Lecture 1


2015 August 10
Lecture 1 Outline

2. Methods of Proof
3. Sets
4. Binary Relations
5. Functions

Class Website: http://www.pitt.edu/~luca/ECON2001/
Who, Where, When, What

- **Luca Rigotti** (me); office WWPW 4905, email luca@pitt.edu
- **Eric Duerr** (opposite end of the room), our Teaching Assistant.
- Classes and Sections meet daily in **WWPH 4716** (this room) at **10:30am** and **3pm** respectively (with some changes).
- This course covers the following vaguely defined topics:
  - Analysis
  - Linear Algebra
  - Multivariate Calculus and Unconstrained Optimization
- The fall math class (Mathematical Methods of Economic Analysis, taught by Roee Teper) will start where we end: constrained optimization.
  - I strongly encourage anyone who wants to take any Ph.D. course (micro, macro, or metrics) to also take that course (econ ph.d. students have no choice). If that is not possible, you should at least sit in for the first few lectures. Without those, you will feel very lost in other courses.
- Math camp is a review course: the amount of material covered is large, and if you are not already somewhat familiar with it, you may be in trouble. Please, talk to me if that is the case.
How

- Two daily meetings.
  - Lectures in the morning.
    - The official schedule says 75 minutes, but I will use more time in many/most cases, so plan for lectures to go for about 90 minutes.
    - I will bring handouts; you should read (last year) posted handouts before class.
  - Interactive review sessions in the afternoon.
    - Eric will present some material and problems, and then you will take turns at the board while working as a group.

- Learning in this class, like other Ph.D. courses, means doing much more than just reading book and notes, and listening to lectures.

  Learning by doing: you can only learn that which you practice.

- Working in groups is strongly encouraged... but always try to work through all problems on your own before meeting with others.
  - A great test of understanding: can you explain it to others?

- This room (4716) is reserved for you all day long. Use it as a place to work together, with or without Eric or me.
Some Friendly Advice

- To learn this sort of mathematics (and Ph.D. level economics) you need to do much more than just read the material and listen to lectures. Learning by doing.

Wait, I said this already...

- Suggestions for this, and other, Ph.D. courses (not in order of importance):
  - problem solving (Problem Sets!) is the most valuable part of any course;
    - DO NOT LOOK FOR/AT ANSWERS. The point is not to get it right, but to learn what you can do by yourself and what you need to work on.
  - active reading: work through each line of a proof, be sure you understand how to get from one line to the next;
    - memorizing is no use: it will not help you do a proof you have not seen before.
  - active listening: follow each step as we work through arguments in class.

Ultimate Objective

- You are here to produce original research, not to repeat what others have done.
- Mastery of the tools necessary to produce original research is the objective.
**Why**

- This class is designed for incoming 1\textsuperscript{st} year Ph.D. students in economics.
- Therefore, we will cover mathematics that is useful during the economics Ph.D.
  - Lots of what happens in the first year classes takes the form: “state a (mathematical) result and then prove it.”
  - The main goal is for you to practice how to think in rigorous terms: assumptions, statement, proof.
- In the next 3 weeks I would you to:
  - develop math skills and knowledge needed to read academic economics papers and produce research as a professional economist;
  - develop the ability to read and evaluate proofs (is it correct?); this is **essential** to understand any branch of economics (theoretical, empirical, experimental);
  - develop the ability to compose simple proofs; this is also essential for contributing to all branches of economics.
- For the most part, I do not expect you to learn new things since this class is intended as a review.
Course requirements:

- Three exams, one at the end of each week:
  - each of these weekly “short” tests will focus on the material covered that week.
- The class is graded pass/fail.

Resources:

- *Mathematical Methods and Models for Economists*, by Angel de la Fuente
- *Mathematics for Economists*, by Carl P. Simon and Lawrence E. Blume
- [http://www.econ.ucsd.edu/~jsobel/205f10/205f10home.htm](http://www.econ.ucsd.edu/~jsobel/205f10/205f10home.htm)
- [http://elsa.berkeley.edu/users/cshannon/e204_11.html](http://elsa.berkeley.edu/users/cshannon/e204_11.html)
- Seek out other references (lots of classes like this one)

Questions?

Ready, Set, Go!
Simple Logic

- $P$ means “$P$ is true”, $\neg P$ means “$P$ is false” (or “not $P$”).
- $P \land Q$ means “$P$ is true and $Q$ is true.”
- $P \lor Q$ means “$P$ is true or $Q$ is true (or possibly both).”
- $\neg P \land Q$ stands for $(\neg P) \land Q$; $\neg P \lor Q$ stands for $(\neg P) \lor Q$.

