A note on the Engelking–Karłowicz theorem

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Abstract

We investigate the chromatic number of infinite graphs whose definition is motivated by the theorem of Engelking and Karłowicz (in [3]). In these graphs, the vertices are subsets of an ordinal, and two subsets X and Y are connected iff for some $a \in X \cap Y$ the order-type of $a \cap X$ is different from that of $a \cap Y$.

In addition to the chromatic number $\chi(G)$ of these graphs we study $\chi_\kappa(G)$, the κ-chromatic number, which is the least cardinal μ with a decomposition of the vertices into μ classes none of which contains a κ-complete subgraph.

1 Introduction

A celebrated theorem of Engelking and Karłowicz [3] states that if θ and μ are cardinals such that $\mu^{<\theta} = \mu$, then there is a family $\mathcal{F}$ of size 2^μ, consisting of functions from μ into μ, with the following property. For every one-to-one sequence $\langle f_i \in \mathcal{F} \mid i \in \theta^* \rangle$ and sequence $\langle \beta_i \in \mu \mid i \in \theta^* \rangle$, where $\theta^* < \theta$, there exists some $\alpha \in \mu$ such that for all $i \in \theta^*$ $f_i(\alpha) = \beta_i$.

An equivalent formulation takes the following form. Let θ and μ be cardinals such that $\mu^{<\theta} = \mu$. Then there are functions $f_\xi : 2^\mu \to \mu$, for $\xi < \mu$, such that if $X \subset 2^\mu$, $|X| < \theta$ and $f : X \to \mu$, then there is $\xi < \mu$ such that $f \subset f_\xi$.

This theorem has diverse applications such as the Hewitt - Marczewski - Pondiczery theorem that the product of 2^μ topological spaces each with a

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dense subset of cardinality μ has itself a dense subset of cardinality μ. We are interested here in the following corollary used by Shelah in [9] and [10]:

Corollary 1.1 *1. If $\mu^{<\theta} = \mu$ and A is any set of cardinality 2^μ, then there is a map $\tau : [A]^{<\theta} \to \mu$ such that whenever $\tau(M_1) = \tau(M_2)$ then M_1 and M_2 have the same order-type and the order-preserving isomorphism $g : M_1 \to M_2$ is the identity on $M_1 \cap M_2$.*

2. Thus, if $\mu^\theta = \mu$ and A is any set of cardinality 2^μ, then there is a map $\tau : [A]^\theta \to \mu$ such that whenever $\tau(M_1) = \tau(M_2)$ then M_1 and M_2 have the same order-type and the order-preserving isomorphism $g : M_1 \to M_2$ is the identity on $M_1 \cap M_2$.

For example, if $\mu^\theta = \mu$ and $\lambda < (2^\mu)^+$ is any ordinal, then there is a map $\tau : [\lambda]^{<\theta} \to \mu$ such that $\tau(M_1) = \tau(M_2)$ implies that M_1 and M_2 are isomorphic with an isomorphism that is the identity on $M_1 \cap M_2$.

Proof. Here $[A]^{<\theta}$ is the collection of all subsets of A of cardinality $< \theta$. Since there are only $\theta \leq \mu$ possible order-types of $M \in [A]^{<\theta}$, it is enough to find a function that works for a specific order-type $\gamma < \theta$ and then to combine these functions into a single τ that works for all $\gamma < \theta$. Note that the requirement that the order-isomorphism $g : M_1 \to M_2$ is the identity on $M_1 \cap M_2$ can be expressed by saying that every $a \in M_1 \cap M_2$ has the same place in M_1 as in M_2 (namely the order-types of $a \cap M_1$ and $a \cap M_2$ are the same).

Fix a sequence of functions $f_\xi : 2^\mu \to \mu$ for $\xi < \mu$, as in the equivalent formulation of the Engelking and Karłowicz theorem. Since A has cardinality 2^μ, we can have such functions be defined over A and with the same properties. Namely, that if $X \in [A]^{<\theta}$ and $f : A \to \mu$ then $f \subset f_\xi$ for some $\xi < \mu$.

For every $M \subset A$ of order-type γ, let $f : M \to \gamma$ be its order-preserving collapse. There is some $\xi < \mu$ such that $f \subset f_\xi$ and we define $\tau(M) = \xi$ (say for the least such ξ). Now if $\tau(M_1) = \tau(M_2) = \xi$ then for $a \in M_1 \cap M_2$, $f_\xi(a)$ is the place of a both in M_1 and in M_2.

The second paragraph of the corollary is obtained by replacing θ with θ^+. It is this case that will interest us in this paper. `q.e.d.`

In this note we want to investigate the extent to which the assumption $\mu^\theta = \mu$ in the second item of the corollary is necessary. We are mainly interested in the case $\mu = \aleph_\omega$ and $\theta = \omega$ and we will prove that if μ is a strong limit singular cardinal of cofinality θ then the conclusion of the corollary does not hold.

