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HENSELIANITY AND THE DENEF-PAS LANGUAGE

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Abstract. We prove that if an equicharacteristic valued field has a $\mathbb{Z}$-group as its value group and admits quantifier elimination in the main sort of the prototypical Denef-Pas style language then it is henselian. In fact the proof of this suggests that a reasonable class of Denef-Pas style languages is natural with respect to henselianity.

§1. Introduction.

Tarski's theorem says that the theory RCF of real closed fields, as formulated in the language $\mathcal{L}_{\mathrm{OR}}$ of ordered rings, admits quantifier elimination (QE). It is natural to ask whether any other ordered fields admit QE in $\mathcal{L}_{\mathrm{OR}}$. There is a good answer to this:

Theorem 1.1 (Macintyre, McKenna, van den Dries). *Let K be an ordered field such that the theory of K in $\mathcal{L}_{\mathrm{OR}}$ admits QE. Then K is real closed.*

This is a prototypical example of a "converse QE" result; it shows that for the class of ordered fields, real-closedness is equivalent to QE.

There are analogous results in the class of valued fields. In the forward direction, the first result is due to Macintyre [4], who showed that the theory of p-adic fields, as formulated in the language $\mathcal{L}_{\mathrm{Mac}}$, admits QE. In this case, we only have a partial converse:

Theorem 1.2 (Macintyre, McKenna, van den Dries). *Let K be a p-field such that the theory of K in $\mathcal{L}_{\mathrm{Mac}}$ admits QE. Then K is p-adically closed.*

The definition of a p-field is rather special: it is a substructure of a p-adically closed field (of p-rank 1) with respect to $\mathcal{L}_{\mathrm{Mac}}$. Let K be a p-field and L a p-adically closed field such that K is an $\mathcal{L}_{\mathrm{Mac}}$-substructure of L. The point is that, as L is henselian, each nth power predicate P_n defines a clopen subset of K in the valuation topology of K, which is essential to the proof of the theorem. This way to interpret each P_n is not very satisfying since an element in P_n may not be an nth power at all in K. Hence it is asked in [5] to extend the result to the class of valued fields where P_n is simply interpreted as the group of nth powers. In [7] this is established for a subclass of such structures.

In this paper we shall prove a converse QE theorem for a different kind of language for valued fields:

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THEOREM 1.3. *Let* $S = \langle K, \overline{K}, \Gamma \cup \{\infty\}, v, \overline{\mathrm{ac}} \rangle$ *be a structure of the Denef-Pas style language* $\mathcal{L}_{\mathrm{RRP}}$ *such that*

1. K *and* $\overline{K}$ *are fields such that* $\operatorname{char} K = \operatorname{char} \overline{K}$*,*
2. $v : K \longrightarrow \Gamma \cup \{\infty\}$ *is a valuation map and* $\overline{\mathrm{ac}} : K \longrightarrow \overline{K}$ *is an angular component map,*
3. *the value group* Γ *is a* $\mathbb{Z}$*-group,*
4. *the theory* $\mathrm{Th}(S)$ *admits QE in the* K*-sort.*

Then the valuation v *is henselian.*

This answers a question mentioned in [1]. This result also holds in slightly more general settings; see Remark 4.11 and Remark 4.13. The relevant definitions will be given in the next section.

§2. Preliminaries. In this paper all valued fields are equicharacteristic and all valuation rings are proper subrings. We use $\mathcal{O}$, $\mathcal{O}_1$, etc. and $\mathcal{M}$, $\mathcal{M}_1$, etc. to denote valuation rings and their maximal ideals, respectively. Valuation maps are denoted by v, v_1, etc. If v is a valuation of K then vK, $\overline{K}$ stand for the corresponding value group and residue field, respectively.

Next we describe the Denef-Pas style language for valued fields.

DEFINITION 2.1. Let K be a valued field and $\overline{K}$ its residue field. An *angular component map* is a function $\overline{\mathrm{ac}} : K \longrightarrow \overline{K}$ such that

1. $\overline{\mathrm{ac}}\, 0 = 0$,
2. the restriction $\overline{\mathrm{ac}} \restriction K^{\times}$ is a group homomorphism $K^{\times} \longrightarrow \overline{K}^{\times}$,
3. the restriction $\overline{\mathrm{ac}} \restriction (\mathcal{O} \setminus \mathcal{M})$ is the projection map, that is, $\overline{\mathrm{ac}}\, u = u + \mathcal{M}$ for all $u \in \mathcal{O} \setminus \mathcal{M}$.

