Due date: Wednesday December 4, 2013

Problem 1: In the graph below use Philip Hall’s theorem to prove that there is no perfect matching.

Consider the set \( A = \{a, c, e, f\} \). Then \( R(A) = \text{set of vertices connected to some vertex in } A = \{2, 4, 6\} \).

But \( |A| = 4 > |R(A)| = 3 \).

So by P. Hall’s thm. there is no perfect matching.

Problem 2: Draw the network associated to the bipartite graph below. Find a maximum matching and a minimum vertex cover in this graph, and find the corresponding maximum flow and minimum cut (by trial and error, or by Ford-Fulkerson algorithm). Show that the flow (respectively cut) you found are indeed maximum (respectively minimum).
Problem 3: Consider the field \( \mathbb{F}_5 = \{0, 1, 2, 3, 4\} \) with addition and multiplication modulo 5. Let \( \mathbb{F}_5^2 \) denote the vector space \( \{(i, j) \mid 0 \leq i, j \leq 4\} \) over \( \mathbb{F}_5 \). Answer the following questions: how many points are on each line \( \ell = \{(x, y) \mid ax + by = c\} \subset \mathbb{F}_5^2 \)? How many lines are there in \( \mathbb{F}_5^2 \)? Write down all the points \((x, y)\) on the line \(2x + 3y = 1\). How many lines are parallel to this line? If \( V = \mathbb{F}_5^2 \) and \( B \) is the collection of all the lines in \( \mathbb{F}_5^2 \), what are the parameters \( k, r, \lambda \) of the block design \((V, B)\)?

Problem 4: Let \((V, B)\) be a block design with parameters \( v = |V|, b = |B|, k, r, \lambda \). Let \( B' \) be the collection of all the complements of the subsets in \( B \), that is:

\[ B' = \{ V \setminus L \mid L \in B \}. \]

Show that \((V, B')\) is a block design with parameters \( v' = v, b' = b, k' = v - k, r' = b - r, \lambda' = \lambda + b - 2r \).
Problem 3

Consider a line $\ell: ax + by = c$. Case 1: Suppose $b \neq 0$, then $y = \frac{c-ax}{b}$.

Thus, for every $x \in \mathbb{F}_5$ there is a unique $y \in \mathbb{F}_5$ s.t. $(x,y)$ is on the line $L$.

So there are $5 = |\mathbb{F}_5|$ points on $L$. Case 2: Suppose $b = 0$. Then $a \neq 0$ and $x = \frac{c}{a}$. So the points on $L$ are $(\frac{c}{a}, y)$ for any $y \in \mathbb{F}_5$. Hence there are $5 = |\mathbb{F}_5|$ points on this line as well.

Now, let's count how many lines pass through a given point $(x_0, y_0) \in \mathbb{F}_5^2$.

The point $(x_0, y_0)$ can be connected to $25 - 1 = 24$ points, but each such line contains $5 - 1 = 4$ points beside $(x_0, y_0)$. Thus there are exactly $24/4 = 6$ lines through $(x_0, y_0)$.

There are 25 points so one can draw $25 \times 6$ lines through the 25 points, but each line contains 5 points so overall we have $25 \times 6 / 5 = 5 \times 6 = 30$ lines.

Points on the line $2x + 3y = 1$: For each $x$, $y = \frac{1-2x}{3}$. Note $-2 = 3 \mod 5$ & (inverse of 3) = 2 $\mod 5$ because $2 \times 3 = 1 \mod 5$. So $y = 2(1+3x) = 2+x$.

So points on the line are: $(0, 2), (1, 3), (2, 4), (3, 0), (4, 1)$.

There are $25 - 5 = 20$ points outside the line $2x + 3y = 1$. Through each of these points passes a line parallel to $2x + 3y = 1$. But each line contains 5 points, so overall $20/5 = 4$ lines parallel to $2x + 3y = 1$ (not counting the line itself).

By above discussion: $v = \# \text{ of points} = 25$

$b = \# \text{ of lines} = 30$

$k = \# \text{ of points on a line} = 5$

$r = \# \text{ of lines through a point} = 6$

$\lambda = \# \text{ of lines through two distinct points} = 1$.

Problem 4

$v' = \# \text{ of points} = v$
\[ b' = \text{# of new blocks} = b, \text{ because for each block } L \in B \text{ we have exactly one new block } L' = V \setminus L \in B'. \]

\[ k' = \# \text{ of points in a block } L' = V \setminus L = v - k. \]

\[ r' = \# \text{ of blocks } L' \in B' \text{ containing a point } x \in V = \# \text{ of blocks } L \in B \text{ not containing } x. \]

\[ = b - r. \]

\[ \lambda' = \# \text{ of } L' \in B' \text{ containing two distinct points } x_1, x_2 \in V. \]

Suppose \( L' = V \setminus L \) contains \( x_1, x_2 \). Thus \( x_1 \notin L \) & \( x_2 \notin L \). Let's count the number of blocks \( L' \in B \) with \( x_1 \notin L \) & \( x_2 \notin L \). Overall there are \( b \) blocks in \( B \). \( r \) of them contain \( x_1 \) & another \( r \) contain \( x_2 \). Moreover, \( \lambda \) of them contain both \( x_1 \) & \( x_2 \). Thus exactly \( 2r - \lambda \) blocks \( L \in B \) contain either \( x_1 \) or \( x_2 \). Hence exactly \( b - 2r + \lambda \) contain none of \( x_1 \) & \( x_2 \).