Structural Properties of Utility Functions
Walrasian Demand

Outline

1. Structural Properties of Utility Functions
   1. Local Non Satiation
   2. Convexity
   3. Quasi-linearity
2. Walrasian Demand
From Last Class

**Definition**
The utility function $u : X \to \mathbb{R}$ represents the binary relation $\succeq$ on $X$ if
$x \succeq y \iff u(x) \geq u(y)$.

**Theorem (Debreu)**
Suppose $X \subseteq \mathbb{R}^n$. A binary relation $\succeq$ on $X$ is complete, transitive, and continuous
if and only if it admits a continuous utility representation $u : X \to \mathbb{R}$.

Today we study connections between a utility function and the underlying preference relation it represents.

**Structural Properties of Utility Functions**
The main idea is to understand the connection between properties of preferences and characteristics of the utility function that represents them.

**NOTATION:**
- We assume $X = \mathbb{R}^n$.
- If $x_i \geq y_i$ for each $i$, we write $x \geq y$. 
Local Non Satiation

**Definition**
A preference relation \( \succeq \) is **locally nonsatiated** if for all \( x \in X \) and \( \varepsilon > 0 \), there exists some \( y \) such that \( \|y - x\| < \varepsilon \) and \( y \succ x \).

- For any consumption bundle, there is always a nearby bundle that is strictly preferred to it. (Draw an example)

**Definition**
A utility function \( u : X \rightarrow \mathbb{R} \) is **locally nonsatiated** if it represents a locally nonsatiated preference relation \( \succeq \); that is, if for every \( x \in X \) and \( \varepsilon > 0 \), there exists some \( y \) such that \( \|y - x\| < \varepsilon \) and \( u(y) > u(x) \).

**Example:** The lexicographic preference on \( \mathbb{R}^2 \) is locally nonsatiated
- Fix \( (x_1, x_2) \) and \( \varepsilon > 0 \).
- Then \( (x_1 + \frac{\varepsilon}{2}, x_2) \) satisfies \( \|(x_1 + \frac{\varepsilon}{2}, x_2) - (x_1, x_2)\| < \varepsilon \)
- and \( (x_1 + \frac{\varepsilon}{2}, x_2) \succ (x_1, x_2) \).
Proposition

If $\succsim$ is strictly monotone, then it is locally nonsatiated.

Proof.

Let $x$ be given, and let $y = x + \frac{\varepsilon}{n} e$, where $e = (1, \ldots, 1)$.

- Then we have $y_i > x_i$ for each $i$.
- Strict monotonicity implies that $y \succ x$.
- Note that

$$
||y - x|| = \sqrt{\sum_{i=1}^{n} \left( \frac{\varepsilon}{n} \right)^2} = \frac{\varepsilon}{\sqrt{n}} < \varepsilon.
$$

Thus $\succsim$ is locally nonsatiated.
Shapes of Functions

Definitions

Suppose $C$ is a convex subset of $X$. A function $f : C \to \mathbb{R}$ is:

- **concave** if
  \[ f(\alpha x + (1 - \alpha)y) \geq \alpha f(x) + (1 - \alpha)f(y) \]
  for all $\alpha \in [0, 1]$ and $x, y \in C$;

- **strictly concave** if
  \[ f(\alpha x + (1 - \alpha)y) > \alpha f(x) + (1 - \alpha)f(y) \]
  for all $\alpha \in (0, 1)$ and $x, y \in X$ such that $x \neq y$;

- **quasiconcave** if
  \[ f(x) \geq f(y) \Rightarrow f(\alpha x + (1 - \alpha)y) \geq f(y) \]
  for all $\alpha \in [0, 1]$;

- **strictly quasiconcave** if
  \[ f(x) \geq f(y) \text{ and } x \neq y \Rightarrow f(\alpha x + (1 - \alpha)y) > f(y) \]
  for all $\alpha \in (0, 1)$. 
Convex Preferences

Definitions

A preference relation $\succeq$ is

- convex if
  \[ x \succeq y \implies \alpha x + (1 - \alpha) y \succeq y \text{ for all } \alpha \in (0, 1) \]

- strictly convex if
  \[ x \succeq y \text{ and } x \neq y \implies \alpha x + (1 - \alpha) y \succ y \text{ for all } \alpha \in (0, 1) \]

- Convexity says that taking convex combinations cannot make the decision maker worse off.
- Strict convexity says that taking convex combinations makes the decision maker better off.