The template for Denef-Pas style language has three sorts: the field sort, the residue field sort, and the value group sort. These are usually denoted by K, $\overline{K}$, and Γ, respectively. Sometimes we shall refer to the K-sort as the "main sort". The K-sort and $\overline{K}$-sort use the language $\mathcal{L}_{\mathrm{R}}$ of rings. The Γ-sort uses the langauge $\mathcal{L}_{\mathrm{OG}}$ of ordered groups, $\{+, <, 0\}$, and an additional symbol ∞ that designates the top element in the ordering. There are two cross-sort function symbols: $v : K \longrightarrow \Gamma \cup \{\infty\}$, which stands for the valuation, and $\overline{\mathrm{ac}} : K \longrightarrow \overline{K}$, which stands for an angular component map.

Any language that expands this template is a Denef-Pas language. A prototypical example is the language $\mathcal{L}_{\mathrm{RRP}}$ used in [6], in which the field sort and the residue field sort use the language $\mathcal{L}_{\mathrm{R}}$ and the Γ-sort uses the language $\mathcal{L}_{\mathrm{Pr}\infty} = \mathcal{L}_{\mathrm{Pr}} \cup \{\infty\}$, where $\mathcal{L}_{\mathrm{Pr}}$ is the Presburger language $\{+, <, 0, 1\} \cup \{D_n : n > 1\}$. Let $S = \langle K, \overline{K}, \Gamma \cup \{\infty\}, v, \overline{\mathrm{ac}} \rangle$ be a structure of $\mathcal{L}_{\mathrm{RRP}}$. One of the main results of [6] is that if K is henselian and both K and $\overline{K}$ are of characteristic 0 then $\mathrm{Th}(S)$ admits QE in the K-sort; that is, for every formula φ in $\mathcal{L}_{\mathrm{RRP}}$ there is a formula φ^* in $\mathcal{L}_{\mathrm{RRP}}$ that does not contain K-quantifiers such that $S \models \varphi \leftrightarrow \varphi^*$. Hence Theorem 1.3 contains a converse of this result with respect to henselianity under the additional assumption that Γ is a $\mathbb{Z}$-group.

The following notions are formulated for any Denef-Pas language $\mathcal{L}$, where we use $\mathcal{L}_K$, $\mathcal{L}_{\overline{K}}$, and $\mathcal{L}_{\Gamma\infty}$ to denote the languages used by the three sorts.

DEFINITION 2.2. A formula φ in $\mathcal{L}$ is *simple* if φ does not contain any K-quantifiers.

DEFINITION 2.3. A formula φ in $\mathcal{L}_K \cup \mathcal{L}_{\Gamma\infty}$ is a Γ-*formula* if it does not contain K-quantifiers or atomic formulas in $\mathcal{L}_K$. Similarly a formula φ in $\mathcal{L}_K \cup \mathcal{L}_{\overline{K}}$ is a $\overline{K}$-*formula* if it does not contain K-quantifiers or atomic formulas in $\mathcal{L}_K$.

Note that Γ-formulas and $\overline{K}$-formulas may contain $\mathcal{L}_K$-terms. We shall simplify our terminology for these formulas. For example, a literal Γ-formula shall be called a "Γ-literal". Similarly for $\overline{K}$-formulas.

§3. Overview of the proof. The proof relies on the approximation technique devised in [5]. In general this technique consists of the following steps. Let $\mathcal{L}$ be a language for valued fields in which henselianity is first-order expressible. The main sort of $\mathcal{L}$ is the field sort. Let (K, v) be a valued field such that $\mathrm{Th}(K)$ admits QE in the main sort of $\mathcal{L}$, where $\mathrm{Th}(K)$ denotes the theory of K as a structure of $\mathcal{L}$. Let $\mathcal{O}, \mathcal{M}$ be its valuation ring and maximal ideal.

- **Step 1**. Show that, except equations in the field, all formulas without quantifier ranging over the main sort define open sets in (the product of) the valuation topology. Note that, for each formula $\varphi(X)$, the assertion that it defines an open set can be expressed by a first-order sentence:

  $$\forall X\ (\varphi(X) \to \exists Y\ (v(Y) > v(X) \wedge \forall Z\ (v(Z) > v(Y) \to \varphi(X+Z)))).$$