$P \Rightarrow Q$ means “whenever $P$ is true, $Q$ also holds.” (if $P$ then $Q$)
- this is sometimes called sufficient condition
- $P \Rightarrow Q$ is equivalent to $\neg P \lor Q$ (do you see why?).

$P \Leftarrow Q$ means “whenever $Q$ is satisfied, $P$ is also satisfied.”
- this is sometimes called necessary condition
- Clearly, $P \Leftarrow Q$ is another way of writing $Q \Rightarrow P$.

$P \iff Q$ means “$P$ is true if and only if (iff) $Q$ is true.”
- this is sometimes called necessary and sufficient condition
- $\iff$ is a fusion of the symbols $\Rightarrow$ and $\Leftarrow$.
- To prove an if and only if statement, one must prove the implication both ways.
Proofs

Mathematicians often collect information and make observations about particular cases or phenomena in an attempt to form a theory (a model) that describes patterns or relationships among quantities and structures. This approach to the development of a theory uses inductive reasoning.

However, the characteristic thinking of the mathematician is deductive reasoning, in which one uses logic to develop and extend a theory by drawing conclusions based on statements accepted as true.

Proofs are essential in mathematical reasoning because they demonstrate that the conclusions are true. Generally speaking, a mathematical explanation for a conclusion has no value if the explanation cannot be backed up by an acceptable proof.

Smith, Eggen, Andre (Replace mathematician with economist)

- A proof is not an example. An example only shows the statement can be true given the assumptions, while a proof shows the statement is always true given the assumptions.

- An example can be used to show that a statement is false; this is called a counterexample.
Methods of Proof

The following are the main methods to approach a proof:

- deduction,
- contraposition,
- induction,
- contradiction.

We will examine each in turn.
Proof by Deduction

Proof by Deduction:
A list of statements, the last of which is the statement to be proven.

Each statement in the list is either

- an axiom: a fundamental assumption about mathematics; or
- part of the definition of the object under study; or
- a previously established theorem; or
- follows from previous statements in the list by a valid rule of inference.
Proof by Deduction: An Example

Prove that the function $f(x) = x^2$ is continuous at $x = 5$.

Recall from one-variable calculus: $f(x) = x^2$ continuous at $x = 5$ means

$$\forall \varepsilon > 0 \ \exists \delta(\varepsilon) > 0 \ \text{s.t.} \ |x - 5| < \delta \Rightarrow |f(x) - f(5)| < \varepsilon$$

In words: “for every strictly positive epsilon, there exists a strictly positive delta (possibly dependent on epsilon) such that if the distance between $x$ and 5 is strictly less than delta, then the distance between $f(x)$ and $f(5)$ is strictly less than epsilon.”

To prove the claim, we must systematically verify that this definition is satisfied.

**Proof.**

Let $\varepsilon > 0$ be given. Pick

$$\delta(\varepsilon) = \min \left\{ 1, \frac{\varepsilon}{11} \right\} > 0$$

Suppose $|x - 5| < \delta$. Since $\delta(\varepsilon) \leq 1$, then $4 < x < 6$, so $9 < x + 5 < 11$ and $|x + 5| < 11$.

$$|f(x) - f(5)| = |x^2 - 25| = |(x + 5)(x - 5)| = |x + 5||x - 5|$$

$$< 11 \cdot \delta(\varepsilon)$$

$$\leq 11 \cdot \frac{\varepsilon}{11}$$

$$= \varepsilon$$

Thus, we have shown that for every $\varepsilon > 0$, there exists $\delta > 0$ such that $|x - 5| < \delta \Rightarrow |f(x) - f(5)| < \varepsilon$, and therefore $f$ is continuous at $x = 5$. \qed
Proof by Contraposition

Contrapositive

The contrapositive of the statement \( P \Rightarrow Q \) is the statement \( \neg Q \Rightarrow \neg P \).

Theorem

\( P \Rightarrow Q \) is true if and only if \( \neg Q \Rightarrow \neg P \) is true.

Proof.

\( \Rightarrow \)

- Suppose \( P \Rightarrow Q \) is true. We need to show that \( \neg Q \Rightarrow \neg P \) is true.
  - \( P \Rightarrow Q \) is equivalent to \( \neg P \lor Q \); so either \( P \) is false, or \( Q \) is true (or both).
  - Therefore, either \( \neg P \) is true, or \( \neg Q \) is false (or both).
  - So, \( \neg(\neg Q) \lor (\neg P) \) is true, that is, \( \neg Q \Rightarrow \neg P \) is true.

