Preference and Utility

Econ 2100          Fall 2017

Lecture 3, 5 September

- Problem Set 1 is due in Kelly’s mailbox by 5pm today

Outline

1. Existence of Utility Functions
2. Continuous Preferences
3. Debreu’s Representation Theorem
Definitions From Last Week

- A binary relation \(\preceq\) on \(X\) is a **preference relation** if it is a weak order, i.e. complete and transitive.

- The **upper contour set** of \(x\) is \(\preceq (x) = \{y \in X : y \preceq x\}\).

- The **lower contour set** of \(x\) is \(\preceq (x) = \{y \in X : x \preceq y\}\).

- The utility function \(u : X \to \mathbb{R}\) **represents** the binary relation \(\succeq\) on \(X\) if
  \[ x \succeq y \iff u(x) \geq u(y). \]

**Question:** Under what assumptions can a preference relation be represented by a utility function?

- We know we need transitivity and continuity. Are they enough?
The following provides an alternative way to show that a preference is represented by some function (very useful).

**Question 1, Problem Set 2 (due Tuesday September 12).**

Let $\succeq$ be a preference relation. Prove that $u : X \to \mathbb{R}$ represents $\succeq$ if and only if:

$$x \succeq y \Rightarrow u(x) \geq u(y); \quad \text{and} \quad x \succ y \Rightarrow u(x) > u(y).$$

This result is useful as it makes it (sometimes) easier to show some utility function represents $\succeq$. 
Existence of a Utility Function When $X$ is Finite

**Theorem**

Suppose $X$ is finite. Then $\succsim$ is a preference relation if and only if there exists some utility function $u : X \to \mathbb{R}$ that represents $\succsim$.

- When the consumption set is finite finding a utility function that represents any given preferences is easy.
- The proof is constructive: a function that works is the one that counts the number of elements that are not as good as the one in question.
  - This function is well defined since there are only finitely many items that can be worse than something.
- In other words, the utility function is
  \[ u(x) = | \succsim(x) | \]
  where $\succsim(x) = \{ y \in X : x \succsim y \}$ is the lower contour set of $x$, and $| \cdot |$ denotes the cardinality of $\cdot$.
- Notice that we only need to prove that this function represents $\succsim(x)$ because from a previous result we know that if that is the case then $\succsim$ is a preference relation.
Existence of a Utility Function When $X$ is Finite

If $X$ is finite and $\succeq$ is a preference relation $\Rightarrow \exists u : X \rightarrow \mathbb{R}$ that represents $\succeq$.

**Proof.**

Let $u(x) = |\succeq (x)|$. Since $X$ is finite, $u(x)$ is finite and therefore well defined.

- Suppose $x \succeq y$. I claim this implies $u(x) \geq u(y)$.
  - Let $z \in \succeq (y)$, i.e. $y \succeq z$.
  - By transitivity, $x \succeq z$, i.e. $z \in \succeq (x)$. Thus $\succeq (y) \subset \succeq (x)$.
  - Therefore $|\succeq (y)| \leq |\succeq (x)|$. By definition, this means $u(y) \leq u(x)$.

- Now suppose $x \succ y$. I claim this implies $u(x) > u(y)$.
  - $x \succ y$ implies $x \succeq y$, so the argument above implies $\succeq (y) \subset \succeq (x)$.
  - $x \preceq x$ by completeness, so $x \in \succeq (x)$. Also, $x \succ y$ implies $x \notin \succeq (y)$.
  - Hence $\succeq (y)$ and $\{x\}$ are disjoint, and both subsets of $\succeq (x)$. Then
    \[
    \succeq (y) \cup \{x\} \subset \succeq (x)
    \]
    \[
    |\succeq (y) \cup \{x\}| \leq |\succeq (x)|
    \]
    \[
    |\succeq (y)| + |\{x\}| \leq |\succeq (x)|
    \]
    \[
    u(y) + 1 \leq u(x)
    \]
    \[
    u(y) < u(x)
    \]

- This proves $x \succeq y \Rightarrow u(x) \geq u(y)$ and $x \succ y \Rightarrow u(x) > u(y)$ and thus we are done.