- **Step 2**. Let $F(X, \bar{a}) \in \mathcal{O}[X]$ be a monic polynomial of degree n, where $\bar{a}$ are the coefficients. Suppose for contradiction that $F(X, \bar{a})$ is a counterexample to a version of Hensel's Lemma: there is an $s \in \mathcal{O}$ such that $F(s, \bar{a}) \in \mathcal{M}$ and $F'(s, \bar{a}) \notin \mathcal{M}$ but $F(X, \bar{a})$ has no root in K. We may assume that $F(X, \bar{a})$ is irreducible over K. Let φ be the formula that defines the nonempty set of the tuples of coefficients of all such counterexamples. By assumption, φ is equivalent to a formula that is quantifier-free in the main sort. Without loss of generality we may assume that φ is in disjunctive normal form. Using the fact that the Vandermonde matrix of $F(X, \bar{a})$ is invertible, we may construct polynomials $F_1(\overline{Y}), \ldots, F_n(\overline{Y}) \in \mathcal{O}[\overline{Y}]$, where $\overline{Y}$ is a tuple of variables $Y_1, \ldots, Y_n$, such that
  1. they are algebraically independent over K,
  2. $F(X, F_1(\bar{b}), \ldots, F_n(\bar{b}))$ has no root in K for every $\bar{b} = (b_1, \ldots, b_n) \in K^n$ with $b_i \neq 0$ for some $i > 1$,
  3. $F(X, F_1(0, 1, 0, \ldots), \ldots, F_n(0, 1, 0, \ldots)) = F(X, \bar{a})$.

  It follows that there is an open neighborhood U of $(0, 1, 0, \ldots)$ in the product topology on K^n such that $(F_1(\bar{b}), \ldots, F_n(\bar{b}))$ satisfies φ for every $\bar{b} \in U$. Since U is not contained in any proper Zariski closed subset of K^n, there must be a disjunct φ_0 of φ that lacks equational conditions and hence, by Step 1, defines a nonempty open subset of K^n. Without loss of generality $\bar{a} \in \varphi_0(K^n)$. For details see [5, Theorem 1, 4].
- **Step 3**. If K is dense in its henselization K^h then the approximation can be carried out as follows: Choose a root $r \in K^h$ of $F(X, \bar{a})$ and write

  $$F(X, \bar{a}) = (X - r)F^*(X, \bar{b}),$$

where $\bar{b} \in K^h$ are the coefficients of F^*. Let $U \subseteq \varphi_0(K^n)$ be an open neighborhood of $\bar{a}$, where φ_0 is as in Step 2. Now we can choose $r', \bar{b}' \in K$ that are arbitrarily close to $r, \bar{b}$ with respect to the valuation. Write

$$F(X, \bar{a}') = (X - r')F^*(X, \bar{b}').$$

So $\bar{a}'$ are arbitrarily close to $\bar{a}$ and hence $\bar{a}' \in U$. But then $F(X, \bar{a}')$ have a root in K, contradicting the choice of U.

- **Step 4**. However, in general K is not dense in its henselization. The solution to this in [5] is rather specialized. Dickmann [2] uses a more general method to get around this problem. Using the Omitting Types Theorem, another valued field (L, w) may be constructed such that (L, w) is elementarily equivalent to (K, v) with respect to $\mathcal{L}$ and w is of rank 1 (that is, wL is a subgroup of the additive group of $\mathbb{R}$ with the canonical ordering). It is well-known that if the valuation w for L is of rank 1 then L is dense in its henselization; see the discussion in [3, p. 53]. Hence the argument outlined above can be used to show that (L, w) is henselian. Consequently (K, v) is henselian.

Note that Step 2 can always be implemented for any valued field that is not henselian. So the bulk of the work in the sequel will concentrate on Step 1, 3, and 4.

§4. Henselianity and Denef-Pas style languages. We shall prove Theorem 1.3 in this section. The proof of this theorem can be adapted for other Denef-Pas style languages as well, provided that the value group satisfies certain mild conditions; see Remark 4.11.

Throughout this section let $S = \langle K, \overline{K}, \Gamma \cup \{\infty\}, v, \overline{\mathrm{ac}} \rangle$ be a structure of $\mathcal{L}_{\mathrm{RRP}}$ that satisfies the assumptions of Theorem 1.3. We shall work in S.

In this section the following notational conventions are adopted. We use X, Y, etc. for K-sort variables, M, N, etc. for Γ-sort variables, and Ξ, Λ, etc. for $\overline{K}$-sort variables. The lowercase of these letters stands for closed terms or elements in the corresponding sorts. Unless indicated otherwise, all these letters stand for tuples of variables whenever they appear in a formula. We use $\mathrm{lh}\, X$ to denote the length of X. Let $\mathbb{Z}$ be the subring generated by 1 in K and $\mathbb{Z}_\Gamma$ the subgroup generated by 1 in Γ.

Remark 4.1. The theory of $\mathbb{Z}$-groups with a top element in $\mathcal{L}_{\mathrm{Pr}\infty}$ admits QE. This follows from a straightforward generalization of [6, Lemma 5.4, 5.5] to $\mathbb{Z}$-groups.

The following lemma is slightly more general than [6, Lemma 5.3].

