Adding and Subtracting Polynomials

When you add polynomials, simply combine all like terms.

When subtracting polynomials, do not forget to use parentheses when needed!

Recall the distributive property:

\[ a(b + c) = ab + ac. \]

**Example:** When multiplying polynomials, do not forget to use parentheses when you multiply polynomials. Simply combine all like terms.

Multiplying Polynomials

**Example:** When subtracting polynomials, change the sign of the terms of the polynomial immediately following the minus sign.

This suggests that when you subtract polynomials, change all signs of the terms of the polynomial immediately following the minus

\[ a(b + c) = ab + ac. \]
Special Products of Polynomials

Take time and learn these formulas. You will need them later and you will be tested on them!

$(F + S)(F - S) = F^2 - S^2$ Difference of squares

$(F + S)^2 = F^2 + 2FS + S^2$ Perfect square trinomial

$(F - S)^2 = F^2 - 2FS + S^2$ Perfect square trinomial

$(F + S)(F^2 - FS + S^2) = F^3 + S^3$ Sum of cubes

$(F - S)(F^2 + FS + S^2) = F^3 - S^3$ Difference of cubes

**WARNING:** Unless the thinking that $(x + y)^2$ equals $x^2 + y^2$ will not be tolerated!!

Table to demonstrate that $(x + y)^2 = x^2 + 2xy + y^2$

### Difference of cubes

$F^3 - S^3 = (F + S)(F^2 - FS + S^2)$

### Sum of cubes

$S + F^3 = (S^2 - FS + F^2)(S + F)$

### Perfect square trinomial

$x + F^2S = (S - F)(S + F)$

$xS + 2FS = (S - F)(S + F)$

$xS - F^2 = (S - F)(S + F)$

_Nonsense like thinking that $(x + y)^2$ equals $x^2 + y^2$ will not be tolerated!!_
We use the table below for reference to help with multiplication.

<table>
<thead>
<tr>
<th>Expression</th>
<th>Result</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(F + S)(F - S)$</td>
<td>$F^2 - S^2$</td>
</tr>
<tr>
<td>$(F + S)^2$</td>
<td>$F^2 + 2FS + S^2$</td>
</tr>
<tr>
<td>$(F - S)^2$</td>
<td>$F^2 - 2FS + S^2$</td>
</tr>
</tbody>
</table>

Example: Perform the indicated products and simplify.

**Difference of squares**

Let $F = 3xy$

Let $S = 4z$

$(3xy - 4z)(3xy + 4z) = (F - S)(F + S) = F^2 - S^2 = 9x^2y^2 - 16z^2$

**Perfect square trinomial**

Let $F = x$

Let $S = 5$

$(x + 5)^2 = (F + S)^2 = F^2 + 2FS + S^2 = x^2 + 10x + 25$

Let $F = 2s$

Let $S = 3t$

$(2s - 3t)^2 = (F - S)^2 = F^2 - 2FS + S^2 = 4s^2 - 12st + 9t^2$

**Factoring**

Factoring is the inverse operation to multiplying polynomials. Remember: In order to be good at factoring polynomials, you must be good at multiplying polynomials. Make sure you are very good at multiplying polynomials and using the special products.

Special Polynomial Factors

<table>
<thead>
<tr>
<th>Factor</th>
<th>Polynomial</th>
</tr>
</thead>
<tbody>
<tr>
<td>Common factor</td>
<td>$ax + ay = a(x + y)$</td>
</tr>
<tr>
<td>Sum/Diff. of squares</td>
<td>$x^2 + y^2$</td>
</tr>
<tr>
<td>Difference of squares</td>
<td>$(x - y)(x + y)$</td>
</tr>
<tr>
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</tr>
<tr>
<td>Difference of cubes</td>
<td>$(x + y)(x^2 - xy + y^2)$</td>
</tr>
<tr>
<td>Perfect square trinomial</td>
<td>$x^2 + 2xy + y^2 = (x + y)^2$</td>
</tr>
<tr>
<td>Perfect square trinomial</td>
<td>$x^2 - 2xy + y^2 = (x - y)^2$</td>
</tr>
</tbody>
</table>

Example: Perform the indicated products and simplify.

$4z^2 - 27 = 31$

Let $F = z$

Let $S = 3z$

$F^2 - 2FS = (S - F)^2 = (3z - z)^2 = 4z^2 - 12z + 9z^2$

$F^2 + 2FS = (S + F)^2 = (3z + z)^2 = 9z^2 + 18z + 9z^2$

$F^2 - 2FS = (S - F)(S + F) = (3z - z)(3z + z) = 4z(4z) = 16z^2$

$F^2 + 2FS = (S + F)(S - F) = (3z + z)(3z - z) = 4z(2z) = 8z^2$

We use the table below for reference to help with multiplication.
Common Factors & Negative Exponents

Anytime you factor, remember to take out the common factor first!! Also, the biggest factor uses the smallest exponent.