In graph theoretic language our problem finds a concise formulation as follows. Let $(A, <)$ be any linearly ordered-set. We say that X and Y subsets of A are *consistent* iff there exists an order isomorphism $f : X \to Y$ that is the identity on $X \cap Y$ (namely $f(x) = x$ for $x \in X \cap Y$). We say that X and Y are inconsistent if they are not consistent. In this paper, we deal only with well-ordered sets $(A, <)$, and in this case X and Y are inconsistent iff either the order-type of X is different from that of Y, or else there exists some $\xi \in X \cap Y$ such that *order-type*$(X \cap \xi) \neq$ *order-type*$(Y \cap \xi)$.

For any ordered-set $(A, <)$, we define a graph with vertices $\mathcal{P}(A)$, the powerset of A, and with edges all pairs (X, Y) where X and Y are inconsistent subsets of A. We will be interested in subgraphs of $\mathcal{P}(\gamma)$ for different ordinals γ's and ask for their chromatic number. In fact, we will be interested here mainly in the case in which we take only subsets of γ of some fixed order-type α.

Recall that the chromatic number $\chi(G)$ of a graph $G = (V, E)$ is the least cardinal κ such that there is a function $\tau : V \to \kappa$ so that $(a, b) \notin E$ whenever $\tau(a) = \tau(b)$. We call such a function "separating". That is, τ is separating iff $\tau(a) \neq \tau(b)$ whenever $(a, b) \in E$. The chromatic number is thus the least cardinality of the range of a separating function.

Let $\text{typ}(\alpha)$ be the class of all sets of ordinals of order-type α. If $X \in \text{typ}(\alpha)$ we say that X is an α-set. (We take sets rather than sequences because we refer to the intersection of two sets when the edges of the graph are defined). For a set B of ordinals, let $\text{typ}(\alpha, B)$ be the collection of all $X \in \text{typ}(\alpha)$ such that $\sup(X) \in B$ (where $\sup(X)$ is the first ordinal greater or equal to all ordinals of X). We will be interested here in two cases: for ordinals $\alpha < \beta$, $\text{typ}(\alpha, \beta)$, is the collection of bounded subsets of β of order-type α, and for a limit ordinal α $\text{typ}(\alpha, \{\beta\})$ is the collection of all unbounded subsets of β of order-type α.

If $a, b \in \text{typ}(\alpha)$ there is a unique order-preserving isomorphism $g : a \to b$, and in this case a and b are consistent iff g is the identity on $a \cap b$. They are inconsistent otherwise. So a and b are inconsistent iff for some $x \in a \cap b$ the order-type of $x \cap a$ differs from that of $x \cap b$.

For two ordinals $\alpha \leq \beta$ let $G(\alpha, \beta)$ be the graph $G = (V, E)$ with set of vertices $V = \text{typ}(\alpha, \beta)$ and edges $(a, b) \in E$ iff a and b are inconsistent. Likewise, $G(< \alpha, \beta)$ consists of subsets of β of order-type $< \alpha$, with edges (a, b) defined whenever the order-type of a is different from that of b, or else they have the same order-type but are inconsistent.

For example, $G(\omega, \omega_1)$ has vertices all ω-sets of countable ordinals, and ω-sets X and Y are connected iff some $a \in X \cap Y$ has different position in X and Y. The graph $G(< \omega, \omega_1)$, has vertices all finite subsets of ω_1, and edges all pairs (a, b) where a and b are inconsistent.

Similarly, $G(\alpha, \{\beta\})$ is the graph with vertices $\mathrm{typ}(\alpha, \{\beta\})$ and edges all pairs (a, b) that are inconsistent α sets (unbounded in β).

The graphs $G(2, \beta)$ were considered by Erdős and Hajnal [4] and called "shift graphs". So a vertex is a pair $\{a, b\}$ (with $a < b$) and two pairs $a_0 < a_1$ and $b_0 < b_1$ are connected in the graph iff $a_1 = b_0$ or $b_1 = a_0$.

We shall be particularly interested in the case $G(\omega, \{\aleph_\omega\})$ which is the graph G with set of vertices all unbounded ω-sets in $\aleph_\omega$, and with edges defined by $(s, t) \in G$ iff there is $x \in s \cap t$ such that $|x \cap s| \neq |t \cap x|$.

Our aim is to investigate the chromatic number of these and similarly defined graphs.

