Problem 1:
(a) Let \( G = (V, E) \) be a graph. For a vertex \( v \in V \) let \( d(v) \) denote the degree of \( v \), i.e. the number of edges connected to \( v \). Show:
\[
\sum_{v \in V} d(v) = 2|E|.
\]
(b) Show that in any graph the number of vertices that have odd degree is even (for example a graph \( G \) cannot have exactly one vertex with odd degree).

Problem 2: Let \( G \) be a graph with \( n \) vertices. The degree sequence of \( G \) is the \( n \)-tuple of degrees of vertices of \( G \) arranged in decreasing ordered.
(a) Write down the degree sequence of the following graph:
(b) Show that the following sequence cannot be the degree sequence of any graph: (4, 3, 2, 2).

(c) Find a connected simple graph with the following degree sequence: (3, 3, 3, 3, 3, 3, 3).

(d) More generally, for any integer $d > 0$ construct a connected (simple) graph with $2^d$ vertices such that its degree sequence is $(d, d, \ldots, d)$ ($2^d$ times).

(e) Prove that if two graphs are isomorphic then they have the same degree sequence.

(f) (Bonus) Find two graphs with the same degree sequence which are not isomorphic. Show why the graphs in your example are not isomorphic.

**Problem 3:** Let $G$ be a connected graph. A spanning tree for $G$ is a subgraph of $G$ that is a tree and contains all the vertices of $G$. (a) Draw a spanning tree for the following graph:

(b) Prove that any connected graph $G$ has at least one spanning tree.

**Problem 4:** (Bonus) Show that a simple graph on $n$ vertices and more than $n^2/4$ edges must contain a triangle, i.e. 3 vertices all connected to each other. If $n = 2k$ is even give an example of a graph with $n$ vertices and exactly $n^2/4$ edges which does not contain any triangle.