Production and inventory planning is the problem of balancing fixed costs of ordering and the costs of holding inventory.

- Economic Lot Size model - Single item, deterministic model
- Model with multiple items
- Model with multiple retailers
- Stochastic problems

Economic Lot Size Model

- Simplest possible model
- Illustrates the tradeoffs between ordering and storage costs.
- One facility
- Single item
- Demand constant $D$ items per unit time
- Order quantities fixed at $Q$ items per order
- Fixed cost $K$ for every time an order is placed
- Inventory holding cost $h$ per item per unit time
- Lead time from order to delivery is 0
- Initial inventory is 0
- Planning horizon infinite

Zero Inventory Ordering Property - Orders should be made when orders drop to zero.
- Why? Consider if you order before they go to zero (what is benefit)
- What happens if you let inventory remain at zero?
Economic Lot Size Model Characteristics

- Zero Inventory Ordering Property
- Illustrate Cycle time
- What is fixed (ordering) costs $K$?
- What is inventory holding costs $\frac{hQ}{2}$?
- Cost per cycle $K + \frac{hQ}{2}$
- Note $Q = TD$
- Cost per time $KQ + \frac{Q}{2}$
- Set fixed costs per unit time = inventory holding cost per unit
- $Q^* = \sqrt{\frac{2KD}{h}}$
- What if Lead time is not zero?

Economic Lot size model assumes an infinite planning horizon. i.e. there are no reasons to constrain the order cycle time.
- But many situations have a planning horizon
- e.g. seasonal goods (fashion)
- In addition, allow the order quantity to change

Finite Horizon Model

Finite Horizon Model Zero Inventory Ordering

- Develop an inventory policy $P$ that places $m$ orders in the interval $[0, t]$.
- Let $T_i$ be the time between the $i^{th}$ order and the $(i+1)^{th}$ order. $T_m$ is the time between the last order and time $t$
- Therefore $t = \sum_{i=1}^{m} T_i$
- Because lead time = 0, $P$ must satisfy the Zero Inventory Ordering Policy

Finite Horizon Costs

- Total costs is fixed cost + inventory holding costs
- $m$ orders during the time horizon
- Let $I(\tau)$ be the inventory level at time $\tau \in [0, t]$
- Total\_cost = $\frac{1}{t} \int_0^t Km + h \int_0^t I(\tau) \, d\tau$.
Finite Horizon Total orders

Total orders:

\[ \text{Total orders} = \sum_{i=1}^{m} T_i \cdot D \cdot T_i = D \sum_{i=1}^{m} T_i^2 \]

Objective:

\[ \text{Min } \sum_{i=1}^{m} T_i^2 \] (1)

subject to:

\[ \sum_{i=1}^{m} T_i = t \] (2)

\[ T_i \geq 0 \quad \forall i = 1, \ldots, m \] (3)

Ordering periods

- For Economic Lot size problems, we found that average cost per unit time:
  \[ \frac{KD}{Q} + \frac{hQ}{2} \text{ or } \frac{K}{T} + \frac{hT}{2} \]
- But the cycle time \( T^* \) can be impractical
  \[ Q^* = \sqrt{\frac{2KD}{h}} \]
  \[ T^* = \frac{Q^*}{D} \]

Power of Two policies

- You can set a restriction on the ordering period so that it is restricted to easily implementable policies
- Note that near the minimum average cost, the curve is flat
- Set a Base order period, and force the order period to be a power of 2 multiple of that
  \[ T = T_B 2^k \]
Finding Power of 2 policy

• Two questions
  • How does one find the best power of two policy?
  • How far from optimal is the best policy of this type?

Define $g = \frac{hD}{2}$
$Q^* = \sqrt{\frac{2KD}{h}}$
$T^* = Q^*/D$

Therefore $f(T) = \frac{T}{2} + gT$

Note that $T^* = \sqrt{T_g}$ and Total cost $f(T^*) = 2\sqrt{Kg}$

Find the smallest integer $k$ such that

$f(T_B2^k) \leq f(T_B2^{(k+1)})$

$\frac{K}{T_B2^k} + gT_B2^k \leq \frac{K}{T_B2^{(k+1)}} + gT_B2^{(k+1)}$

So $k$ is the smallest integer where

$\sqrt{\frac{K}{2g}} = \frac{T^*}{\sqrt{2}} \leq T_B2^k = T$

And step through values of $k$

How good are power of 2 policies?

• Question 2: How far from optimal is the best policy of this type?
• The optimal power of 2 policy must be in the interval

$\left[\frac{1}{\sqrt{2}} + T^*, \sqrt{2} + T^*\right]$  

We can verify $f\left(\frac{1}{\sqrt{2}} + T^*\right) = f\left(\sqrt{2} + T^*\right) = \frac{1}{2} \left(\frac{1}{\sqrt{2}} + \sqrt{2}\right) f(T^*)$

Then compare $\frac{f(T)}{f(T^*)} \leq \frac{1}{2} \left(\frac{1}{\sqrt{2}} + \sqrt{2}\right) \approx 1.06$

Multi-Item Inventory Models

• In reality, warehouses have to coordinate inventory of multiple items to minimize cost without exceeding the shared warehouse capacity.
• Economic Warehouse Lot Scheduling Problem (EWLSP)
• Staggering - schedule placement of orders so that the warehouse capacity is not violated
• Independent solutions - Items ordered independently, with expectation that order periods will not coincide
• Rotation - Items share order interval, but orders are staggered.