In this graph theoretic terminology, Corollary 1.1 can be restated as follows:

Corollary 1.2 *Suppose that $\mu^{<\theta} = \mu$, and let λ be any ordinal of cardinality $\leq 2^\mu$. Let G be the graph with vertices all subsets of λ of cardinality $< \theta$ and edges connecting two vertices if and only if they are inconsistent. Then the chromatic number of G is $\leq \mu$.*

In particular, we get the following when we consider θ^+. Suppose that $\mu^\theta = \mu$, and let λ be any ordinal of cardinality 2^μ. Let G be the graph with vertices all subsets of λ of cardinality θ and edges connecting two vertices if and only if they are inconsistent. Then the chromatic number of G is $\leq \mu$.

Here are a couple of illustrations of the corollary. Since $\mu^{<\aleph_0} = \mu$ for every infinite cardinal μ, we have that $\chi(G(< \omega, 2^\mu)) \leq \mu$. Another example: $\chi(G(< \omega_1, 2^{(2^{\aleph_0})}) \leq 2^{\aleph_0}$. In fact, $\chi(G(< \omega_1, 2^{(2^{\aleph_0})}) = 2^{\aleph_0}$, because already $G(\omega, \{\omega\})$ contains a clique of size $2^{\aleph_0}$. To see this, form for every subset X of the even numbers the set $S(X)$ which is the union of X with the set of odd numbers. Then $\{S(X) \mid X \subseteq \text{even}\}$ is a clique.

Corollary 1.2 thus says that if $\mu^\theta = \mu$ then for every $\alpha < \theta^+$ there is a separating function from $G(\alpha, 2^\mu)$ into μ and hence $\chi(G(\alpha, 2^\mu)) \leq \mu$. A simple but quotable result of this note is that if $\aleph_\omega$ is strong limit, then $\chi(G(\omega, \{\aleph_\omega\}) > \aleph_\omega$. Hence Corollary 1.1 does not hold in case $\mu = \aleph_\omega$, $\theta = \omega$, and the cardinal assumption in that corollary is needed. This is the content of the following section.

2 $\chi(G(\omega, \{\beth_\omega\})) > \beth_\omega$ (and similar results)

It is convenient to define, for any set of ordinals B, a function $\pi_B : B \to$ *order-type*(B) by

$$\pi_B(a) = \textit{order-type}(a \cap B).$$

If $A \subseteq B$, then $\pi_B \restriction A$ is denoted $\pi_{A,B}$. That is, $\pi_{A,B}$ gives the position of a within B for every $a \in A$. So for arbitrary sets of ordinals X and Y, X and Y are consistent iff $\pi_{X \cap Y, X} = \pi_{X \cap Y, Y}$.

The following lemma is obvious.

Lemma 2.1 *Suppose A and B are α sets of ordinals and $X_0 \subseteq A \cap B$. Then $\pi_{X_0,A} = \pi_{X_0,B}$ iff the (unique) order isomorphism $g : A \to B$ is the identity on X_0.*

Our first result is

Theorem 2.2 $\chi(G(\omega, \{\beth_\omega\})) > \beth_\omega$. *More generally, if λ is a strong limit singular cardinal and* $\mathrm{cf}(\lambda) = \kappa$*, then $\chi(G(\kappa, \{\lambda\})) > \lambda$.*

Proof. For a simpler exposition we present the proof for the particular case of $\beth_\omega$, but the reader will have no problems in making the obvious changes. Recall that $\beth_0 = \aleph_0$, $\beth_{n+1} = 2^{\beth_n}$, and $\beth_\omega$ is the limit of the $\beth_n$ sequence. Recall also that the graph $G(\omega, \{\beth_\omega\})$ consists of all ω-sets that are unbounded in $\beth_\omega$. Suppose τ is a separating function from $\mathrm{typ}(\omega, \{\beth_\omega\})$ into $\beth_\omega$, and we shall reach a contradiction. So we assume that for every two ω-sets $M_1 \neq M_2$ unbounded in $\beth_\omega$, if $\tau(M_1) = \tau(M_2)$ then M_1 and M_2 are consistent.

Given any $M \in \mathrm{typ}(\omega, \{\beth_\omega\})$ we define the trace of M, denoted t_M, as the following (partial) function on $\beth_\omega$. For every $\alpha < \beth_\omega$, pick (if there is one) some $N \in \mathrm{typ}(\omega, \{\beth_\omega\})$ such that $M \subseteq N$ and $\tau(N) = \alpha$. Then define

$$t_M(\alpha) = \mathrm{range}\ \pi_{M,N} = \{|x \cap N| \mid x \in M\}. \quad (1)$$

In words, $t_M(\alpha)$ is the set of positions occupied by M in N (it is a subset of ω). Notice that $t_M(\alpha)$ does not depend on N: if N' is some other member of $\mathrm{typ}(\omega, \{\beth_\omega\})$ with $M \subseteq N'$ and $\tau(N') = \alpha$, then M occupies the same positions in N as in N'. In fact, the isomorphism between N and N' is the identity on $N \cap N'$ and hence on M.