$\blacksquare$
Existence of a utility function can also be proven for a countable space.

**Theorem**

Suppose $X$ is countable. Then $\succsim$ is a preference relation if and only if there exists some utility function $u : X \rightarrow \mathbb{R}$ that represents $\succsim$.

**Proof.**

Question 2, Problem Set 2

- Notice that the previous construction cannot be applied directly here because the cardinality of the lower contour sets can be infinite.
- You will have to come up with a “trick” that works.
Lexicographic Preferences

- We want conditions on a preference relation which guarantee the existence of a utility function representing those preferences even if $X$ is uncountable.
  - Typically, we assume that $X$ lives in $\mathbb{R}^n$, but ideally we would want as few restrictions on $X$ as we can get away with.
- We know that without completeness and transitivity a utility function does not exist, so we cannot dispense of those.
- What else is needed?
- Next, a counterexample of the existence of a representation.

**Exercise**

Show that the lexicographic ordering on $\mathbb{R}^2$ defined by

$$(x_1, x_2) \preceq (y_1, y_2) \iff \begin{cases} x_1 > y_1 \\ x_1 = y_1 \text{ and } x_2 \geq y_2 \end{cases}$$

is complete, transitive, and antisymmetric (i.e. if $x \preceq y$ and $y \preceq x$, then $x = y$).

- The lexicographic ordering is a preference relation (it is complete and transitive), yet admits no utility representation.
The Lexicographic Ordering in $\mathbb{R}^2$ admits no utility representation

Let $X = \mathbb{R}^2$ and define $\succeq$ by $(x_1, x_2) \succeq (y_1, y_2) \iff \begin{cases} x_1 > y_1 \\ or \\ x_1 = y_1 \text{ and } x_2 \geq y_2 \end{cases}$

- Suppose there exists a utility function $u$ representing $\succeq$ (find a contradiction).
- For any $x_1 \in \mathbb{R}$, $(x_1, 1) \succeq (x_1, 0)$ and thus $u(x_1, 1) > u(x_1, 0)$ since $u$ represents $\succeq$.
- The rational numbers are dense in the real line (for any open interval $(a, b)$, there exists a rational number $r$ such that $r \in (a, b)$); hence, there exists a rational number $r(x_1) \in \mathbb{R}$ such that:

$$u(x_1, 1) > r(x_1) > u(x_1, 0)$$

(A)

- Define $r : \mathbb{R} \to \mathbb{R}$ by selecting $r(x_1)$ to satisfy $u(x_1, 1) > r(x_1) > u(x_1, 0)$.
- CLAIM: $r(\cdot)$ is a one-to-one function.
  - Suppose $x_1 \neq x_1'$; and without loss of generality assume $x_1 > x_1'$.
  - Then: $r(x_1) \succeq u(x_1, 0) \succeq u(x_1', 1) \succeq r(x_1')$

Thus $r(\cdot)$ is an a one-to-one function from the real numbers to the rational numbers, which contradicts the fact the real line is uncountable.
Continuous Preferences

- Continuity will get rid of this example.

**Definition**

A binary relation \( \preceq \) on the metric space \( X \) is **continuous** if, for all \( x \in X \), the upper and lower contour sets, \( \{ y \in X : y \preceq x \} \) and \( \{ y \in X : x \preceq y \} \), are closed.

**Examples**

- Define \( \preceq \) by \( (x_1, x_2) \preceq (y_1, y_2) \iff \begin{cases} x_1 > y_1 \\
\text{or} \\
x_1 = y_1 \text{ and } x_2 \geq y_2 \end{cases} \)
  - Draw the upper contour set of \( (1, 1) \); this preference on \( \mathbb{R}^2 \) is not continuous.