LEMMA 4.2. *Let φ be a simple formula in $\mathcal{L}_{\mathrm{RRP}}$. Then φ is equivalent to a formula of the form*

$$\bigvee_i (\sigma_i \wedge \chi_i \wedge \theta_i)$$

where σ_i is a quantifier-free formula in $\mathcal{L}_K$, χ_i a $\overline{K}$-formula, and θ_i a Γ-formula.

PROOF. We can write φ in its prenex normal form $Q_1 \dots Q_k\ \psi$ where each Q_j is either a Γ-quantifier or a $\overline{K}$-quantifier and ψ is a quantifier-free formula. We proceed by induction on the number k of quantifiers.

If $k = 0$ then φ is quantifier-free. Since there are no symbols in $\mathcal{L}_{\mathrm{RRP}}$ relating the $\overline{K}$-sort and the Γ-sort, φ can be written in its disjunctive normal form

$$\bigvee_i (\sigma_i \wedge \chi_i \wedge \theta_i)$$

where σ_i is a conjunction of literals in $\mathcal{L}_K$, χ_i a conjunction of $\overline{K}$-literals, and θ_i a conjunction of Γ-literals. This proves the base case.

Suppose now $k = l + 1$ and Q_1 is $\exists N$. So by the inductive hypothesis φ can be written in the form

$$Q_1 \ \bigvee_i (\sigma'_i \wedge \chi'_i \wedge \theta'_i)$$

where σ'_i is a quantifier-free formula in $\mathcal{L}_K$, χ'_i a $\overline{K}$-formula, and θ'_i a Γ-formula. Now we can simply push the quantifier in and write φ as

$$\bigvee_i (\sigma'_i \wedge \chi'_i \wedge \exists N\ \theta'_i).$$

If Q_1 is $\forall N$ then we can rewrite $\bigvee_i (\sigma'_i \wedge \chi'_i \wedge \theta'_i)$ in its conjunctive normal form and then push the quantifier in. The other two cases of Q_1 being $\exists \Xi$ or $\forall \Xi$ are treated in the same way. $\dashv$

Simple formulas play an important role in this section. Let φ be a simple formula. By Lemma 4.2, φ can be written as a disjunction of conjunctions of formulas of the following forms:

- Type I: $F(X) \,\square\, 0$, where $\square$ is either $=$ or $\neq$ and $F(X) \in \mathbb{Z}[X]$.
- Type II: Γ-formulas. Suppose that $F_i(X) \in \mathbb{Z}[X]$ run through all the distinct polynomials that appear in a formula of this type in the form $vF_i(X)$. For every i, since the formulas $vF_i(X) = \infty$ and $vF_i(X) \neq \infty$ are equivalent to the formulas $F_i(X) = 0$ and $F_i(X) \neq 0$ respectively and the latter ones can be assimilated into Type I, we may assume that $vF_i(X) = \infty$ and $vF_i(X) \neq \infty$ do not occur in φ and $F_i(X) \neq 0$ is a conjunct in each disjunct of φ in which $F_i(X)$ appears in a formula of this type.
- Type III: $\overline{K}$-formulas. Suppose that $F_i(X) \in \mathbb{Z}[X]$ run through all the distinct polynomials that appear in a formula of this type in the form $\overline{\mathrm{ac}}\, F_i(X)$. Similar to Type II, for every i, since the formulas $\overline{\mathrm{ac}}\, F_i(X) = 0$ and $\overline{\mathrm{ac}}\, F_i(X) \neq 0$ are equivalent to the formulas $F_i(X) = 0$ and $F_i(X) \neq 0$, we may assume that $\overline{\mathrm{ac}}\, F_i(X) = 0$ and $\overline{\mathrm{ac}}\, F_i(X) \neq 0$ do not occur in φ and $F_i(X) \neq 0$ is a conjunct in each disjunct of φ in which $F_i(X)$ appears in a formula of this type.

4.1. Step 1: Open sets. Since Step 2, 3, and 4 in Section 3 do not involve formulas that contain free $\overline{K}$-variables or free Γ-variables, we may limit our attention to such formulas of Type I, II, and III. We shall show that such formulas, except equalities in the K-sort, define open sets in the corresponding product of the valuation topology. This takes care of Step 1.

Since polynomials are continuous maps with respect to the valuation topology, it is clear that disequalities in the K-sort define open sets.

LEMMA 4.3. *Let $\varphi(X)$ be a formula of Type II. Then φ defines an open set.*

PROOF. First note that, for $m \in \Gamma$, sets of the forms $\{x : v(x) \square m\}$, where $\square$ is one of the symbols $=$, $\neq$, $<$, $\geq$, are all open in the valuation topology. See [3, Remark 2.3.3].