Examples:

- Factor the following and use mathematics writing style:
  - **GCF:** \(4z^2\)
    - \(z\) smallest exponent: 2
  - **GCF:** \(6A^{-8}B^{-9}\)
    - \(A\) smallest exponent: -8
    - \(B\) smallest exponent: -9

- **a)** \(12x^6 - 16x^2\)
  \(= 4z^2 (3x^4 - 4)\)

- **h)** \(30A^{-5}B^{-9} - 24A^{-8}B^{-5}\)
  \(= 6A^{-8}B^{-9} (5A^{-3} - 4B^{-4})\)

- **Factoring By Grouping**

Examples:

- **a)** \(ax + ay + bx + by\)
  \(= a(x + y) + b(x + y)\)
  \(= (a + b) (x + y)\)

- **c)** \(x^2 + 4xy - 2x - 8y\)
  \(= x(x + 4y) - 2(x + 4y)\)
  \(= (x - 2)(x + 4y)\)

- **e)** \(x^3 - 4x^2 + 3x - 12\)
  \(=x^2(x - 4) + 3(x - 4)\)
  \(= (x^2 + 3)(x - 4)\)

- **b)** \(ax + ay - x - y\)
  \(= a(x + y) - 1(x + y)\)
  \(= (a - 1) (x + y)\)

- **d)** \(xy - 3x - 4y + 12\)
  \(= x(y - 3) - 4(y - 3)\)
  \(= (x - 4)(y - 3)\)

- **f)** \(2x^2 - 7xy + 6y^2\)
  \(=2 x^2 - 4xy - 3xy + 6y^2\)
  \(= 2x(x - 2y) - 3y(x - 2y)\)
  \(= (2x - 3y)(x - 2y)\)

- **Factoring First!! Also, the biggest factor uses the smallest exponent.**
  When factoring, remember to take out the common factors first. Whenever you factor, remember to take out the common factors first.
Factoring Polynomials

Using The ac–Method

\[ ax^2 + bx + c \]

We assume that \( a, b, \) and \( c \) are integers.

Step 1. Multiply \( a \cdot c \)

Step 2. If you can, find two integers such that:

a. Their product is \( ac \)
b. Their sum is \( b \)

Step 3. If two integers don't exist, STOP because the problem cannot be factored. Otherwise, move on to Step 4.

Step 4. Rewrite the middle term \( (bx) \) using the two integers from Step 2 as coefficients.

Step 5. Factor by grouping (like when there are 4 terms).

Examples: Factor the following using the ac–method:

\[ a) \quad 4x^2 – x – 18 = 4x^2 – 9x + 8x – 18 = x(4x – 9) + 2(4x – 9) = (x + 2)(4x – 9) \]

\[ b) \quad 12x^2 – 23x – 24 = 12x^2 – 32x + 9x – 24 = 4x(3x – 8) + 3(3x – 8) = (4x + 3)(3x – 8) \]

Sum / Difference of Squares & Cubes

You are required to determine if the sum or difference of squares or cubes. If so, then factor the polynomial using the following relationships:

- Sum of Squares: \( F^2 + S^2 = (F + S)(F^2 – FS + S^2) \)
- Difference of Squares: \( F^2 – S^2 = (F – S)(F + S) \)
- Difference of Cubes: \( F^3 – S^3 = (F – S)(F^2 + FS + S^2) \)
- Sum of Cubes: \( F^3 + S^3 = (F + S)(F^2 – FS + S^2) \)

Examples: Is the polynomial is the sum of squares, difference of squares, difference of cubes, or the sum of cubes? If so, then factor it.

\[ a) \quad A^2 + 64 \text{ does not factor} \]

\[ b) \quad w^2 – 25 \text{ d.s. } (w – 5)(w + 5) \]

\[ c) \quad z^2 – 8 \text{ no} \]

\[ d) \quad z^3 – 8 \text{ d.c. } (z – 2)(z^2 + 2z + 4) \]

\[ e) \quad 4x^2 – 9y^4 \text{ d.s. } (2x – 3y^2)(2x + 3y^2) \]

\[ f) \quad z^3 + 125 \text{ s.c. } (z + 5)(z^2 – 5z + 25) \]

\[ g) \quad B^2 + 9 \text{ s.s. does not factor} \]

\[ h) \quad c^2 – 16 \text{ d.s. } (c – 4)(c + 4) \]

\[ i) \quad c^3 – 16 \text{ no} \]

\[ j) \quad x^2y^6 – 81z^{10} \text{ d.s. } (xy^3 – 9z^5)(xy^3 + 9z^5) \]

\[ k) \quad 8y^3z^6 + 27w^3 \text{ s.c. } (2yz^2 + 3w)(4y^2z^4 – 6yz^2w + 9w^2) \]
Perfect Square Trinomials

You are required to determine at a quick glance any special product polynomial. If so, then factor it.