- Define \( \preceq \) by \( (x_1, x_2) \preceq (y_1, y_2) \iff \begin{cases} x_1 \geq y_1 \\
\text{and} \\
x_2 \geq y_2 \end{cases} \)
  - Draw the upper contour set of \( (1, 1) \); this preference on \( \mathbb{R}^2 \) is continuous.
**Continuous Preferences**

**Definition**

A binary relation $\succeq$ on the metric space $X$ is **continuous** if, for all $x \in X$, the upper and lower contour sets, $\{y \in X : y \succeq x\}$ and $\{y \in X : x \succeq y\}$, are closed.

**Proposition**

A binary relation $\succeq$ is continuous if and only if:

1. If $x_n \succeq y$ for all $n$ and $x_n \to x$, then $x \succeq y$; and
2. If $x \succeq y_n$ for all $n$ and $y_n \to y$, then $x \succeq y$.

**Proof.**

This follows from the fact that a set is closed if and only if it contains all of its limit points.
Debreu’s Representation Theorem

The main result of today is due to Gerard Debreu, and it provides necessary and sufficient conditions for the existence of a continuous utility function.

**Theorem (Debreu)**

Suppose $X \subset \mathbb{R}^n$. The binary relation $\succeq$ on $X$ is complete, transitive, and continuous if and only if there exists a continuous utility representation $u : X \rightarrow \mathbb{R}$.

- We will prove sufficiency next (under a couple of simplifying assumptions, but you should check what happens without those).
- You will do necessity as homework.

**Question 4, Problem Set 2.**

Suppose $X \subset \mathbb{R}^n$. Prove that if $u : X \rightarrow \mathbb{R}$ is a continuous utility function representing $\succeq$, then $\succeq$ is a complete, transitive, and continuous preference relation.
Debreu’s Representation Theorem

**Theorem (Debreu)**

Suppose $X \subset \mathbb{R}^n$. The binary relation $\succeq$ on $X$ is complete, transitive, and continuous if and only if there exists a continuous utility representation $u : X \rightarrow \mathbb{R}$.

- Two simplifying assumptions:
  - $X = \mathbb{R}^n$; and
  - $\succeq$ is strictly monotone, (i.e. if $x_i \geq y_i$ for all $i$ and $x \neq y$, then $x \succ y$).
    - When strict monotonicity holds we have
      $$\alpha \geq \beta \iff \alpha e \succeq \beta e.$$ (mon)
    - Where $e = (1, 1, \ldots, 1)$ (make sure you check this).

To prove:

if $\succeq$ is a continuous and strictly monotone preference relation on $\mathbb{R}^n$, then there exists a continuous utility representation of $\succeq$.

- How do we find a utility function? Look at the point on the $45^\circ$ line that is indifferent:
  $$u(x) = \alpha^*(x)$$ where $\alpha^*(x)$ is the real number $\alpha^*$ such that $\alpha^* e \sim x$

- The proof is in 3 steps.
Step 1: There exists a unique $\alpha^*(x)$ such that $\alpha^* e \sim x$.

**Proof.**

Let $B = \{ \beta \in \mathbb{R} : \beta e \succcurlyeq x \} \subset \mathbb{R}$ and define

$$
\alpha^* = \inf \left\{ \beta \in \mathbb{R} : \beta e \succeq x \right\}. 
$$

- Oviously, $(\max_i x_i)e \geq x$, so by strict monotonicity $(\max_i x_i)e \succeq x$ which implies $B$ is nonempty and $\alpha^*$ is well-defined.

- Now show that $\alpha^* e \succeq x$ and $\alpha^* e \preceq x$, so that $\alpha^* e \sim x$.
  - Since $\alpha^*$ is the infimum of $B$, there exists a sequence $\beta_n \in B$ s.t. $\beta_n \to \alpha^*$.
  - Then $\beta_n e \to \alpha^* e$ (in $\mathbb{R}^n$) and $\beta_n e \succeq x$ because $\beta \in B$.
  - By continuity, one gets: $\alpha^* e \succeq x$ as desired.
  - Since $\alpha^*$ is a lower bound of $B$, if $\alpha \in B \Rightarrow \alpha \geq \alpha^*$.
  - Using the contrapositive:
    $$
    \alpha < \alpha^* \Rightarrow \alpha e < x. \quad \text{(A)}
    $$
  - Let $\alpha_n = \alpha^* - \frac{1}{n}$. By (A), $\alpha_n e \prec x$, so $\alpha_n e \preceq x$.
  - Also, $\alpha_n e \to \alpha^* e$ (in $\mathbb{R}^n$). Hence, by continuity, $\alpha^* e \preceq x$ as desired.