Question

- What does convexity imply for the utility function representing $\succeq$?
Let $\preceq$ on $\mathbb{R}^2$ be defined as $x \preceq y$ if and only if $x_1 + x_2 \geq y_1 + y_2$ is convex.

Proof: Suppose $x \preceq y$, i.e. $x_1 + x_2 \geq y_1 + y_2$, and fix $\alpha \in (0, 1)$.

- Then
  \[ \alpha x + (1 - \alpha)y = (\alpha x_1 + (1 - \alpha)y_1, \alpha x_2 + (1 - \alpha)y_2) \]

- So,
  \[ [\alpha x_1 + (1 - \alpha)y_1] + [\alpha x_2 + (1 - \alpha)y_2] = \alpha [x_1 + x_2] + (1 - \alpha) [y_1 + y_2] \geq y_1 + y_2 \]

This is not strictly convex, because $(1, 0) \preceq (0, 1)$ and $(1, 0) \neq (0, 1)$ but

\[ \frac{1}{2} (1, 0) + \frac{1}{2} (0, 1) = \left( \frac{1}{2}, \frac{1}{2} \right) \preceq (0, 1). \]
Convexity is equivalent to quasi concavity of the corresponding utility function.

**Proposition**

If $u$ represents $\succsim$, then:

1. $\succsim$ is convex if and only if $u$ is quasiconcave;
2. $\succsim$ is strictly convex if and only if $u$ is strictly quasiconcave.

Convexity of $\succsim$ implies that any utility representation is quasiconcave, but not necessarily concave.

**Proof.**

Question 5b. Problem Set 2, due next Tuesday.
**Proposition**

Let \( \succeq \) be a preference relation on \( X \) represented by \( u : X \rightarrow \mathbb{R} \). Then, the upper contour set is a convex subset of \( X \) if and only if \( u \) is quasiconcave.

**Proof.**

- Suppose that \( u \) is quasiconcave.
  - Fix \( z \in X \), and take any \( x, y \in \succeq (z) \).
  - Wlog, assume \( u(x) \geq u(y) \), so that \( u(x) \geq u(y) \geq u(z) \), and let \( \alpha \in [0, 1] \).
  - By quasiconcavity of \( u \),
    \[
    u(z) \leq u(y) \leq u(\alpha x + (1 - \alpha)y),
    \]
    so \( \alpha x + (1 - \alpha)y \succeq z \).
  - Hence \( \alpha x + (1 - \alpha)y \) belongs to \( \succeq (z) \), proving it is convex.

- Now suppose the better-than set is convex.
  - Let \( x, y \in X \) and \( \alpha \in [0, 1] \), and suppose \( u(x) \geq u(y) \).
  - Then \( x \succeq y \) and \( y \succeq y \), and so \( x \) and \( y \) are both in \( \succeq (y) \).
  - Since \( \succeq (y) \) is convex (by assumption), then \( \alpha x + (1 - \alpha)y \succeq y \).
  - Since \( u \) represents \( \succeq \),
    \[
    u(\alpha x + (1 - \alpha)y) \geq u(y)
    \]
    Thus \( u \) is quasiconcave.
**Proposition**

If $\succeq$ is convex, then $C_{\succeq}(A)$ is convex for all convex $A$.

If $\succeq$ is strictly convex, then $C_{\succeq}(A)$ has at most one element for any convex $A$.