\( \Leftarrow \)

- Suppose \( \neg Q \Rightarrow \neg P \) is true. We need to show that \( P \Rightarrow Q \) is true.
  - \( \neg Q \Rightarrow \neg P \) is equivalent to \( \neg(\neg Q) \lor \neg P \);
  - so either \( \neg Q \) is false, or \( \neg P \) is true (or both),
  - so either \( Q \) is true, or \( P \) is false (or both),
  - therefore \( \neg P \lor Q \) is true, so \( P \Rightarrow Q \) is true.
Proof by Contraposition

Theorem

Given two real numbers \( x \) and \( y \),

\[
\forall \varepsilon > 0, \ x \leq y + \varepsilon \Rightarrow x \leq y
\]

\[
\forall \varepsilon > 0 \quad x \leq y + \varepsilon \quad \Rightarrow \quad x \leq y
\]

First, find the contrapositive:

\[
\neg Q \quad \Rightarrow \quad \neg P
\]

\[
x > y \quad \Rightarrow \quad \exists \varepsilon > 0 \text{ such that } x > y + \varepsilon
\]

Proof.

Suppose \( x > y \). Then \( x - y > 0 \); let \( \varepsilon = \frac{x - y}{6} \) so that we have

\[
y + \varepsilon < y + (x - y) = x
\]

as desired.
Proof by Induction: An Example

**Theorem**

For every \( n \in \mathbb{N}_0 = \{0, 1, 2, 3, \ldots\} \), \[ \sum_{k=0}^{n} k = \frac{n(n+1)}{2} \] equivalently: \( 0 + 1 + 2 + \cdots + n = \frac{n(n+1)}{2} \).

**Proof.**

1. **Base step** check \( n = 0 \):
   
   \[ \text{LHS} = \sum_{k=0}^{0} k = \text{the empty sum} = 0. \quad \text{RHS} = \frac{0 \cdot 1}{2} = 0. \] So, the claim holds for \( n = 0 \).

2. **Induction step** (also called induction hypothesis): Suppose the claim is true for \( n \)
   
   \[ \sum_{k=0}^{n} k = \frac{n(n+1)}{2} \] for some \( n > 0 \).

3. **Show that the claim holds for** \( n + 1 \); we need to show that: \[ \sum_{k=0}^{n+1} k = \frac{(n+1)((n+1)+1)}{2} \]

   Note that: \( \frac{(n+1)((n+1)+1)}{2} = (n + 1)\frac{(n+2)}{2} \). Now
   
   \[ \sum_{k=0}^{n+1} k = \sum_{k=0}^{n} k + (n + 1) \]
   
   \[ = \frac{n(n+1)}{2} + (n + 1) \]
   
   \[ = (n + 1)\left(\frac{n}{2} + 1\right) = (n + 1)\frac{(n + 2)}{2} \]

   by the Induction hypothesis

So, by mathematical induction, \( \sum_{k=0}^{n} k = \frac{n(n+1)}{2} \) for all \( n \in \mathbb{N}_0 \). \( \square \)
Idea of Proof by Contradiction

Assume the negation of what is claimed, and work toward a contradiction with some of the assumptions (or some already established result).

- This is used a lot, almost as much as deduction.
- These proofs start with “suppose not”, so one must understand how to negate the statement.


**Proof by Contradiction: An Example**

### Theorem

There is no rational number \(q\) such that \(q^2 = 2\).

### Proof.

**Negation of the Claim**

Suppose not. Then \(q \in \mathbb{Q}\) (\(q\) is rational) and \(q^2 = 2\).

- By definition of \(\mathbb{Q}\), we can write \(q = \frac{m}{n}\) for some integers \(m, n \in \mathbb{Z}\).
  - Without loss of generality, assume that \(m\) and \(n\) have no common factor (why wlog? if they have a common factor, one can divide it out).
- Hence: \(2 = q^2 = \frac{m^2}{n^2}\) and therefore, \(m^2 = 2n^2\), so \(m^2\) is even.
- **Claim:** if \(m^2\) is even \(m\) must also be even.
  - **Proof (by contradiction):** Problem Set 1.
- Since \(m\) is even, \(m = 2r\) for some \(r \in \mathbb{Z}\). Thus
  
  \[
  4r^2 = (2r)^2 = m^2 = 2n^2 \quad \text{or} \quad n^2 = 2r^2
  \]

- So \(n^2\) is even; therefore, \(n\) is also even.
- Hence, \(n = 2s\) for some \(s \in \mathbb{Z}\).
- Thus, \(m\) and \(n\) have a common factor, contradicting the assumption that \(q = \frac{m}{n}\) where \(m\) and \(n\) have no common factor.