Let $F_i(X) \in \mathbb{Z}[X]$ run through all the distinct polynomials that appear in $\varphi(X)$ in the form $vF_i(X)$. Let $\varphi^*(M)$ be the formula obtained from $\varphi(X)$ by replacing each $vF_i(X)$ with a new variable M_i. Let B be the set

$$\left\{\langle m_1, \dots, m_d\rangle \in \Gamma^d : S \models \varphi^*(m_1, \dots, m_d)\right\},$$

where $d = \operatorname{lh} M$. For each $m = \langle m_1, \dots, m_d\rangle \in \Gamma^d$ let

$$A_m = \left\{ x \in K^e : \bigwedge_{i=1}^{d} vF_i(x) = m_i \right\},$$

where $e = \operatorname{lh} X$. Since polynomial maps are continuous, each A_m is open in the valuation topology. So $\varphi(K^e) = \bigcup_{m \in B} A_m$ is open. ⊣

Let $\mathcal{O}, \mathcal{M}$ be the valuation ring and its maximal ideal that correspond to v. The following lemma establishes a crucial relation between the valuation and the angular component map.

LEMMA 4.4. *For nonzero $x, y \in K$ with $v(x) = v(y) = m \in \Gamma$, $\overline{\mathrm{ac}}\, x = \overline{\mathrm{ac}}\, y$ if and only if $v(x - y) > m$.*

PROOF. If $x = y$ then the lemma is trivial. So we assume further that $x \neq y$.

For the "only if" direction, suppose for contradiction that $\overline{\mathrm{ac}}\, x = \overline{\mathrm{ac}}\, y$ but $v(x - y) = m$. So $(x - y)/x$ is a unit. So

$$\begin{aligned}
\overline{\mathrm{ac}}\, \frac{x-y}{x} &= (1 - \frac{y}{x}) + \mathcal{M} \\
&= (1 + \mathcal{M}) - \left(\frac{y}{x} + \mathcal{M}\right) \\
&= (1 + \mathcal{M}) - \overline{\mathrm{ac}}\, \frac{y}{x} \\
&= (1 + \mathcal{M}) - \frac{\overline{\mathrm{ac}}\, y}{\overline{\mathrm{ac}}\, x} \\
&= 0.
\end{aligned}$$

So $(x - y)/x = 0$, so $x = y$, contradiction.

For the "if" direction, suppose for contradiction that $v(x - y) > m$ but $\overline{\mathrm{ac}}\, x \neq \overline{\mathrm{ac}}\, y$. If $m = 0$, that is, x and y are units in the valuation ring, then

$$x + \mathcal{M} = \overline{\mathrm{ac}}\, x \neq \overline{\mathrm{ac}}\, y = y + \mathcal{M}\,.$$

So $x - y$ is a unit in the valuation ring, that is, $v(x - y) = 0$, contradiction. In general we may consider $1 - y/x$: since $v(1 - y/x) > 0$ and y/x is a unit, we get $\overline{\mathrm{ac}}\, 1 = \overline{\mathrm{ac}}(y/x)$ by the previous two sentences, so $\overline{\mathrm{ac}}\, x = \overline{\mathrm{ac}}\, y$. ⊣

LEMMA 4.5. *Let $\lambda \in \overline{K}^{\times}$ and $F(X) \in \mathbb{Z}[X]$. The set*

$$A_\lambda = \{x \in K^e : \overline{\mathrm{ac}}\, F(x) = \lambda\}$$

is open, where $e = \mathrm{lh}\, X$.

PROOF. For any $x \in K^e$ such that $F(x) \neq 0$ we consider the open set

$$U = \{F(x) + z : z \in K \text{ and } v(z) > vF(x)\}.$$

Since F is continuous, there is an open neighborhood U_x of x such that $F(U_x) \subseteq U$. Since $vF(y) = vF(x)$ and $v(F(y) - F(x)) > vF(x)$ for every $y \in U_x$, by Lemma 4.4, $\overline{\mathrm{ac}}\, F(y) = \overline{\mathrm{ac}}\, F(x)$. So $A_\lambda = \bigcup_{x \in A_\lambda} U_x$ is open. ⊣

LEMMA 4.6. *Let $\varphi(X)$ be a formula of Type III. Then φ defines an open set.*

PROOF. Let $F_i(X) \in \mathbb{Z}[X]$ run through all the distinct polynomials that appear in $\varphi(X)$ in the form $\overline{\mathrm{ac}}\, F_i(X)$. Let $\varphi^*(\Lambda)$ be the formula obtained from $\varphi(X)$ by replacing each $\overline{\mathrm{ac}}\, F_i(X)$ with a new variable Λ_i. Let B be the set

$$\left\{ \langle \lambda_1, \ldots, \lambda_d \rangle \in (\overline{K}^{\times})^d : S \models \varphi^*(\lambda_1, \ldots, \lambda_d) \right\},$$

where $d = \mathrm{lh}\, \Lambda$. For each $\lambda = \langle \lambda_1, \ldots, \lambda_d \rangle \in (\overline{K}^{\times})^d$ let

$$A_\lambda = \left\{ x \in K^e : \bigwedge_{i=1}^{d} \overline{\mathrm{ac}}\, F_i(x) = \lambda_i \right\},$$

where $e = \mathrm{lh}\, X$. By Lemma 4.5 each A_λ is open. So $\varphi(K^e) = \bigcup_{\lambda \in B} A_\lambda$ is open. ⊣