**Proof.**

- Let $A$ be convex and $x, y \in C_{\succeq}(A)$.
  - By definition of $C_{\succeq}(A)$, $x \succeq y$.
  - Since $A$ is convex: $\alpha x + (1 - \alpha)y \in A$ for any $\alpha \in [0, 1]$.
  - Convexity of $\succeq$ implies $\alpha x + (1 - \alpha)y \succeq y$.
  - By definition of $C_{\succeq}$, $y \succeq z$ for all $z \in A$.
  - Using transitivity, $\alpha x + (1 - \alpha)y \succeq y \succeq z$ for all $z \in A$.
  - Hence, $\alpha x + (1 - \alpha)y \in C_{\succeq}(A)$ by definition of induced choice rule.
  - Therefore, $C_{\succeq}(A)$ is convex for any convex $A$.

- Now suppose there exists a convex $A$ for which $|C_{\succeq}(A)| \geq 2$.
  - Then there exist $x, y \in C_{\succeq}(A)$ with $x \neq y$.
  - Since $A$ is convex, $\alpha x + (1 - \alpha)y \in A$ for all $\alpha \in (0, 1)$.
  - Since $x \succeq y$ and $x \neq y$, strict convexity implies $\alpha x + (1 - \alpha)y \succ y$, but this contradicts the fact that $y \in C_{\succeq}(A)$.

$\square$
Quasi-linear Utility

**Definition**

The function \( u : \mathbb{R}^n \to \mathbb{R} \) is quasi-linear if there exists a function \( v : \mathbb{R}^{n-1} \to \mathbb{R} \) such that \( u(x, m) = v(x) + m \).

We usually think of the \( n \)-th good as money (the numeraire).

** Proposition**

The preference relation \( \succeq \) on \( \mathbb{R}^n \) admits a quasi-linear representation if and only if:

1. \((x, m) \succeq (x, m') \) if and only if \( m \geq m' \), for all \( x \in \mathbb{R}^{n-1} \) and all \( m, m' \in \mathbb{R} \);

2. \((x, m) \succeq (x', m') \) if and only if \( (x, m + m'') \succeq (x', m' + m''), \) for all \( x \in \mathbb{R}^{n-1} \) and \( m, m', m'' \in \mathbb{R} \);

3. for all \( x, x' \in \mathbb{R}^{n-1} \), there exist \( m, m' \in \mathbb{R} \) such that \( (x, m) \sim (x', m') \).

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1. Given two bundles with identical goods, the consumer always prefers the one with more money.

2. Adding (or subtracting) the same monetary amount does not change rankings.

3. Monetary transfers can always be used to achieve indifference.

**Proof.**

Question 5c. Problem Set 2, due next Tuesday.
Proposition

Suppose that the preference relation \( \succeq \) on \( \mathbb{R}^n \) admits two quasi-linear representations: \( v(x) + m \) and \( v'(x) + m \), where \( v, v' : \mathbb{R}^{n-1} \rightarrow \mathbb{R} \). Then there exists \( c \in \mathbb{R} \) such that \( v'(x) = v(x) - c \) for all \( x \in \mathbb{R}^{n-1} \).

Proof.

Exercise
Homothetic Preferences and Utility

- Homothetic preferences are also useful in many applications, in particular for aggregation problems and macroeconomics.

**Definition**

The preference relation $\succsim$ on $X$ is homothetic if for all $x, y \in X$,

$$x \sim y \implies \alpha x \sim \alpha y \text{ for each } \alpha > 0$$

**Proposition**

The continuous preference relation $\succsim$ on $\mathbb{R}^n$ is homothetic if and only if it is represented by a utility function that is homogeneous of degree 1.

- A function is homogeneous of degree $r$ if $f(\alpha x) = \alpha^r f(x)$ for any $x$ and $\alpha > 0$.

**Proof.**

Question 5d. Problem Set 2, due next Tuesday.
Suppose the consumer uses her income to purchase goods (commodities) at the exogenously given prices:

- What are the optimal consumption choices?
- How do they depend on prices and income?

Typically, we answer this questions solving a constrained optimization problem using calculus.

That means the utility function must be not only continuous, but also differentiable.

- Differentiability is not a property we can derive from preferences.

That is, however, not necessary and we can talk about optimal choices even when preferences are not necessarily represented by a utility function.
First, we define what a consumer can buy.

