The details for those who want them:

Compare corresponding lengths on two curves and find
\[
ds' - ds = (\dot{x} + \dot{\xi} + \dot{\eta} + \dot{\zeta}) \, ds
\]
where \( \dot{\cdot} = d/ds \) since
\[
(ds')^2 = [(x + \xi + dx + d\xi) - (x + \xi)]^2 + \cdots = (dx + d\xi)^2 + \cdots
\]
\[
= (dx^2 + 2 dx \, d\xi + d\xi^2) + \cdots
\]
\[
= (dx^2 + dy^2 + dz^2) + 2 dx \, d\xi + 2 dy \, d\eta + 2 dz \, d\zeta
\]
in 2\text{nd} order quantities
\[
ds^2 = ds^2 \left( 1 + 2 \frac{dx \, d\xi}{ds} + \cdots \right) = ds^2 (1 + 2 \dot{x} \dot{\xi} + \cdots)
\]
\[
\therefore ds' = ds(1 + \dot{x} \dot{\xi} + \cdots) \quad \therefore ds' - ds = (\dot{x} \dot{\xi} + \cdots) \, ds
\]
The condition for the motion to be force free, excepting the constraint to a surface \( f(x, y, z) \),
\[
m(a_x, a_y, a_z) = m \left( \frac{d^2x}{dt^2}, \frac{d^2y}{dt^2}, \frac{d^2z}{dt^2} \right) = \lambda \begin{pmatrix}
\frac{\partial f}{\partial x}, & \frac{\partial f}{\partial y}, & \frac{\partial f}{\partial z}
\end{pmatrix}
\]
\[
\text{orthogonal to surface}
\]
is
\[
(x(s), y(s), z(s)) \Leftrightarrow (a_x, a_y, a_z) \text{ is orthogonal to the surface tangent vector } (\xi, \eta, \zeta)
\]
The condition follows since the variation \( \delta \int ds \) must vanish if the curve is extremal in length (a geodesic);
\[
\delta \int ds = \int ds' - \int ds = \int (\dot{x} \dot{\xi} + \dot{y} \dot{\eta} + \dot{z} \dot{\zeta}) \, ds
\]
\[
= \int \left( \frac{d}{ds} (\dot{x} + \cdots) \right) ds \quad \text{same start and finish}
\]
\[
(\dot{x} + \cdots)_{\text{end}}^{\text{start}} = 0
\]
vanishes in general if \( (\dot{x}, \dot{y}, \dot{z}) \) orthogonal to \( (\xi, \eta, \zeta) \)
\[
(\xi, \eta, \zeta)(\text{start}) = (\xi, \eta, \zeta)(\text{end}) = 0
\]
The orthogonality of \( (\dot{x}, \dot{y}, \dot{z}) \) to \( (\xi, \eta, \zeta) \) implies the orthogonality of \( (a_x, a_y, a_z) \) to \( (\xi, \eta, \zeta) \) because constrained motion has constant kinetic energy, and hence \( s \propto t \).

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