

# QUANTIFIER ELIMINATION IN ADELIC STRUCTURES OVER ALGEBRAICALLY CLOSED VALUED FIELDS

YIMU YIN

ABSTRACT. Applying Weispfenning’s fundamental work on boolean products, we deduce that the theory of adelic structures over algebraically closed valued fields in the language  $\mathcal{L}_{AA}$  admits quantifier elimination and is complete.

To call a boolean product over algebraically closed valued fields “adelic” one needs to express the product formula in a first-order way. This is not achieved. Here, by an “adelic structure” we just mean a boolean product over algebraically closed valued fields.

We use the Basarab-Kuhlmann style language  $\mathcal{L}_{RV}$  for algebraically closed valued fields of mixed characteristic introduced in [1]. The theory ACVF of algebraically closed valued fields admits quantifier elimination in  $\mathcal{L}_{RV}$ , a short proof of which may be found in [3].

**Theorem 0.1.** *The theory ACVF admits quantifier elimination.*

We first describe an expanded language  $\mathcal{L}_{BF}$  for boolean algebras, which includes the following:

- (1) the language of boolean algebras  $\mathcal{L}_{BA} = \{0, 1, \cap, \cup, \sim, \leq\}$ ;
- (2) a set of unary relations  $\{0 <_n : n \geq 1\}$ ;
- (3) a unary relation  $\mathcal{F}$ ;
- (4) a constant  $\mathbf{a}$ .

The theory of infinite atomic boolean algebras with the distinguished set of finite elements in  $\mathcal{L}_{BF}$  (hereafter abbreviated as IABF) states the following:

- (1) the usual axioms for boolean algebras;
- (2) for every  $\xi > 0$  there is an atom  $\eta$  such that  $\xi \geq \eta$ ;
- (3)  $\mathbf{a}$  is an atom;
- (4) axioms for  $\mathcal{F}$ :
  - (a)  $\mathcal{F}(0)$  and  $\neg\mathcal{F}(1)$ ;
  - (b)  $\mathcal{F}(\xi \cup \eta)$  if and only if  $\mathcal{F}(\xi)$  and  $\mathcal{F}(\eta)$ ;
  - (c) if  $\neg\mathcal{F}(\xi)$  then there is an  $\eta$  such that  $\neg\mathcal{F}(\xi \cap \eta)$  and  $\neg\mathcal{F}(\xi \cap \sim \eta)$ ;
- (5)  $0 <_n \xi$  if and only if there are  $\eta_1, \dots, \eta_n$  such that  $0 \leq \eta_1 < \dots < \eta_n < \xi$  and  $\mathcal{F}(\eta_i)$  for each  $i \leq n$ .

**Theorem 0.2** (Weispfenning [2], Part II, 1.4(ii, iii)). *The theory IABF admits quantifier elimination and is complete.*

So IABF axiomatizes the theory of powerset algebras of infinite set, where  $\mathcal{F}$  ranges over finite subsets.

In order to formulate a first-order language for adelic structures over algebraically closed valued fields we treat  $(\mathcal{L}_{RV}, \mathcal{L}_{BF})$  as a 2-sorted language (these sorts shall be called the first-order sort, or FO-sort for short, and the boolean algebra sort, or

BA-sort for short) and further expand it as follows. For each  $n$ -ary relation symbol  $R$  (including equality and functions) in  $\mathcal{L}_{\text{RV}}$  we add an  $n$ -ary function  $\mathcal{V}_R$  from the FO-sort to the BA-sort. For example, if  $a, b$  are two  $\mathcal{L}_{\text{RV}}$ -terms then  $\mathcal{V}_=(a, b)$  is considered an  $\mathcal{L}_{\text{BF}}$ -term. In fact, since the boolean value of each quantifier-free formula in  $\mathcal{L}_{\text{RV}}$  is determined by the functions  $\mathcal{V}_R$ , for notational simplicity we may think of one function  $\mathcal{V}$  that assigns a boolean value  $\mathcal{V}\phi$  to each quantifier-free  $\mathcal{L}_{\text{RV}}$ -formula  $\phi$ . Let  $\mathcal{L}_{\text{AA}}$  denote this expansion of  $(\mathcal{L}_{\text{RV}}, \mathcal{L}_{\text{BF}})$ .

For each  $\forall\exists$ -formula  $\phi$  in  $\mathcal{L}_{\text{RV}}$  of the form  $\forall\bar{x}\exists\bar{y}\psi(\bar{x}, \bar{y}, \bar{z})$  with  $\psi$  quantifier-free, let  $\phi^{\mathcal{V}}$  be the  $\mathcal{L}_{\text{AA}}$ -formula  $\forall\bar{x}\exists\bar{y}\mathcal{V}(\psi(\bar{x}, \bar{y}, \bar{z})) = 1$ . Now it is routine to check that ACVF is a  $\forall\exists$ -theory in  $\mathcal{L}_{\text{RV}}$ . Let  $\text{ACVF}^{\mathcal{V}} = \{\phi^{\mathcal{V}} : \phi \text{ is an axiom of ACVF}\}$ .

The theory of adelic structures over algebraically closed valued field in  $\mathcal{L}_{\text{AA}}$  (hereafter abbreviated as AACF) states the following:

- (1)  $\text{ACVF}^{\mathcal{V}}$  and IABF for the corresponding sorts;
- (2)  $\mathcal{V}(\text{char } \bar{K} = p)$  is an atom for each prime number  $p$ ;
- (3)  $\mathbf{a} < \mathcal{V}(\text{char } \bar{K} > p)$  for each prime number  $p$ ;
- (4) the axioms for abstract boolean products (hereafter abbreviated as ABP):
  - (a)  $\phi \leftrightarrow \mathcal{V}\phi = 1$  for each atomic formula  $\phi$  in  $\mathcal{L}_{\text{RV}}$ ;
  - (b)  $\mathcal{V}(x = y) = \mathcal{V}(y = x)$ ;
  - (c)  $\bigcap_{i=1}^n \mathcal{V}(x_i = y_i) \cap \mathcal{V}\phi(x_1, \dots, x_n) \leq \mathcal{V}\phi(y_1, \dots, y_n)$  for each atomic formula  $\phi$  in  $\mathcal{L}_{\text{RV}}$ ;
  - (d) **Finitary gluing:** For all  $x, y \in \text{FO}$  and  $\alpha, \beta \in \text{BA}$ , if  $\alpha \cap \beta = 0$  and  $\alpha \cup \beta = 1$  then there is a  $z$  of the first sort such that  $\mathcal{V}(z = x) \geq \alpha$  and  $\mathcal{V}(z = y) \geq \beta$ .

**Proposition 0.3.** *The theory AACF admits quantifier elimination in all sorts and is complete.*

*Proof.* Quantifier elimination is immediate by Theorem 0.1, Theorem 0.2, and [2, Part II, 3.7(ii)]. Completeness follows from the representation theorem [2, Part I, 3.27]. (Details to be added later.)  $\square$

## REFERENCES

1. Ehud Hrushovski and David Kazhdan, *Integration in valued fields*, math.AG/0510133.
2. Volker Weispfenning, *Model theory of lattice products*, Habilitationsschrift, Universität Heidelberg, 1978.
3. Yimu Yin, *Grothendieck homomorphisms in algebraically closed valued fields*, arXiv:0809.0473v1, 2008.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF PITTSBURGH, 301 THACKERAY HALL, PITTSBURGH, PA 15260

*E-mail address:* yimu.yin@gmail.com