**Definition**

The **Budget Set** $B(p, w) \subseteq \mathbb{R}^n$ at prices $p$ and income $w$ is the set of all affordable consumption bundles and is defined by

$$B(p, w) = \{ x \in \mathbb{R}^n_+ : p \cdot x \leq w \}.$$

This is the set of consumption bundles the consumer can choose from. She cannot purchase consumption bundles outside of this set.

Implicit assumptions: goods are perfectly divisible; consumption is non-negative; the total price of consumption cannot exceed income; prices are linear. Think of possible violations.

**Exercise**

Suppose $w = $100. There are two commodities, electricity and food. Each unit of food costs $1. The first 20 Kwh electricity cost $1 per Kwh, but the price of each incremental unit of electricity is $1.50 per Kwh. Write the consumer’s budget set formally and draw it.
Walrasian Demand

Main Idea
- The optimal consumption bundles are those that are preferred to all other affordable bundles.

Definition
Given a preference relation $\succeq$, the Walrasian demand correspondence $x^*: \mathbb{R}_+^n \times \mathbb{R}_+ \rightarrow \mathbb{R}_+^n$ is defined by
$$x^*(p, w) = \{x \in B(p, w) : x \succeq y \text{ for any } y \in B(p, w)\}.$$

By definition, for any $x^* \in x^*(p, w)$
$$x^* \succeq x \text{ for any } x \in B(p, w).$$

Walrasian demand equals the induced choice rule given the preference relation $\succeq$ and the available set $B(p, w)$:
$$x^*(p, w) = C_\succeq(B(p, w)).$$

More implicit assumptions: income is non-negative; prices are strictly positive.
Walrasian Demand With Utility

- Although we do not need the utility function to exist to define Walrasian demand, if a utility function exists there is an equivalent definition.

**Definition**

Given a utility function \( u : \mathbb{R}_+^n \to \mathbb{R} \), the Walrasian demand correspondence \( x^* : \mathbb{R}_+^{n+} \times \mathbb{R}_+ \to \mathbb{R}_+^n \) is defined by

\[
x^*(p, w) = \arg \max_{x \in B(p, w)} u(x) \quad \text{where} \quad B(p, w) = \{ x \in \mathbb{R}_+^n : p \cdot x \leq w \}.
\]

- As before,

\[
x^*(p, w) = C_{\succsim} (B(p, w)).
\]

and for any \( x^* \in x^*(p, w) \)

\[
u(x^*) \geq u(x) \quad \text{for any } x \in B(p, w).
\]

- We can derive some properties of Walrasian demand directly from assumptions on preferences and/or utility.
Walrasian Demand Is Homogeneous of Degree Zero

Proposition

Walrasian demand is homogeneous of degree zero: for any \( \alpha > 0 \)

\[
x^*(\alpha p, \alpha w) = x^*(p, w)
\]

Proof.

For any \( \alpha > 0 \),

\[
B(\alpha p, \alpha w) = \{x \in \mathbb{R}_+^n : \alpha p \cdot x \leq \alpha w\} = \{x \in \mathbb{R}_+^n : p \cdot x \leq w\} = B(p, w)
\]

because \( \alpha \) is a scalar

- Since the constraints are the same, the optimal choices must also be the same.
The Consumer Spends All Her Income
This is sometimes known as Walras’ Law for individuals

**Proposition (Full Expenditure)**

If $\succeq$ is locally nonsatiated, then

$$p \cdot x = w \quad \text{for any } x \in x^*(p, w)$$

**Proof.**

Suppose not.

- Then there exists an $x \in x^*(p, w)$ with $p \cdot x < w$
- Find some $y$ such that
  $$\|y - x\| < \varepsilon \text{ with } \varepsilon > 0 \quad \text{and} \quad p \cdot y \leq w$$
  (why does such a $y$ always exist?)
- By local non satiation, this implies $y \succ x$ contradicting $x \in x^*(p, w)$.  

**Proposition**

If $u$ is quasiconcave, then $x^*(p, w)$ is convex.

If $u$ is strictly quasiconcave, then $x^*(p, w)$ is unique.