**Contradiction**

Therefore, the claim holds: there is no rational number \(q\) such that \(q^2 = 2\).
Sets

**Definition**

A set $X$ is a collection of objects.

- We write $x \in X$ to mean “$x$ is an element of $X$ ”.
- If $x$ is not a member of $X$, we write $x \notin X$.
- The symbol $\emptyset$ denotes the empty set.
- If a set contains at least one element, it is called nonempty.

**Definition**

If $A$ and $B$ are sets, and if for every $x \in A$ we have $x \in B$, we say that $A$ is a subset of $B$, and write $A \subseteq B$, or $B \supseteq A$.

If, in addition, $\exists x \in B$ such that $x \notin A$, then $A$ is said to be a proper subset of $B$.

- Note that $A \subseteq A$ for every set $A$. 
Set Operations

**Definition (Set Equality)**

Given two sets $A$ and $B$:

\[ A = B \iff A \subseteq B \text{ and } A \supseteq B \]

Equivalently:

\[ x \in A \iff x \in B \]

**Definitions**

The **union** of two sets $A$ and $B$ is a new set defined as follows:

\[ A \cup B = \{ x : x \in A \text{ or } x \in B \} \]

The **intersection** of two sets $A$ and $B$ is a new set defined as follows:

\[ A \cap B = \{ x \in A : x \in B \} \]

The **difference** of two sets $A$ and $B$ is a new set defined as follows:

\[ A \setminus B = \{ x \in A : x \notin B \} \]
Power Set

Definition
Given a set $X$, the power set of $X$, denoted $2^X$, is the collection of all subsets of $X$.

Example
Given the set $\mathbb{N}$ of all natural numbers, $2^\mathbb{N}$ is the set of all subsets of $\mathbb{N}$, Thus, $\emptyset$, \{1, 2\}, \{2, 4, 6, \ldots\} are elements of $2^\mathbb{N}$.

- We can also form $2^{2^\mathbb{N}} = 2(2^\mathbb{N})$, the set of all subsets of the set of all subsets of the natural numbers; $2^{2^{2^\mathbb{N}}}$ and so on.

Question
- Is the power set always a set?

Russell’s Paradox: if the collection of all sets is always a set $\Omega$, then one can form $E = \{X \in \Omega : X \notin X\}$, the set of all sets which are not elements of themselves. Is $E \in E$? If so, then $E \notin E$, contradiction; if not, then $E \in E$, again a contradiction.

- Moral of the story: one has to be careful in forming sets (see Zermelo–Fraenkel set theory).
Describing Sets

Let $X$ be any set, and $\mathcal{P}(x)$ a mathematical statement about a variable $x$. Then

$$\{x \in X : \mathcal{P}(x)\}$$

is also set; it is the collection of all elements $x$ of $X$ such that the statement $\mathcal{P}(x)$ is true.

- The statement $\mathcal{P}$ can be complex. In particular, it can include quantifiers.

Examples

- $\{x \in [0, 1] : \forall y \in [0, 1] \ x \geq y\}$ is a valid set; it equals $\{1\}$.
- $\{x \in (0, 1) : \forall y \in (0, 1) \ x > y\}$ is also a valid set; it equals the empty set.
- The set of all upper bounds for $X \subseteq 2^\mathbb{R}$ is $U = \{u \in \mathbb{R} : u \geq x \ \forall x \in X\}$

In practice, the things we economists do are always legal. You can apply the power set construction an arbitrary finite number of times, and use quantifiers of the form $\forall x \in X$ as long as $X$ is a set formed by taking at most a finite number of applications of the power set operation. Thus, $2^{(2^\mathbb{R})}$ is fine.
A function $f$ from a set $X$ to a set $Y$ is a mapping that assigns to each element of $X$ exactly one element of $Y$.

- We write $f(x)$ as the point in $Y$ that the function associates with the point $x \in X$.
- We write that $f$ is a function from $X$ to $Y$ by writing $f : X \rightarrow Y$.

- The set $X$ (the points one “plugs into” the function) is called the function’s **domain**;
- the set $Y$ (the items that one can get out of the function) is called the function’s **codomain** or **range**.

