Properties of Real Numbers

Types of Numbers

Natural Numbers (Counting Numbers) \( \mathbb{N} \)

\[ \mathbb{N} = \{1, 2, 3, 4, 5, \ldots \} \]

Whole Numbers \( \mathbb{W} \)

\[ \mathbb{W} = \{0, 1, 2, 3, 4, 5, \ldots \} \]

Integers \( \mathbb{Z} \)

\[ \mathbb{Z} = \{\ldots, -4, -3, -2, -1, 0, 1, 2, 3, 4, \ldots \} \]

Rational Numbers \( \mathbb{Q} \)

\[ \mathbb{Q} = \left\{ \frac{a}{b} \mid a, b, \in \mathbb{Z}, b \neq 0 \right\} \]

Irrational Numbers \( \mathbb{I} \)

Numbers that can be written as an infinite nonrepeating decimal

Real Numbers \( \mathbb{R} \)

Any number that is rational or irrational \( \mathbb{R} = \mathbb{Q} \cup \mathbb{I} \)
Real Number Line

Visualize a line with equally spaced markers each of which is associated with the integers. If the integers have their natural order, then the real numbers can be visualized as points on the line.

We notice that

1. Every real number corresponds to a unique point on the line.
2. Every point on the line corresponds to a unique real number.

This is why the set of real numbers is sometimes referred to as the real number line.
## Types of Intervals

<table>
<thead>
<tr>
<th>Interval Notation</th>
<th>Graph</th>
<th>Algebraic Notation</th>
<th>Interval Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>(a, b)</td>
<td>![Open Interval Graph](a to b, open)</td>
<td>a &lt; x &lt; b</td>
<td>Open, finite</td>
</tr>
<tr>
<td>[a, b]</td>
<td>![Closed Interval Graph](a to b, closed)</td>
<td>a ≤ x ≤ b</td>
<td>Closed, finite</td>
</tr>
<tr>
<td>[a, b)</td>
<td>![Half-Open Interval Graph](a to b, half-open)</td>
<td>a ≤ x &lt; b</td>
<td>Half–open, finite</td>
</tr>
<tr>
<td>(a, b]</td>
<td>![Half-Open Interval Graph](a to b, half-open)</td>
<td>a &lt; x ≤ b</td>
<td>Half–open, finite</td>
</tr>
<tr>
<td>(a, ∞)</td>
<td>![Open Interval Graph](a to infinity, open)</td>
<td>x &gt; a</td>
<td>Open, infinite</td>
</tr>
<tr>
<td>(−∞, b)</td>
<td>![Open Interval Graph](negative infinity to b, open)</td>
<td>x &lt; b</td>
<td>Open, infinite</td>
</tr>
<tr>
<td>[a, ∞)</td>
<td>![Closed Interval Graph](a to infinity, closed)</td>
<td>x ≥ a</td>
<td>Closed, infinite</td>
</tr>
<tr>
<td>(−∞, b]</td>
<td>![Closed Interval Graph](negative infinity to b, closed)</td>
<td>x ≤ b</td>
<td>Closed, infinite</td>
</tr>
</tbody>
</table>

If your answer is composed of two (or more) distinct intervals, then the algebraic form of your answer must contain the conjunction 'OR'.
**Example 1:** Describe algebraically the following intervals:

a) \[ -3 < x < 4 \]

b) \[ x \leq -1 \]

c) \[ 2 \leq x < 8 \]

d) \[ x > 5 \]

e) \[ x \leq -4 \text{ or } x > 2 \]

**Example 2:** Describe the region(s) containing the indicated sign(s):

<table>
<thead>
<tr>
<th>Sign</th>
<th>Graph</th>
<th>Answer</th>
</tr>
</thead>
<tbody>
<tr>
<td>+</td>
<td></td>
<td>[ x &lt; -4 \text{ or } x &gt; -1 ]</td>
</tr>
<tr>
<td>+, 0</td>
<td></td>
<td>[ 2 &lt; x \leq 4 \text{ or } x \geq 5 ]</td>
</tr>
<tr>
<td>-</td>
<td></td>
<td>[ x &lt; 2 \text{ or } 4 \leq x \leq 5 ]</td>
</tr>
<tr>
<td>-</td>
<td></td>
<td>[ -2 &lt; x &lt; 0 \text{ or } x &gt; 2 ]</td>
</tr>
</tbody>
</table>
Operations with Real Numbers

Absolute Value

\[ |x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases} \]

Rule for Order of Operations

(Please Excuse My Dear Aunt Sally)