4.2. Step 3 and 4: Omitting a type. If K is dense in its henselization then, combining the argument in Step 3 and the results in the last section, we see that the conclusion of Theorem 1.3 holds. If K is not dense in its henselization then we need to carry out Step 4. Thus, we shall show:

THEOREM 4.7. *There is a structure $S_1 = \langle K_1, \overline{K}_1, \Gamma_1 \cup \{\infty\}, v_1, \overline{\mathrm{ac}}_1 \rangle$ of $\mathcal{L}_{\mathrm{RRP}}$ such that $S_1 \equiv S$ and v_1 is of rank 1.*

For the rest of this section let X, Y be two single variables. For $r, t \in \mathcal{O}$ we say that they are *comparable*, written as $r \asymp t$, if there is a natural number n such that either $v(r^n) \leq v(t) \leq v(r^{n+1})$ or $v(t^n) \leq v(r) \leq v(t^{n+1})$. They are *incomparable* if they are not comparable. We write $r \ll t$ if r, t are incomparable and $v(r) < v(t)$.

By the Omitting Types Theorem, Theorem 4.7 may be proved by omitting the 2-type

$$\Phi(X, Y) = \left\{ 0 < v(X^l) < v(Y) \wedge Y \neq 0 : l \geq 1 \right\}.$$

Thus it suffices to show that this type is not isolated modulo $\mathrm{Th}(S)$. To that end, suppose for contradiction that there is a formula $\pi(X, Y)$ such that the sentence $\exists X, Y\ \pi(X, Y)$ is in $\mathrm{Th}(S)$ and $\pi(X, Y) \vdash \Phi(X, Y)$ modulo $\mathrm{Th}(S)$. Since $\mathrm{Th}(S)$ admits QE in the main sort, by Lemma 4.2, $\pi(X, Y)$ is equivalent to a disjunction of conjunctions of formulas of Type I, II, and III. Without loss of generality we may assume that $\pi(X, Y)$ is just a conjunction of formulas of those three types.

The following lemma shows that in fact $\pi(X, Y)$ does not contain equations in the K-sort.

LEMMA 4.8. *For any nonzero $r, t \in \mathcal{M}$ with $r \ll t$ and any nonzero polynomial $F(X, Y) \in \mathbb{Z}[X, Y]$, $F(r, t) \neq 0$.*

PROOF. Suppose for contradiction $F(r, t) = 0$. Write $F(X, Y)$ as

$$Y^d(F_l(X)Y^l + \ldots + F_0(X)), \tag{4.1}$$

where $F_0(X), \ldots, F_l(X) \in \mathbb{Z}[X]$. If $F(X, Y)$ is a monomial in Y then it can be written as

$$(e_k X^k + \ldots + e_0)Y^i \tag{4.2}$$

for some $i \geq 0$, where $e_0, \ldots, e_k \in \mathbb{Z}$. But no two summands in $e_k r^k + \ldots + e_0$ have the same valuation, for otherwise we would have $v(r) = 0$. Hence $v(e_k r^k + \ldots + e_0) < \infty$, contradiction.

So we may assume that $F(X, Y)$ has at least two nonzero monomial summands. Now for some $i > j \geq 0$ we have $v(F_i(r)t^i) = v(F_j(r)t^j)$. So

$$v(t^{i-j}) = v(F_j(r)/F_i(r)).$$

But again, in either $F_i(r)$ or $F_j(r)$, no two summands have the same valuation, so either $F_j(r)/F_i(r) \ll r$ or $F_j(r)/F_i(r) \asymp r$. So either $t \ll r$ or $t \asymp r$, contradiction again. ⊣

Remark 4.9. This lemma is well-known. It is a corollary of the fundamental dimension inequality in the theory of valued fields; see [3, Theorem 3.4.3]. We prefer to give an elementary proof here to make clear that its failure in valued fields of mixed characteristics is the main reason that Theorem 1.3 has not been extended to such fields in general. On the other hand, the above lemma clearly may be applied to the case $r = \mathrm{char}(\overline{K}) > 0$. So Theorem 1.3 does hold for a particular subclass of valued fields of mixed characteristics, namely tight valued fields; see Remark 4.13.