- To prove uniqueness, suppose $\hat{\alpha} e \sim x$.
  - By transitivity, $\hat{\alpha} e \sim \alpha^* e$. Then, by (mon), $\hat{\alpha} = \alpha^*$. \qed
Step 2: The $u(x)$ that represents $\succeq$ is defined as follows:

$$u(x) = \alpha^*(x) \quad \text{where } \alpha^*(x) \text{ is the unique } \alpha^* \text{ such that } \alpha^*e \sim x.$$ 

**Proof.**

Show that $u(x)$ represents $\succeq$.

- Suppose $x \succeq y$.
  - By construction of $\alpha^*$, we have:
    $$x \sim \alpha^*(x)e \quad \text{and} \quad \alpha^*(y)e \sim y$$
  - By transitivity, we have:
    $$x \sim \alpha^*(x)e \succeq \alpha^*(y)e \sim y$$
  - By (mon) this is equivalent to
    $$\alpha^*(x) \geq \alpha^*(y)$$

- Repeat the same argument to show that $x \succ y$ implies $\alpha^*(x) > \alpha^*(y)$.

- Since we have shown that $x \succeq y \Rightarrow u(x) \geq u(y)$ and $x \succ y \Rightarrow u(x) > u(y)$, we are done.
Step 3: The defined $u(x)$ is continuous.

$u(x) = \alpha^*(x)$ where $\alpha^*(x)$ is the unique $\alpha^*$ such that $\alpha^*e \sim x$

**Proof.**

To prove that a function $f : \mathbb{R}^n \to \mathbb{R}$ is continuous, it suffices to show that $f^{-1}((a, b))$ is open for all $a, b \in \mathbb{R}$.

- Since $ae \sim ae$, we have $u(ae) = \alpha^*(ae) = a$ for any $a \in \mathbb{R}$.
- Notice that
  
  $u^{-1}((a, b)) = u^{-1}((a, \infty) \cap (-\infty, b)) = u^{-1}((a, \infty)) \cap u^{-1}((-\infty, b))$.

- But $u(ae) = a$, so
  
  $u^{-1}((a, \infty)) = u^{-1}((u(ae), \infty)) = \{x \in \mathbb{R}^n : x \succ ae\}$, this is open because the strict upper contour set of $ae$ is open whenever $\succ$ is complete and continuous (why?).

- An entirely symmetric argument proves that $u^{-1}((-\infty, b))$ is the strict lower contour set of $be$, hence it is also an open set.

- Since $u^{-1}((a, b))$ is the intersection of two open sets, it is open. □
Debreu’s Representation Theorem

- Back to the theorem

**Theorem (Debreu)**

Suppose $X \subset \mathbb{R}^n$. The binary relation $\succeq$ on $X$ is complete, transitive, and continuous if and only if there exists a continuous utility representation $u : X \to \mathbb{R}$.

**Conclusion**

- When her preference relation is complete, transitive, and continuous the consumer’s taste is entirely described by some continuous utility function that represents that preference relation.

- The theorem asserts that one of the utility representations for $\succeq$ must be continuous, not that all of the utility representations must be continuous.

- Continuity is a cardinal feature of the utility function, not an ordinal feature, since it is not robust to strictly increasing transformations.

**Exercise**

Construct a preference relation on $\mathbb{R}$ that is not continuous, but admits a utility representation.
The induced choice rule for $\succeq$ is $C_\succeq(A) = \{x \in A : x \succeq y \text{ for all } y \in A\}$.