1. **P**arentheses: Simplify all groupings first.
2. **E**xponents: Calculate exponential powers and radicals.
3. **M**ultiplication and **D**ivision: Perform all multiplications and divisions as they occur from left to right.
4. **A**ddition and **S**ubtraction: Perform all additions and subtractions as they occur from left to right.
Evaluate the following expressions and note the use of the equal signs because we use mathematics writing style:

Ex 1: \[(4 - 6)^2 + 6(-4) + 5\]

\[
(4 - 6)^2 + 6(-4) + 5 = (-2)^2 + 6(-4) + 5
\]

(P) \[= (-2)^2 + 6(-4) + 5\]

(E) \[= 4 + 6(-4) + 5\]

(M) \[= 4 - 24 + 5\]

(S) \[= -20 + 5\]

(A) \[= -15\]

Ex 2: \[6 + 24 \div 3 \cdot 2 + 3\sqrt{16}\]

\[
6 + 24 \div 3 \cdot 2 + 3\sqrt{16} = 6 + 24 \div 3 \cdot 2 + 3 \cdot 4
\]

(E) \[= 6 + 24 \div 3 \cdot 2 + 3 \cdot 4\]

(D) \[= 6 + 8 \cdot 2 + 3 \cdot 4\]

(M) \[= 6 + 16 + 3 \cdot 4\]

(M) \[= 6 + 16 + 12\]

(A) \[= 22 + 12\]

(A) \[= 34\]
Solving First Degree Equations

Addition Property of Equality

If you add or subtract the same quantity to both sides of an equation, it does not affect the solution.

For any real numbers $a$, $b$, and $c$,

if $a = b$, then $a + c = b + c$,
if $a = b$, then $a - c = b - c$.

**Example:** Solve $3x - 4 = 2x + 3$

We can add the quantity $(-2x + 4)$ to both sides of the equation:

$3x - 4 = 2x + 3$

$3x - 4 - 2x + 4 = 2x + 3 - 2x + 4$

$x = 7$
Multiplication Property of Equality

If you multiply or divide the same nonzero quantity to both sides of an equation, it does not affect the solution.

For any real numbers $a$, $b$, and $c$, with $c \neq 0$,

if $a = b$, then $a \cdot c = b \cdot c$,
if $a = b$, then $a \div c = b \div c$.

**Example 2**: Solve $-4x = 12$

We can divide both sides of the equation by $-4$.

$$-4x = 12$$

$$\frac{-4x}{-4} = \frac{12}{-4}$$

$$x = -3$$
Solving First Degree Equations in
One Variable (Be GLAD)

**Remember:** You can always **simplify** any side of the equation at any time.

1. Simplify the expressions on both sides of the equation. You must first eliminate all of the **G**rouping symbols and simplify if necessary.

2. If there are fractions, you may multiply both sides of the equation by the **L**CM. Simplify if necessary.

3. You want the variable terms on one side of the equation and the constant terms on the other side. If this is not the case, you should **A**dd or **s**ubtract to both sides of the equation a variable term and/or a numerical value in order to get the variable terms on one side and the constant values on the other side. Simplify if necessary.

4. You want the variable term coefficient to be 1. If this is not the case, you should **D**ivide both sides of the equation by the variable term coefficient. Simplify if necessary.

5. **Check** your answer by substituting it into the original equation.

**Note:** Always make sure your final answer has the variable on the left side of the equation!!!
Evaluating and Solving Formulas

Solving for any term in a formula is similar

**Remember:** You can always simplify any side of the equation at any time.

1. Simplify the expressions on both sides of the equation. You must first eliminate all of the Grouping symbols and simplify if necessary.

2. If there are fractions, you may multiply both sides of the equation by the LCD. Simplify if necessary.

3. You want all of the terms with the desired variable on one side of the equation and all other term on the other side. If this is not the case, you should Add or subtract to both sides of the equation various variable terms and/or a numerical value in order to get desired variable terms on one side and everything else on the other side. Simplify if necessary.

4. You want one variable term with a coefficient of 1. If this is not the case, you might have to factor out the desired variable and Divide both sides of the equation by the resulting coefficient. Simplify if necessary.

**Note:** Always make sure your final answer has the desired variable on the left side of the equation!!!
Solving Word Problems
(Super Solvers Use C.A.P.E.S.)

1. **Read** the problem carefully. (Reread it several times if necessary)

2. **Categorize** the problem type if possible. (Is it a problem of numerical expression, distance–rate–time, cost–profit, or simple interest type?)

3. Decide what is asked for, and **Assign** a variable to the unknown quantity. Label the variable so you know exactly what it represents.

