) = \arg \max_{F(x, q) = 0} \phi(x; q)$$

- Assume constraint qualifications are met and form the Lagrangian:

$$L(\lambda, x; q) = \phi(x; q) - \lambda^T F(x; q).$$

- The derivative of the Lagrangian is:

$$f(\lambda, x; q) \equiv D(\lambda, x) L(\lambda, x; q) = \begin{pmatrix} 1 \times k \\ -F(x; q) \end{pmatrix} + \begin{pmatrix} D_x \phi(x; q) - \lambda^T D_x F(x; q) \end{pmatrix}_{1 \times (k+n)}$$

- Fix $\bar{q} \in \mathbb{R}^m$ and let $\lambda^*$ and $x^*$ be the corresponding maximum;

- We can now use IFT on the FOC $f(\lambda^*, x^*; q) = 0_{k+n}^T$ (it works because the Jacobian is non singular).
Comparative Statics With Constraints

- By IFT, $x^*$ and $\lambda^*$ are implicit functions of $q$ in a neighborhood of $\bar{q}$, and
  \[ D_q(\lambda^*(\bar{q}), x^*(\bar{q})) = - \left[ D_{(\lambda,x)} f(\lambda^*, x^*; \bar{q}) \right]^{-1} D_q f(\lambda^*, x^*; \bar{q}) \]

- We know $f$ is the derivative of the Lagrangian:
  \[
  f(\lambda^*, x^*; \bar{q}) = \begin{pmatrix}
  -F(x; q) \\
  D_x \phi(x; q) - \lambda^T D_x F(x; q)
  \end{pmatrix}
  \]

- So, we can figure out that

  \[
  D_{(\lambda,x)} f(\lambda^*, x^*; \bar{q}) = \begin{pmatrix}
  k \times k \\
  0 \\
  (k+n) \times (k+n)
  \end{pmatrix}
  \begin{pmatrix}
  k \times n \\
  -D_x F(x^*; \bar{q}) \\
  n \times k \\
  D_x^2 \phi(x^*; \bar{q}) - D_x [\lambda^T D_x F(x^*; \bar{q})] \\
  n \times n
  \end{pmatrix}
  \]

  and

  \[
  D_q f(\lambda^*, x^*; \bar{q}) = \begin{pmatrix}
  k \times m \\
  -D_q F(x^*(\bar{q}); \bar{q}) \\
  (k+n) \times m
  \end{pmatrix}
  \begin{pmatrix}
  k \times m \\
  -D_q F(x^*(\bar{q}); \bar{q}) \\
  n \times m \\
  D_{qx} \phi(x^*; \bar{q}) - D_q [\lambda^T D_x F(x^*; \bar{q})^T] \\
  n \times m
  \end{pmatrix}
  \]
Envelope Theorem With Constraints

Now we can figure out the change in the objective function.

- Using the Chain Rule, the change of $\phi^*(q) = \phi(x^*(q); q)$ is:
  \[
  D_q \phi(x^*(\bar{q}); \bar{q}) = D_q \phi(x, q)|_{q=q, x=x^*(q)} + D_x \phi(x, q)|_{q=\bar{q}, x=x^*(\bar{q})} \ D_q x^*(q)
  \]

- Because the constraint hold, $-F(x; q) = 0_k$ and thus
  \[
  -D_x F(x; q) D_q x^*(q) = D_q F(x; q)
  \]

- From FOC we get $D_x \phi(x; q) = \lambda^T D_x F(x; q)$

- Thus,
  \[
  D_x \phi(x, q)|_{q=\bar{q}, x=x^*(\bar{q})} \ D_q x^*(q) = \lambda^T D_x F(x; q) D_q x^*(q) = -\lambda^T D_q F(x; q)
  \]

Then, the change in the objective function due to a change in $q$ is:

\[
D_q \phi(x^*(\bar{q}); \bar{q}) = D_q \phi(x, q)|_{q=\bar{q}, x=x^*(\bar{q})} - (\lambda^*(\bar{q}))^T D_q F(x, q)|_{q=\bar{q}, x=x^*(\bar{q})}
\]