Problem 1: The binomial expansion theorem states that \((a+b)^n = \sum_{i=0}^{n} \binom{n}{i} a^i b^{n-i}\).

(a) Extend this theorem to sum of three numbers, that is, prove that:
\[
(a + b + c)^n = \sum_{i+j+k=n; \ i,j,k \geq 0} \frac{n!}{i!j!k!} a^i b^j c^k.
\]

(b) (Bonus problem). More generally, prove:
\[
(a_1 + \cdots + a_m)^n = \sum_{i_1+\cdots+i_m=n; \ i_1,\ldots,i_m \geq 0} \frac{n!}{i_1!\cdots i_m!} a_1^{i_1} \cdots a_m^{i_m}.
\]

The numbers \(\frac{n!}{i_1!\cdots i_m!}\) are sometimes called multinomial numbers. (Hint: proof is similar to the proof of binomial theorem.)

Problem 2: Consider the sequence \((b_i)\) defined by the recurrence relation \(b_{i+2} = b_{i+1} + 2b_i\) and \(b_0 = 1, b_1 = 1\). Find a generating function for this sequence. Use this to give a non-recursive formula for \(b_i\) in terms of \(i\).

Problem 3: Find a sequence \((c_i)\) defined by a recurrence relation whose generating function is \(4 - x^2 - 2x + 3\).

Problem 4: Let \(a_i\) be the number of ways you can write the number \(i\) as a sum \(n_1 + n_2 + n_3 + n_4\) where the integers \(n_1, \ldots, n_4\) are subject to the following restrictions:

- \(n_1\) can only have values between 0 and 3.
- \(n_2\) can only have values bigger than 0.
• $n_3$ can only have values 2 and 4.

• $n_4$ can only have values bigger than 1.

Write a generating function for the sequence $a_i$. Represent this function as a rational function in $x$. Justify your answer. (Hint: recall the multiplication rule of power series. The desired generating function is product of 4 power series corresponding to $n_1, \ldots, n_4$.)

**Problem 5:** Give a power series expansion for $(1+x)^{-5}$. That is, determine the $a_i$ where $(1+x)^{-5} = \sum_{i=0}^{\infty} a_ix^i$.

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**Problem 1:** We will prove (b) by induction on $n$. The statement for $n=1$ is obvious. Suppose the statement is true for $n$, we will show it holds for $n+1$. We have $(a_1+\cdots+a_k)^n = \sum_{i_1+\cdots+i_k=n} \frac{n!}{i_1! \cdots i_k!} a_1^{i_1} \cdots a_k^{i_k}$.

\[
(a_1+\cdots+a_k)^{n+1} = (a_1+\cdots+a_k) \left( \sum_{i_1+\cdots+i_k=n} \frac{n!}{i_1! \cdots i_k!} a_1^{i_1} \cdots a_k^{i_k} \right)
\]

\[
= \sum_{i_1+\cdots+i_k=n} \frac{n!}{i_1! \cdots i_k!} a_1^{i_1} \cdots a_k^{i_k} + \cdots + \sum_{i_1+\cdots+i_k=n} \frac{n!}{i_1! \cdots i_k!} a_1^{i_1} \cdots a_k^{i_k+1}
\]

\[
= \sum_{\substack{i_1+\cdots+i_k=n+1 \\
j_1, \ldots, j_k \geq 0}} \frac{n!}{j_1! \cdots j_k!} a_1^{j_1} \cdots a_k^{j_k}
\]

It remains to show \( \frac{n!}{(j_1-1)! \cdots j_k!} + \cdots + \frac{n!}{j_1! \cdots (j_k-1)!} = \frac{(n+1)!}{j_1! \cdots j_k!} \). To show this note that lefthand side is equal to:

\[
\frac{j_1 \cdot n!}{j_1! \cdots j_k!} + \cdots + \frac{j_k \cdot n!}{j_1! \cdots j_k!} = \frac{(j_1+\cdots+j_k) \cdot n!}{j_1! \cdots j_k!} = \frac{(n+1)!}{j_1! \cdots j_k!}
\]

as required.
Problem 2: \( b_{i+2} = b_{i+1} + 2b_i \), \( b_0 = b_1 = 1 \).