4. **Draw a Picture**, diagram, or chart whenever possible!!

5. **Form an Equation** (or inequality) that relates the information provided.

6. **Solve** the equation (or inequality).

7. **Check your solution** with the wording of the problem to be sure it makes sense.

8. Write the solution of the problem as asked.

     distance–rate–time: distance = rate • time (d = r • t)
     cost–profit: profit = revenue – cost (P = R – C)
     simple interest: interest = principal * rate (i = P • r)
     compound interest: A = P (1 + \( \frac{r}{n} \))^{nt}
Example: A grocery store bought ice cream for 59¢ a half gallon and stored it in two freezers. During the night, one freezer “defrosted” and ruined 14 half gallons. If the remaining ice cream is sold for 98¢ a half gallon, how many half gallons did the store buy if it made a profit of $42.44?

\[
\begin{array}{ccc}
\text{charge} & = & \text{price} \cdot \text{quantity} \\
\hline
\text{bought} & & .59 \quad x \\
\text{sold} & & .98 \quad x - 14 \\
\end{array}
\]

Example: Last summer, Ernie sold surfboards. One style sold for $70 and the other sold for $50. He sold a total of 48 surfboards. How many of each style did he sell if the receipts from each style were equal?

\[
\begin{array}{ccc}
\text{charge} & = & \text{price} \cdot \text{quantity} \\
\hline
\text{one style} & & 70 \quad x \\
\text{other style} & & 50 \quad 48 - x \\
\end{array}
\]
Example: Sellit Realty Company gets a 6% fee for selling improved properties and 10% for selling unimproved land. Last week, the total sales were $220,000 and the total fees were $14,000. What were the sales from each of the two types of properties?

\[
\begin{array}{lll}
\text{fees} & = & \text{rate} \times \text{sales} \\
\hline
\text{improved} & : & 0.06 \times x \\
\text{unimproved} & : & 0.10 \times (220000 - x)
\end{array}
\]

Example: Maria jogs to the country at a rate of 10 mph. She returns along the same route at 6 mph. If the total trip took 1 hour 36 minutes, how far did she jog?

\[
\begin{array}{lll}
\text{distance} & = & \text{rate} \times \text{time} \\
\hline
\text{going} & : & x \times 10 \\
\text{coming} & : & x \times 6
\end{array}
\]
Addition Property of Inequalities

You can always add or subtract the same quantity to both sides of an inequality without affecting the solution.

**Example 1**: Solve $3x - 4 \geq 2x + 3$
We add the quantity $(-2x + 4)$ to both sides of the inequality:

$3x - 4 \geq 2x + 3$

$3x - 4 - 2x + 4 \geq 2x + 3 - 2x + 4$

$x \geq 7$

Multiplication Property of Inequalities

You can always multiply or divide the same positive quantity to both sides of an inequality without affecting the solution.

You can always multiply or divide the same negative quantity to both sides of an inequality without affecting the solution provided you immediately change the direction of the inequality!

**Example 2**: Solve $-4x \geq 12$
We divide both sides of the inequality by $-4$. Since $-4$ is negative, we must immediately reverse the direction of the inequality:

$-4x > 12$

$\frac{-4x}{-4} < \frac{12}{-4}$

$x < -3$
Solving Linear Inequalities:

**Remember:** You can always simplify any side of the inequality at any time. You can solve linear inequalities similar to linear equations, as long as you remember to reverse the direction whenever multiplying or dividing by a negative quantity.

1. Simplify the expressions on both sides of the inequality. You must first eliminate all of the Grouping symbols and simplify if necessary.

2. If there are numerical fractions, you may multiply both sides of the inequality by the LCD. If the LCD is negative, make sure you reverse the direction! Simplify if necessary.

3. You want the variable terms on one side of the inequality and the constant terms on the other side. If this is not the case, you should Add or subtract to both sides of the inequality a variable term and/or a numerical value in order to get the variable terms on one side and the constant values on the other side. Simplify if necessary.

4. You want the variable term coefficient to be 1. If this is not the case, you should Divide both sides of the equation by the variable term coefficient. If the coefficients negative, make sure you reverse the direction! Simplify if necessary.

**Note:** Always make sure your final answer has the variable on the left side of the inequality!!!
Absolute Value Inequalities

\[ |x| = \begin{cases} 
  x & \text{if } x \geq 0 \\
  -x & \text{if } x < 0 
\end{cases} \]

Solving Absolute Value Inequalities:

You must put the absolute value on a side by itself!

1. If \(|x| < c\) (\(c > 0\)), then \(-c < x < c\)

Notice that you have three sides to work with.