Example (One can use functions to define sets)

Let $[a, b] \subset \mathbb{R}$, and suppose $f : [a, b] \rightarrow \mathbb{R}$, then

$$\{ t \in [a, b] : f(t) < 7 \}$$

is a set; it consists of all those elements $t$ in the interval $[a, b]$ such that $f(t) < 7$. 
Classes of Functions

**Definition**

A function $f : X \rightarrow Y$ is **onto** if $\forall y \in Y$, $\exists x \in X$ such that $y = f(x)$.

- The fact that a function is onto is sometimes written as $f(X) = Y$ (image=range).

**Definition**

A function $f : X \rightarrow Y$ is said to be **one-to-one** if $f(x) = f(x')$ implies that $x = x'$.

**Result**

An equivalent definition of one-to-one function is that for every two points $x, x' \in X$ such that $x \neq x'$, it is the case that $f(x) \neq f(x')$.

Prove this as an exercise.

**Definition**

A **bijection** is a function $f : A \rightarrow B$ that is one-to-one, and onto.
Cardinality

**Definition**

Two sets $A, B$ are **numerically equivalent** (or have the same cardinality) if there is a bijection $f : A \rightarrow B$.

- Remember, a bijection is a function $f : A \rightarrow B$ that is one-to-one $(a \neq a' \Rightarrow f(a) \neq f(a'))$, and onto $(\forall b \in B \exists a \in A$ such that $f(a) = b$).

**Example**

$A = \{2, 4, 6, \ldots, 50\}$ is numerically equivalent to the set $\{1, 2, \ldots, 25\}$ under the function $f(n) = 2n$.

**Example**

$B = \{1, 4, 9, 16, 25, 36, 49 \ldots\} = \{n^2 : n \in \mathbb{N}\}$ is numerically equivalent to $\mathbb{N}$. 
Sets can be either finite or infinite.

**Definition**

A set is *finite* if it is numerically equivalent to \( \{1, \ldots, n\} \) for some \( n \).

A set that is not finite is called *infinite*.

**Example**

\( A = \{2, 4, 6, \ldots, 50\} \) is finite while \( B = \{1, 4, 9, 16, 25, 36, 49, 64, \ldots\} \) is infinite.
Finiteness

- A set is either countable or uncountable.

**Definition**

A set is **countable** if it is numerically equivalent to the set of natural numbers \( \mathbb{N} = \{1, 2, 3, \ldots \} \).

A set that is not countable is called **uncountable**.

**Example**

The set of integers \( \mathbb{Z} = \{0, 1, -1, 2, -2, \ldots \} \) is countable. Define \( f : \mathbb{N} \rightarrow \mathbb{Z} \) by

\[
\begin{align*}
  f(1) &= 0 \\
  f(2) &= 1 \\
  f(3) &= -1 \\
  & \vdots \\
  f(n) &= (-1)^n \left\lfloor \frac{n}{2} \right\rfloor
\end{align*}
\]

where \( \lfloor x \rfloor \) is defined as “the greatest integer less than or equal to \( x \)”. One can easily verify that \( f \) is one-to-one and onto (Exercise).
Theorem
The set of rational numbers $\mathbb{Q}$ is countable.

Theorem (Cantor)
$2^{\mathbb{N}}$, the set of all subsets of $\mathbb{N}$, is not countable.

Prove these results as exercise.
Cardinality Facts

Notation

We let $|A|$ denote the cardinality of $A$.

- Note that $|A|$ cannot be the absolute value as since it contains a set (same symbols can have different meanings depending on the context).

REMARKS

- If $A$ is numerically equivalent to $\{1, 2, \ldots, n\}$ for some $n \in \mathbb{N}$, then $|A| = n$.
- $A$ and $B$ are numerically equivalent if and only if $|A| = |B|$.
- If $|A| = n$ and $A$ is a proper subset of $B$, $|B| > n = |A|$.
**Definition**

Given two sets $X$ and $Y$, their **Cartesian product** is defined as

\[ X \times Y = \{(x, y) : x \in X, y \in Y\} \]

- The Cartesian product is the set of all possible pairs.

**Definition**

$R \subset X \times Y$ is a **binary relation** from $X$ to $Y$.

We write “$xRy$” if $(x, y) \in R$ and “not $xRy$” if $(x, y) \notin R$.

- In this definition, the order matters ($xRy$ is different from $yRx$).