LEMMA 4.10. *Let $\varphi(X, Y)$ be a conjunction of formulas of Type II, where X, Y are the only free variables. Let $x, y \in \mathcal{M}$ be nonzero such that $x \ll y$ and $S \models \varphi(x, y)$. Then for every natural number k there is an $m \in \Gamma$ with $v(x^k) < m < v(x^l)$ for some $l > k$ such that for every $t \in \mathcal{M}$ with $v(t) = m$ we have $S \models \varphi(x, t)$.*

PROOF. Let $F_i(X, Y) \in \mathbb{Z}[X, Y]$ run through all the distinct polynomials that appear in $\varphi(X, Y)$ in the form $vF_i(X, Y)$. We may assume that each $F_i(X, Y)$ is written in the form (4.1) in Lemma 4.8. It is not hard to see that if we choose a $k_0 > 0$ that is larger than the sum of all the exponents of X that appear in all the polynomials $F_i(X, Y)$, then for each $F_i(X, Y)$ there are integers e_i, d_i with $e_i < k_0$ such that for each $t \in \mathcal{M}$, if $v(t) > v(x^{k_0})$ then

$$vF_i(x, t) = v(x^{e_i} t^{d_i}). \tag{4.3}$$

Now substituting $v(x^{e_i} t^{d_i})$ for $vF_i(x, t)$ in $\varphi(x, t)$ and then substituting two variables N_1, N_2 for $v(x), v(t)$ respectively in the resulting formula we obtain an

$\mathcal{L}_{\mathrm{Pr}\infty}$-formula $\varphi^*(N_1, N_2)$ from $\varphi(x,t)$ such that for all $t \in \mathcal{M}$ with $v(t) > v(x^{k_0})$

$$S \models \varphi(x,t) \text{ if and only if } \Gamma \cup \{\infty\} \models \varphi^*(v(x), v(t)). \tag{4.4}$$

In particular we have $\Gamma \cup \{\infty\} \models \varphi^*(v(x), v(y))$. Let $v(x) = n$. Let $\Gamma(n)$ be the smallest $\mathbb{Z}$-group (smallest submodel) generated by n in Γ. It is easy to see that the set $\{kn : k \in \mathbb{N}\}$ is cofinal in $\Gamma(n)$. By Remark 4.1, $\Gamma(n) \cup \{\infty\}$ is an elementary substructure of $\Gamma \cup \{\infty\}$. So for every natural number $k \geq k_0$ we have

$$\Gamma(n) \cup \{\infty\} \models \exists N \, (kn < N < \infty \wedge \varphi^*(n, N)).$$

So for some $m \in \Gamma(n)$ and some $l > k$ we have

$$\Gamma(n) \cup \{\infty\} \models kn < m < ln \wedge \varphi^*(n, m).$$

So for every $t \in \mathcal{M}$ with $v(t) = m$ we have $\Gamma \cup \{\infty\} \models \varphi^*(n, v(t))$. By (4.4) we have $S \models \varphi(x,t)$, as desired. ⊣

Remark 4.11. A close examination of the proof of Lemma 4.3 and Lemma 4.6 shows that, regardless of what languages the group Γ and the field $\overline{K}$ use and what additional structures they have, formulas of Type II and III always define open sets. Therefore Lemma 4.10 is actually the only place where we need to use some special properties that hold in $\mathbb{Z}$-groups, namely

1. for any element n in the Γ-sort the set $\{kn : k \in \mathbb{N}\}$ is cofinal in the submodel generated by n;
2. the theory of the Γ-sort in $\mathcal{L}_{\Gamma\infty}$ is model-complete.

So our converse QE result holds for any group Γ, any field $\overline{K}$, and any languages $\mathcal{L}_{\Gamma\infty}, \mathcal{L}_{\overline{K}}$, provided that these two properties are satisfied.

LEMMA 4.12. *Let $\varphi(X,Y)$ be a formula of Type III, where X, Y are the only free variables. Let $x, y \in \mathcal{M}$ be nonzero such that $x \ll y$ and $S \models \varphi(x,y)$. For every $t \in \mathcal{M}$, if $v(t)$ is sufficiently large and $\overline{\mathrm{ac}}\, t = \overline{\mathrm{ac}}\, y$, then $S \models \varphi(x,t)$.*