F2 + 2FS + S2 = (F + S)2
F2 – 2FS + S2 = (F – S)2

Examples: Is the polynomial a perfect square trinomial? If so, then factor it.

x2 – x + 1 no
x2 – 2x + 1 yes (x – 1)2
x2 + 6x + 9 yes (x + 3)2
x2 – 4x – 4 no
x2 – 4x + 4 yes (x – 2)2
a2 – 10a + 25 yes (a – 5)2
9B2 – 12BC + 4C2 yes (3B – 2C)2
z2 + 13z + 36 no
25z2 + 60z + 36 yes (5z + 6)2

Factoring With Substitution

Examples: Factor the following and use mathematics writing style:

a) x3 + 8 = (x + 2)(x2 – 2x + 4)
b) x6 – 81z10 = (x3 + 3z5)(x3 – 3z5)
c) x2 – (y + 2)2 = (x – y – 2)(x + y + 2)
d) (x + 2y)2 – 25 = (x + 2y – 5)(x + 2y + 5)
e) (2x – 5)2 – 2(2x – 5) – 8 = (2x – 9)(2x – 3)
f) (x – 3)2 – 9(y + 2)2 = (x – 3y – 9)(x + 3y + 3)
**Factoring Completely**

You are always required to continue factoring until every expression can be factored no further.

**Steps For Factoring Polynomials**

1. Always take out the greatest common factor first!
   (This includes the case with negative exponents.)

2. See if you can factor by grouping.

   A. If there are 2 terms:
      a. See if it is a difference of squares
      b. See if it is a sum or difference of cubes

   B. If there are 3 terms:
      See if it is a perfect square trinomial

   C. If there are 4 terms:
      See if you can factor by grouping.

   Try factoring by any other method you can.

3. Try factoring by any other method you can.
   (Substitution, trial and error, ac method, etc.)

Hint: Whenever you have to factor, DO NOT MULTIPLY OUT the terms unless you have to order! First consider using a substitution, trial and error, or any other method you can.

Examples: Factor completely the following and use mathematicians' writing style:

a) \( w - 7 - 4w^{-9} = w^{-9}(w^2 - 4) = w^{-9}(w - 2)(w + 2) \)

b) \( 4 + 4z^{-1} + z^{-2} = z^{-2}(4z^2 + 4z + 1) = z^{-2}(2z + 1)^2 \)

c) \( (x^2 - x)^2 - 18(x^2 - x) + 72 = (x^2 - x - 6)(x^2 - x - 12) = (x - 3)(x + 2)(x - 4)(x + 3) \)

d) \( x^8 - y^8 = (x^4 - y^4)(x^4 + y^4) = (x^2 - y^2)(x^2 + y^2)(x^4 + y^4) = (x - y)(x + y)(x^2 + y^2)(x^4 + y^4) \)

e) \( (x^2 + 3x - 10)^2 - (x - 2)^2 = (x^2 + 3x - 10 + x - 2)(x^2 + 3x - 10 - x + 2) = (x^2 + 4x - 12)(x^2 + 2x - 8) = (x + 6)(x - 2)(x + 4)(x - 2) = (x + 6)(x + 4)(x - 2)^2 \)
Solving Polynomial Equations

Zero Product Property

If a and b are any real numbers whose product is zero, then we know that a is zero or b is zero. That is

\[ a \cdot b = 0 \implies a = 0 \text{ or } b = 0 \]

This property allows us to solve many different types of equations.

Example 1:

\[
(x - 3)(x + 7) = 0
\]

\[
x - 3 = 0 \quad x + 7 = 0
\]

\[
x = 3 \quad x = -7
\]

Example 2:

\[
5x(x + 1)(x - 2) = 0
\]

\[
5x = 0 \quad x + 1 = 0 \quad x - 2 = 0
\]

\[
x = 0 \quad x = -1 \quad x = 2
\]

Notice that if \( a \cdot b = 1 \), then we know nothing about a or b except that neither one is equal to zero. This is a unique property of zero.

**WARNING:**

Improper use of the Zero Product Property will not be tolerated.

**Zero Product Property**

This property is only true for factors whose product is zero.

Solving Equations

By Factoring

(SFS Method)

1. Put the equation in Standard form. This means you must get a zero on one side of the equation.
2. Factor the one side.
3. Solve by using the Zero Product Property. (Set each variable factor equal to zero and solve.)
4. Check your answers.
Examples:

Solve the following equations by factoring:

Step 1: Standard form
Step 2: Factor completely
Step 3: Set factors to 0, solve

Don't need \( z + 5 = 0 \) twice

Don't need \( 2z + 5 = 0 \) twice

a) \[ 4x^3 = 100x \]

\[ 4x^3 - 100x = 0 \]

\[ 4x(x^2 - 25) = 0 \]

\[ 4x(x - 5)(x + 5) = 0 \]

\[ 4x = 0 \quad x - 5 = 0 \quad x + 5 = 0 \]

\[ x = 0 \quad x = 5 \quad x = -5 \]

b) \[ (3z + 6)(4z + 12) = -3 \]

\[ 12z^2 + 60z + 72 = -3 \]

\[ 12z^2 + 60z + 75 = 0 \]

\[ 3(4z^2 + 20z + 25) = 0 \]

\[ 3(2z + 5)^2 = 0 \]

\[ 2z + 5 = 0 \]

\[ z = -\frac{5}{2} \]