Let \( f(x) = \sum_{i=0}^{\infty} b_i x^i \). Then \( \sum_{i=0}^{\infty} b_{i+2} x^{i+2} = \sum_{i=0}^{\infty} b_{i+1} x^{i+2} + \sum_{i=0}^{\infty} b_i x^{i+2} \).

\[ f(x) - x - 1 = x (f(x) - 1) + 2x^2 f(x) \]

\[ f(x) - x f(x) - 2x^2 f(x) = 1 + x - x = 1 \]

\[ f(x) = \frac{1}{1 - x - 2x^2} \text{ generating function for } \{b_i\}_{i=0}^{\infty}. \]

\[ 1 - x - 2x^2 = -2 \left( x - \frac{1+3}{-4} \right) \left( x - \frac{1-3}{-4} \right) = -2 \left( x + 1 \right) \left( x - \frac{1}{2} \right). \]

\[ \frac{A}{-2(x+1)} + \frac{B}{x - \frac{1}{2}} = \frac{1}{-2(x+1)(x - \frac{1}{2})} \Rightarrow A = -\frac{2}{3}, \quad B = -\frac{1}{3} \]

\[ f(x) = \frac{(-2/3)}{-2(x+1)} + \frac{(-1/3)}{x - \frac{1}{2}} = \frac{1}{3} \left( \frac{1}{1-(-x)} \right) + \left( \frac{2}{3} \right) \frac{1}{1-2x} = \]

\[ \frac{1}{3} \sum_{i=0}^{\infty} (-x)^i + \frac{2}{3} \sum_{i=0}^{\infty} (2x)^i = \sum_{i=0}^{\infty} \left( \left( \frac{1}{3} \right) (-1)^i + \left( \frac{2}{3} \right) 2^i \right) x^i \]

\[ \Rightarrow b_i = \left( \frac{1}{3} \right) (-1)^i + \left( \frac{2}{3} \right) 2^i. \]

We verify: \( b_0 = \left( \frac{1}{3} \right) + \left( \frac{2}{3} \right) = 1 \) \( b_1 = \left( \frac{1}{3} \right) (-1) + \left( \frac{2}{3} \right) (2) = 1 \)
Problem 3:

\[
\frac{4}{x^2 - 2x + 3} = \sum_{i=0}^{\infty} c_i x^i \Rightarrow 4 = (-x^2 - 2x + 3) \sum_{i=0}^{\infty} c_i x^i
\]

\[
= \sum_{i=0}^{\infty} -c_i x^{i+2} + \sum_{i=0}^{\infty} (-2) c_i x^{i+1} + \sum_{i=0}^{\infty} 3c_i x^i \Rightarrow
\]

\[
4 = \sum_{i=2}^{\infty} -c_{i-2} x^i + \sum_{i=1}^{\infty} (2) c_{i-1} x^i + \sum_{i=0}^{\infty} 3c_i x^i
\]

\[
4 = 4 + 0x + 0x^2 + \ldots , \text{ equating coeff. of } x^i \text{ in both sides:}
\]

\[
i \geq 2 \quad -c_{i-2} + 2c_{i-1} + 3c_i = 0
\]

(By looking at coeff. of \(x^0\) and \(x^1\) we can compute \(c_0\) and \(c_1\).)

Problem 4:

\[
(x^0 + x^3)(x^1 + x^2 + \ldots)(x^2 + x^4)(x^2 + x^3 + \ldots) =
\]

\[
(1+x^3)(x^2 + x^4) x^3 \frac{1}{(1-x)^2}
\]
Problem 5:
\[
\frac{1}{1-x} = \sum_{i=0}^{\infty} x^i \quad \Rightarrow \quad \frac{1}{(1-x)^5} = \left( \sum_{i=0}^{\infty} x^i \right)^5 = \sum_{i=0}^{\infty} \binom{5+i-1}{5} x^i
\]

\[
\Rightarrow \quad \frac{1}{(1+x)^5} = \sum_{i=0}^{8} (-1)^i \binom{5+i-1}{5} x^i
\]

number of sol. of 
\[
i = x_1 + x_2 + x_3 + x_4 + x_5
\]
where \( x_i \geq 0 \) integer