Same as before ($u$ (strictly) quasiconcave means $\succeq$ (strictly) convex).

**Proof.**

Suppose $x, y \in x^*(p, w)$ and pick $\alpha \in [0, 1]$.

- First convexity: need to show $\alpha x + (1 - \alpha) y \in x^*(p, w)$.
  - $x \succeq y$ by definition of $x^*(p, w)$.
  - $u$ is quasiconcave, thus $\succeq$ is convex and $\alpha x + (1 - \alpha) y \succeq y$.
  - $y \succeq z$ for any $z \in B(p, w)$ by definition of $x^*(p, w)$.
  - Transitivity implies $\alpha x + (1 - \alpha) y \succeq z$ for any $z \in B(p, w)$; thus $\alpha x + (1 - \alpha) y \in x^*(p, w)$.

- Now uniqueness.
  - $x, y \in x^*(p, w)$ and $x \neq y$ imply $\alpha x + (1 - \alpha) y \succ y$ for any $\alpha \in (0, 1)$ because $u$ is strictly quasiconcave ($\succeq$ is strictly convex).
  - Since $B(p, w)$ is convex, $\alpha x + (1 - \alpha) y \in B(p, w)$, contradicting $y \in x^*(p, w)$.

$\square$
Walrasian Demand Is Non-Empty and Compact

**Proposition**

If $u$ is continuous, then $x^*(p, w)$ is nonempty and compact.

- We already proved this as well.

**Proof.**

Define $A$ by

$$A = B(p, w) = \{x \in \mathbb{R}_+^n : p \cdot x \leq w\}$$

- This is a closed and bounded (i.e. compact, set) and

  $$x^*(p, w) = C_\succeq (A) = C_\succeq (B(p, w))$$

  where $\succeq$ are the preferences represented by $u$.

- Then $x^*(p, w)$ is the set of maximizers of a continuous function over a compact set.
Walrasian Demand: Examples

How do we find the Walrasian Demand?

- Need to solve a constrained maximization problem, usually using calculus.

Question 6, Problem Set 2; due next Tuesday.

For each of the following utility functions, find the Walrasian demand correspondence. (Hint: pictures may help)

1. \( u(x) = \prod_{i=1}^{n} x_i^{\alpha_i} \) with \( \alpha_i > 0 \) (Cobb-Douglas).

2. \( u(x) = \min \{ \alpha_1 x_1, \alpha_2 x_2, \ldots, \alpha_n x_n \} \) with \( \alpha_i > 0 \) (generalized Leontief).

3. \( u(x) = \sum_{i=1}^{n} \alpha_i x_i \) for \( \alpha_i > 0 \) (generalized linear).

4. \( u(x) = \left[ \sum_{i=1}^{n} \alpha_i x_i^p \right]^{\frac{1}{p}} \) (generalized CES).

Can we do the second one using calculus?

How about the third? Do we need calculus?

Constant elasticity of substitution (CES) preferences are the most commonly used homothetic preferences. Many preferences are a special case of CES.
An Optimization Recipe

How to solve $\max f(x)$ subject to $g_i(x) \leq 0$ with $i = 1, \ldots, m$

1. Write the Lagrange function $L : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ as

$$L(x, \lambda) = f(x) - \sum_{i=1}^{m} \lambda_i g_i(x)$$

2. Write the First Order Conditions:

$$\nabla L(x, \lambda) = \nabla f(x) - \sum_{i=1}^{m} \lambda_i \nabla g_i(x) = 0$$

$$\frac{\partial f(x)}{\partial x_j} - \sum_{i=1}^{m} \lambda_i \frac{\partial g_i(x)}{\partial x_j} = 0 \text{ for all } j = 1, \ldots, n$$

3. Write constraints, inequalities for $\lambda$, and complementary slackness conditions:

$$g_i(x) \leq 0 \text{ with } i = 1, \ldots, m$$

$$\lambda_i \geq 0 \text{ with } i = 1, \ldots, m$$

$$\lambda_i g_i(x) = 0 \text{ with } i = 1, \ldots, m$$

4. Find the $x$ and $\lambda$ that satisfy all these and you are done...hopefully.
Compute Walrasian demand when the utility function is \( u(x_1, x_2) = x_1^\alpha x_2^{1-\alpha} \)