**Proposition**

If $\succeq$ is a continuous preference relation and $A \subset \mathbb{R}^n$ is nonempty and compact, then $C_\succeq(A)$ is nonempty and compact.

**Proof.**

Suppose $\succeq$ is continuous.

- By Debreu’s Theorem, there exists some continuous function $u$ representing the preferences.
- One can show (do it as exercise) that
  
  $$C_\succeq(A) = \arg \max_{x \in A} u(x).$$

- Nonemptiness then follows from continuity of $u$ and the Extreme Value Theorem.
- Compactness follows from the fact $u^{-1}(\cdot)$ is bounded (it is a subset of the bounded set $A$), and closed (since the inverse image of a closed set under a continuous function is closed).
The main idea is to understand the connection between properties of preferences and characteristics of the utility function that represents them.

**NOTATION:**
- From now on, assume $X = \mathbb{R}^n$.
- Remember the notation for vectors: if $x_i \geq y_i$ for each $i$, we write $x \succeq y$. 
Monotonicity

- Monotonicity says more is better.

**Definitions**

- A preference relation $\succsim$ is **weakly monotone** if $x \geq y$ implies $x \succsim y$.
- A preference relation $\succsim$ is **strictly monotone** if $x \geq y$ and $x \neq y$ imply $x \succ y$.

In our notation, $x \geq y$ and $x \neq y$ imply $x_i > y_i$ for some $i$.

**Question**

What does monotonicity imply for the utility function representing $\succsim$?
Suppose $\succeq$ is the preference relation on $\mathbb{R}^2$ defined by
\[ x \succeq y \text{ if and only if } x_1 \geq y_1. \]

- $\succeq$ is weakly monotone, because if $x \succeq y$, then $x_1 \geq y_1$.
- It is not strictly monotone, because
  \[ (1, 1) \succeq (1, 0) \text{ and } (1, 1) \nprec (1, 0) \]
- but
  \[ \text{not } [(1, 1) \succ (1, 0)] \]
  since $(1, 0) \preceq (1, 1)$.
The lexicographic preference on $\mathbb{R}^2$ is strictly monotone

**Proof:** Suppose $x \geq y$ and $x \neq y$. Then there are two possibilities:

(a) : $x_1 > y_1$ and $x_2 \geq y_2$ or (b) : $x_1 \geq y_1$ and $x_2 > y_2$.

- If (a) holds, then
  - $x \succcurlyeq y$ because $x_1 \geq y_1$, and
  - not ($y \succcurlyeq x$) because neither $y_1 > x_1$ (excluded by $x_1 > y_1$) nor $y_1 = x_1$ (also excluded by $x_1 > y_1$).

- If (b) holds, then
  - $x \succcurlyeq y$ because either $x_1 > y_1$ or $x_1 = y_1$ and $x_2 \geq y_2$, and
  - not ($y \succcurlyeq x$) because not ($y_1 > x_1$) (excluded by $x_1 \geq y_1$) and not ($y_2 \geq x_2$) (excluded by $x_2 > y_2$).

- In both cases, $x \succcurlyeq y$ and not ($y \succcurlyeq x$) thus $x \succ y$. 

Strict Monotonicity: An Example
Monotonicity and Utility Functions

**Definitions**

A function \( f : \mathbb{R}^n \rightarrow \mathbb{R} \) is

- **nondecreasing** if \( x \geq y \) implies \( f(x) \geq f(y) \);
- **strictly increasing** if \( x \geq y \) and \( x \neq y \) imply \( f(x) > f(y) \).

Monotonicity is equivalent to the corresponding utility function being nondecreasing or increasing.

**Proposition**

If \( u \) represents \( \preceq \), then:

1. \( \preceq \) is weakly monotone if and only if \( u \) is nondecreasing;
2. \( \preceq \) is strictly monotone if and only if \( u \) is strictly increasing.

**Proof.**

Question 5a. Problem Set 2.