1(a) Prove \( f(x) = |x| \) is continuous at \( x = 0 \).

\[ \text{Proof:} \]

\[ \text{N.T.S.} \quad \lim_{x \to 0} |x| = f(0) = 0. \]

Let \( \varepsilon > 0 \) be arbitrary.

Choose \( \delta = \varepsilon > 0 \).

Then \( \forall x \in \mathbb{R} \text{ s.t. } x \neq 0 \) and \( |x - 0| < \delta \), we have

\[ |x| - 0| = |x| < \delta = \varepsilon, \quad \text{and thus} \]

\[ \lim_{x \to 0} |x| = 0 \] \( \Rightarrow \) \( f(x) = |x| \) is continuous at \( x = 0 \).

(b) Consider the function

\[ g(x) = \begin{cases} 
|x|, & \text{if } x \neq 0 \\
3, & \text{if } x = 0
\end{cases} \]

Is \( g \) continuous at \( x = 0 \)? Does \( \lim_{x \to 0} g(x) \) exist?

\[ \text{Answer: No, } g \text{ is not continuous at } x = 0, \text{ but} \]

\[ \lim_{x \to 0} g(x) \text{ does exist and is equal to } 0. \]

\[ \text{Proof: Let } \varepsilon > 0 \text{ be arbitrary.} \]

Choose \( \delta = \varepsilon > 0 \).

Then \( \forall x \in \mathbb{R} \text{ s.t. } x \neq 0 \) and \( |x - 0| < \delta \),

\[ |g(x) - 0| = |x| - 0| = |x| < \delta = \varepsilon. \]

Thus \( \lim_{x \to 0} g(x) = 0 \neq 3 = g(0) \), and so \( g \) is not cts at \( x = 0 \).
2) Prove that the function
\[ h(x) = \begin{cases} 
1, & \text{if } x > 0 \\
0, & \text{if } x \leq 0 
\end{cases} \]
does not have a limit at \( x = 0 \).

**Proof:** Define \( x_n = \frac{1}{n} \) & \( y_n = -\frac{1}{n} \) for \( n \in \mathbb{N} \).

Then \( \lim_{n \to \infty} x_n = 0 = \lim_{n \to \infty} y_n \), and

\[
\lim_{n \to \infty} h(x_n) = \lim_{n \to \infty} h\left(\frac{1}{n}\right) = \lim_{n \to \infty} 1, \text{ since } \frac{1}{n} > 0 \text{ for all } n \\
= 1
\]

& \( \lim_{n \to \infty} h(y_n) = \lim_{n \to \infty} h\left(-\frac{1}{n}\right) = \lim_{n \to \infty} 0, \text{ since } -\frac{1}{n} < 0 \text{ for all } n \\
= 0 \).

Thus \( \lim_{n \to \infty} h(x_n) \neq \lim_{n \to \infty} h(y_n) \) & so by the sequential characterization of a limit, \( \lim_{n \to \infty} h(x) \) does not exist.
(3) Suppose \( f: A \rightarrow \mathbb{R} \) is continuous where \( A \) is any subset of \( \mathbb{R} \). Decide if each of the following steps are correct, and prove the correct ones.

(a) Suppose \( f \) is not bounded. Then \( \exists \) a sequence \( \{x_n\} \subseteq A \) s.t. \( f(x_n) > n \ \forall n \in \mathbb{N} \).

True.

By the definition of a bounded function, \( f \) is bounded if and only if \( \exists N \in \mathbb{N} \) s.t. \( f(x) \leq N \ \forall x \in A \).

Negating this definition, we get that \( f \) is not bounded on \( A \) if and only if \( \forall N \in \mathbb{N}, \exists x_N \in A \) s.t. \( f(x_N) > N \).

Thus, we may construct a sequence \( \{x_n\} \) s.t. \( f(x_n) > n \ \forall n \in \mathbb{N} \). \( \Box \)

(b) The sequence \( \{x_n\} \) has a convergent subsequence \( \{x_{n_k}\} \).

False. This result is only guaranteed if we know that \( \{x_n\} \) is bounded (or if we know \( A \) is bounded), by Bolzano-Weierstrass.

For instance, if \( A = \mathbb{R} \) & \( f:A \rightarrow \mathbb{R} \) is defined by \( f(x) = x \), we can choose \( x_n = n + 1 \ \forall n \) as a sequence s.t. \( f(x_n) > n \ \forall n \), and \( \{x_n\} \) has no convergent subsequence (since \( \lim_{n \rightarrow \infty} x_n = \infty \)). \( \Box \)
3 (c) This subsequence converges to some $x$ in $A$.

False. If such a subsequence exists, in order to guarantee that its limit is in $A$, we must assume $A$ is closed. If $A$ is not closed, then there exists a cluster point of $A$ which is not an element of $A$, to which $\{x_{n_k}\}$ might converge.

For example, $A = (0,1)$, $x_n = \frac{1}{n}$ $\forall n \in \mathbb{N}$,

$$\lim_{n \to \infty} x_n = \lim_{n \to \infty} \frac{1}{n} = 0 \notin A.$$

\[ \square \]

(d) One has

$$\lim_{i \to \infty} f(x_{n_i}) = f(x).$$

True. Assuming $\lim_{i \to \infty} x_{n_i} = x$, if since $f$ is continuous by assumption,

$$\lim_{i \to \infty} f(x_{n_i}) = f(x)$$ by the sequential characterization of continuity (Proposition 3.2.2 (iii)). \[ \square \]

(e) (a) - (d) gives a contradiction.

Since $n_i > i \forall j \in \mathbb{N}$, $f(n_i) > n_j \geq i \forall j \in \mathbb{N}$.

We know that any convergent sequence must be bounded, and since $\{f(x_{n_i})\}_{i=1}^{\infty}$ is not bounded, it cannot be convergent, contradicting (d).

However, (b) & (c) are not necessarily true for general $A$, so no contradiction is reached.