**Example**

Suppose $X = \{1, 2, 3\}$ and $R \subset X \times X$ is the binary relation given by

\[ R = \{(1, 1), (2, 1), (2, 2), (3, 1), (3, 2), (3, 3)\} \]

This describes the binary relation “weakly greater than,” or $\geq$.
Equivalence Relations

**Definition**
When $X = Y$ and $R \subseteq X \times X$, we say $R$ is a binary relation on $X$.

**Definitions**
A binary relation $R$ on $X$ is
- **reflexive** if $\forall x \in X$, $xRx$.
- **symmetric** if $\forall x, y \in X$, $xRy \Leftrightarrow yRx$.
- **transitive** if $\forall x, y, z \in X$, $(xRy \land yRz) \Rightarrow xRz$.

**In economics**, binary relations are used to describe a consumer’s ranking of consumption bundles (preferences).

**Definition**
A binary relation $R$ on $X$ that is reflexive, symmetric and transitive is called an **equivalence relation**.
Definitions

The **image** of a function $f$ from a set $X$ to a set $Y$ is the set

$$f(X) = \{ y \in Y : f(x) = y \text{ for some } x \in X \}$$

The **graph** of function $f : X \to Y$ is defined as

$$G_f = \{(x, y) \in X \times Y : y = f(x) \}$$

Remark

The fact that $f$ is a function tells us that

$$(x, y) \in G_f \land (x, z) \in G_f \Rightarrow y = z$$

remember that $\land$ means “and”.

Example

Suppose $f : X \to Y$ is a function from $X$ to $Y$. The binary relation $R \subseteq X \times Y$ defined by

$$xRy \iff f(x) = y$$

is the graph of the function $f$. Thus, one can think of a function as a binary relation $R$ from $X$ to $Y$ such that for each $x \in X$ there exists exactly one $y \in Y$ such that $(x, y) \in R$. 
We can write quantifiers and other operations over functions.

**Example**

The statement

$$\forall f : \mathbb{N} \to \mathbb{R} \ \exists x \in \mathbb{R} \text{ such that } \nexists n \in \mathbb{N} \text{ such that } f(n) = x$$

says there is no function mapping $\mathbb{N}$ onto $\mathbb{R}$.
Inverse Image

**Definition**

Let \( f : X \to Y \) be a function from \( X \) to \( Y \). The **inverse image** of a set \( W \subseteq Y \) is defined as

\[
f^{-1}(W) \equiv \{ x \in X \mid f(x) \in W \}.
\]

- This is the set of points in \( X \) that map to points in \( W \).
- The inverse image does not necessarily define a function from \( Y \) to \( X \);
  - it could be that there are two distinct elements of \( X \) that map to the same point in \( Y \).
Invertible Functions

**Definition**
A function $f : X \rightarrow Y$ is called invertible if the inverse image mapping is a function from $Y$ to $X$.

**Notation**
If $f$ is invertible, then for every $y \in Y$ one defines $f^{-1}(y)$ as the point $x \in X$ for which $f(x) = y$.

**Theorem**
A function is invertible if and only if it is both one-to-one and onto.

Prove this as an exercise (notice, this is a $\Leftrightarrow$ statement).
Operations on Functions

Definition
Given two functions \( g : X \rightarrow Y \) and \( f : Y \rightarrow Z \) the composition function \( f \circ g \) is a mapping from \( X \) to \( Z \) defined as \( f \circ g = f(g(x)) \) for every \( x \in X \).

Notation
Since \( f \circ g : X \rightarrow Z \), one can write it as a function \( h : X \rightarrow Z \) defined by \( h(x) \equiv f(g(x)) \) for every \( x \in X \).

Definition
If \( f : X \rightarrow Y \) and \( g : X \rightarrow Y \), then we can form new functions from \( X \) to \( Y \)

- The **sum**, denoted \( f + g \) and defined by \((f + g)(x) = f(x) + g(x)\).
- **Multiplication by a constant**, denoted \( \lambda f \) and defined by \((\lambda f)(x) = \lambda f(x)\).
- The **product**, denoted \( fg \) and defined by \((fg)(x) = f(x)g(x)\).
- The **quotient**, denoted \( f/g \) and defined by \((f/g)(x) = \frac{f(x)}{g(x)}\).

These definitions make sense only when the range has special structure so that one can add, multiply by a constant, multiply, or divide elements of \( Y \).
Tomorrow

- We being real analysis.

1. Vector Spaces
2. Real Numbers
3. Sup and Inf, Max and Min
4. Intermediate Value Theorem