PROOF. Let $F_i(X,Y) \in \mathbb{Z}[X,Y]$ run through all the distinct polynomials that appear in $\varphi(X,Y)$ in the form $\overline{\mathrm{ac}}\, F_i(X,Y)$. As in the previous lemma we choose a $k_0 > 0$ that is larger than the sum of all the exponents of X that appear in all the polynomials $F_i(X,Y)$. So for each $t \in \mathcal{M}$, if $v(t) > v(x^{k_0})$ then the equation (4.3) in Lemma 4.10 holds for each $F_i(X,Y)$. For such a $t \in \mathcal{M}$, if $F_i(X,Y)$ is written in the form (4.1) in Lemma 4.8, then we have

$$v(F_l(x)t^l + \ldots + F_0(x)) = vF_0(x)$$

and

$$v(F_l(x)t^l + \ldots + F_1(x)t) > vF_0(x)$$

if $l > 0$. Let $F_0(X)$ be written as $X^b(e_j X^j + \ldots + e_0)$, with $e_0, \ldots, e_j \in \mathbb{Z}$ and e_0 nonzero. So by Lemma 4.4 we have

$$\overline{\mathrm{ac}}\, F_i(x,t) = (\overline{\mathrm{ac}}\, t)^d \cdot \overline{\mathrm{ac}}\, F_0(x) = (\overline{\mathrm{ac}}\, t)^d \cdot (\overline{\mathrm{ac}}\, x)^b \cdot \overline{\mathrm{ac}}\, e_0.$$

In particular, since $x \ll y$, we have

$$\overline{\mathrm{ac}}\, F_i(x,y) = (\overline{\mathrm{ac}}\, y)^d \cdot (\overline{\mathrm{ac}}\, x)^b \cdot \overline{\mathrm{ac}}\, e_0.$$

Now if $\overline{\mathrm{ac}}\, t = \overline{\mathrm{ac}}\, y$ then we have

$$\overline{\mathrm{ac}}\, F_i(x,t) = (\overline{\mathrm{ac}}\, t)^d \cdot (\overline{\mathrm{ac}}\, x)^b \cdot \overline{\mathrm{ac}}\, e_0 = (\overline{\mathrm{ac}}\, y)^d \cdot (\overline{\mathrm{ac}}\, x)^b \cdot \overline{\mathrm{ac}}\, e_0 = \overline{\mathrm{ac}}\, F_i(x,y).$$

So clearly $S \models \varphi(x,t)$, as desired. ⊣

PROOF OF THEOREM 4.7. Let $x \ll y$ be such that $S \models \pi(x,y)$. We shall show that there is a $t \in \mathcal{M}$ with $x \asymp t$ such that $S \models \pi(x,t)$. This shows that the type $\Phi(X,Y)$ is not isolated by $\pi(X,Y)$ modulo $\mathrm{Th}(S)$.

By Lemma 4.8, $\pi(X,Y)$ cannot contain equalities in the K-sort. Clearly, for sufficiently large k, if $t \in \mathcal{M}$ is nonzero and $v(t) \geq v(x^k)$ then the pair (x,t) satisfies the disequalities in the K-sort that appear in $\pi(X,Y)$. Finally, by Lemma 4.10 and 4.12 we can choose a sufficiently large k and a $t \in \mathcal{M}$ with $v(x^k) < v(t) < v(x^l)$ for some $l > k$ and $\overline{\mathrm{ac}}\, t = \overline{\mathrm{ac}}\, y$ such that $S \models \pi(x,t)$, as desired. ⊣

Remark 4.13. If we replace the first assumption of Theorem 1.3 with $\mathrm{char}\, K = 0$ and $\mathrm{char}(\overline{K}) = p > 0$ and then add another assumption that the valued field is *tight*, that is, $v(p) \in \mathbb{Z}_\Gamma$, then the argument above can be quite easily adapted to show that the theorem still holds. To see this, first note that for some $n \in \mathbb{Z}_\Gamma$ the sentence $v(p) = n$ is in $\mathrm{Th}(S)$. Next, we leave Step 1, 2, and 3 unchanged. For Step 4, it is enough to show that the 1-type

$$\Phi(Y) = \left\{0 < v(p^l) < v(Y) \wedge Y \neq 0 : l \geq 1\right\}.$$

is not isolated modulo $\mathrm{Th}(S)$. To that end, suppose for contradiction that there is a formula $\pi(Y)$ such that the sentence $\exists Y\ \pi(Y)$ is in $\mathrm{Th}(S)$ and $\pi(Y) \vdash \Phi(Y)$ modulo $\mathrm{Th}(S)$. By QE in the K-sort, $\pi(Y)$ is equivalent to, without loss of generality, a conjunction of formulas of Type I, II, and III with only one free K-sort variable Y. Clearly Lemma 4.8 holds with $r = p$ and Lemma 4.10, Lemma 4.12 hold with $x = p$. Now the contradiction is that we can find an element $t \in \mathcal{M}$ such that $v(t) \in \mathbb{Z}_\Gamma$ and $S \models \pi(t)$. Finally, observe that the tightness condition is necessary for Step 4, since, otherwise, the sentences $v(p) > 1, v(p) > 2, \ldots$ are all in $\mathrm{Th}(S)$.

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