Here \( x^*(p, w) \) is the solution to

\[
\max_{x_1, x_2 \in \{p_1 x_1 + p_2 x_2 \leq w, \ x_1 \geq 0, \ x_2 \geq 0\}} x_1^\alpha x_2^{1-\alpha}
\]

1. The Langrangian is

\[
L(x, \lambda) = x_1^\alpha x_2^{1-\alpha} - \lambda_w (p_1 x_1 + p_2 x_2 - w) - (-\lambda_1 x_1) - (-\lambda_2 x_2)
\]

2. The First Order Conditions for \( x \) is:

\[
\nabla L(x, \lambda) = \begin{pmatrix}
\alpha x_1^{\alpha-1} x_2^{1-\alpha} - \lambda_w p_1 + \lambda_1 \\
(1 - \alpha) x_1^\alpha x_2^{-\alpha} - \lambda_w p_2 + \lambda_2
\end{pmatrix} = \begin{pmatrix}
\alpha \frac{u(x_1, x_2)}{x_1} - \lambda_w p_1 + \lambda_1 \\
(1 - \alpha) \frac{u(x_1, x_2)}{x_2} - \lambda_w p_2 + \lambda_2
\end{pmatrix} = 0
\]

3. The constraints, inequalities for \( \lambda \), and complementary slackness are:

\[
p_1 x_1 + p_2 x_2 - w \leq 0 \quad -x_1 \leq 0, \quad \text{and} \quad -x_2 \leq 0
\]

\[
\lambda_w \geq 0, \quad \lambda_1 \geq 0, \quad \text{and} \quad \lambda_2 \geq 0
\]

\[
\lambda_w (p_1 x_1 + p_2 x_2 - w) = 0, \quad \lambda_1 x_1 = 0, \quad \text{and} \quad \lambda_2 x_2 = 0
\]

4. Find a solution to the above (easy for me to say).
Compute Walrasian demand when the utility function is \( u(x_1, x_2) = x_1^\alpha x_2^{1-\alpha} \)

We must solve:

\[
\alpha \frac{u(x_1, x_2)}{x_1} - \lambda_w p_1 + \lambda_1 = 0 \quad \text{and} \quad (1 - \alpha) \frac{u(x_1, x_2)}{x_2} - \lambda_w p_2 + \lambda_2 = 0
\]

\[
p_1 x_1 + p_2 x_2 - w \leq 0
\]

\[-x_1 \leq 0, \quad -x_2 \leq 0
\]

\[
\lambda_w \left( p_1 x_1 + p_2 x_2 - w \right) = 0 \quad \text{and} \quad \lambda_w \geq 0, \quad \lambda_1 \geq 0, \quad \lambda_2 \geq 0
\]

- \( x^*(p, w) \) must be strictly positive (why?), hence \( \lambda_1 = \lambda_2 = 0 \).
- The budget constraint must bind (why?), hence \( \lambda_w \geq 0 \).
- Therefore the top two equalities become

\[
\alpha u(x_1, x_2) = \lambda_w p_1 x_1 \quad \text{and} \quad (1 - \alpha) u(x_1, x_2) = \lambda_w p_2 x_2
\]

- Summing both sides and using Full Expenditure we get

\[
u(x_1, x_2) = \lambda_w (p_1 x_1 + p_2 x_2) = \lambda_w w
\]

- Substituting back then yields

\[
x_1^*(p, w) = \frac{\alpha w}{p_1}, \quad x_2^*(p, w) = \frac{(1 - \alpha) w}{p_2}, \quad \text{and} \quad \lambda_w = \left( \frac{\alpha}{p_1} \right)^\alpha \left( \frac{1 - \alpha}{p_2} \right)^{1-\alpha}
\]
Next Week

- More Properties of Walrasian Demand.
- Indirect Utility.
- Comparative Statics.
- Expenditure Minimization.