2. If \(|x| > c\) (\(c > 0\)), then \(x < -c\) or \(x > c\)

Notice that you have to solve two different inequalities at the same time.

3. If \(|x| < -c\) (\(c > 0\)), then there is no solution

4. If \(|x| > -c\) (\(c > 0\)), then \(x\) can be any real number

Note: Always make sure your final answer has the variable on the left side of the inequality, if possible!!!
# Properties of Exponents

<table>
<thead>
<tr>
<th>Property:</th>
<th>Example:</th>
</tr>
</thead>
<tbody>
<tr>
<td>$b^0 = 1$</td>
<td>$5^0 = 1$</td>
</tr>
<tr>
<td>$b^{-n} = \frac{1}{b^n}$</td>
<td>$3^{-2} = \frac{1}{3^2} = \frac{1}{9}$</td>
</tr>
<tr>
<td>$b^m \cdot b^n = b^{m+n}$</td>
<td>$2^3 \cdot 2^5 = 2^8$</td>
</tr>
<tr>
<td>$\frac{b^m}{b^n} = b^{m-n}$</td>
<td>$\frac{5^9}{5^7} = 5^2 = 25$</td>
</tr>
<tr>
<td>$(b^m)^n = b^{m \cdot n}$</td>
<td>$(2^3)^4 = 2^{12} = 4096$</td>
</tr>
<tr>
<td>$(a \cdot b)^n = a^n \cdot b^n$</td>
<td>$(4x)^3 = 4^3 \cdot x^3 = 64x^3$</td>
</tr>
<tr>
<td>$\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$</td>
<td>$\left(\frac{3}{5}\right)^2 = \frac{3^2}{5^2} = \frac{9}{25}$</td>
</tr>
</tbody>
</table>

## Some Additional Properties

\[
\left(\frac{a}{b}\right)^{-n} = \frac{b^n}{a^n} \quad \frac{1}{b^{-n}} = b^n \quad \frac{a^{-n}}{b^{-m}} = \frac{b^m}{a^n}
\]
**Remember:** Only a factor may be moved from the numerator to the denominator or from the denominator to the numerator. When that happens, its exponent changes signs!

\[
x^{-n} = \frac{1}{x^n} \quad \frac{1}{x^{-n}} = x^n
\]

\[
x^n = \frac{1}{x^{-n}} \quad \frac{1}{x^n} = x^{-n}
\]

\[
\frac{x^2x^{-3}}{x^{-7}x^4} = \frac{x^2}{x^4x^3} = \frac{x^9}{x^7} = x^2
\]

**Remember:** All nonzero terms with a zero exponent equal 1. All exponents (including the zero exponent) only refer to the single factor immediately below and to the left of the exponent.

**Examples:** Simplify the following using mathematics writing style:

a) \[ -7^0 + 3x^0 \]
   \[ = -(7^0) + 3(x^0) \]
   \[ = -1 + 3(1) \]
   \[ = 2 \]

b) \[ 2x^{-3} + (3x)^{-1} \]
   \[ = 2(x^{-3}) + (3x)^{-1} \]
   \[ = \frac{2}{x^3} + \frac{1}{3x} \]
Scientific Notation

A positive number $x$ in scientific notation has the form

$$x = s \times 10^n$$

where $n$ is any integer and $s$ is a number such that $1 \leq s < 10$. The trick is to count how many places it takes to move the decimal point to the appropriate position. Also, remember that “it's positive to move to the left.”

<table>
<thead>
<tr>
<th>Decimal Notation</th>
<th>Scientific Notation</th>
</tr>
</thead>
<tbody>
<tr>
<td>15</td>
<td>$1.5 \times 10^1$</td>
</tr>
<tr>
<td>0.003</td>
<td>$3.0 \times 10^{-3}$</td>
</tr>
<tr>
<td>254</td>
<td>$2.54 \times 10^2$</td>
</tr>
<tr>
<td>0.00000342</td>
<td>$3.42 \times 10^{-6}$</td>
</tr>
<tr>
<td>2</td>
<td>$2.0 \times 10^0$</td>
</tr>
</tbody>
</table>

**Example:** Simplify $\frac{(75000000)(200)}{(800000)(0.00025)}$ using scientific notation.

$$\frac{(75000000)(200)}{(800000)(0.00025)} = \frac{(7.5 \times 10^7)(2 \times 10^2)}{(8 \times 10^5)(2.5 \times 10^{-4})}$$

$$= \frac{3}{4} \times 10^8$$

$$= 0.75 \times 10^8$$

$$= 7.5 \times 10^7$$