Now, for every $n < \omega$, the set of all functions from $\beth_n$ to $\mathcal{P}(\omega)$ has cardinality $2^{\beth_{n+1}}$, but the cardinality of the set of ω sequences unbounded in

$\beth_\omega$ is $2^{\beth_\omega}$. Hence there are two distinct sets M_n and M'_n in $\mathrm{typ}(\omega, \{\beth_\omega\})$ that begin after $\beth_n$ and are such that

$$t_{M_n} \restriction \beth_n = t_{M'_n} \restriction \beth_n. \tag{2}$$

Let $K = \bigcup(\{M_n \mid n \in \omega\} \cup \{M'_n \mid n \in \omega\})$. Then K has order-type ω (its intersection with any $\beth_n$ is finite) so that $K \in \mathrm{typ}(\omega, \{\beth_\omega\})$, and $\tau(K) = \alpha$ is defined. Pick $n \in \omega$ with $\alpha < \beth_n$. Then $\alpha \in \mathrm{dom}(t_{M_n}) \cap \mathrm{dom}(t_{M'_n})$, and $t_{M_n}(\alpha) = t_{M'_n}(\alpha)$ by the choice of M_n and M'_n in (2). Hence M_n occupies in K the same positions as M'_n does, which is impossible since $M_n \neq M'_n$. q.e.d.

The following definition motivates much of the research reported in this paper.

Definition 2.3 *Let κ be a (finite or infinite) cardinal; we say that a graph G has κ-chromatic number μ iff μ is the least cardinal such that there is a function τ from the vertices into μ such that $\tau^{-1}\{\alpha\}$ does not contain a clique of cardinality κ. That is, if $\{a_i \mid i \in \kappa\}$ is a set of size κ of vertices with $\tau(a_i) = \tau(a_j)$ for all i and j, then there are $a_i \neq a_j$ in this collection that are not edge connected in the graph. We say that such a function τ is a κ- separating coloring of the graph. We denote with $\chi_\kappa(G)$ the κ chromatic number of G.*

For example, $\chi_2(G)$ is the chromatic number of G. $\chi_3(G) = 1$ is the statement that G is triangle free, and $\chi_3(G)$ is the least cardinality of a partition of G into triangle free subsets. So, $\chi_\kappa(G) > \mu$ is equivalent to the statement that any function F from the set of vertices of G to μ has some $\gamma \in \mu$ such that $F^{-1}\{\gamma\}$ contains a clique of cardinality κ.

Clearly, $\chi_2(G) \geq \chi_3(G) \geq \cdots \geq \chi_{\aleph_0}(G) \geq \cdots$.

A well-known question of Erdős and Hajnal [6] can be expressed in these terms as follows: is there a graph G with $\chi_4(G) = 1$ and $\chi_3(G) > \aleph_0$? (That is, does there exist a graph with no subgraph isomorphic to K_4 which cannot be expressed as a union of $\aleph_0$ triangle free graphs?)

In an email, A. Hajnal noted that a result in [6] is (in our terminology) that for every regular κ and $2 \leq n < \omega$ there is a graph G such that $\chi_n(G) = \kappa$ but $\chi_{n+1}(G) = 1$. This result was used by our referee to answer a question that we had in a previous draft and to construct, for every n, a graph G such that $\chi_2(G) > \cdots > \chi_n(G) > \omega$. The construction of graphs G_i and uncountable cardinals κ_i, for $i = n, n-1, \ldots, 1$ is done backwards and so that $\chi_i(G_i) = \kappa_i$ holds. First $\kappa_n = \aleph_1$ (for example) and G_n is chosen so that $\chi_n(G_n) = \kappa_n$ but $\chi_{n+1}(G_n) = 1$. If G_{i+1} and κ_{i+1} are defined, then $\kappa_i > |G_{i+1}|$ is chosen and G_i is defined so that $\chi_i(G_i) = \kappa_i$ and $\chi_{i+1}(G_i) = 1$. Then G is defined as the vertex disjoint union of the G_i's.

An obvious application of the Erdős-Rado theorem is the following.

Theorem 2.4 *For every cardinal λ, for the graph $G = (\omega, \lambda)$, $\chi_{(2^{\aleph_0})^+}(G) = 1$.*

Proof. Suppose that $A \subseteq G$ is a clique of cardinality $(2^{\aleph_0})^+$. Define for $X \neq Y$ in A which are inconsistent $f(X,Y) = \langle n, m \rangle$ if the nth member of X is equal to the mth member of Y and $n \neq m$. As there is no homogenous triple, a contradiction to the Erdős-Rado theorem is obtained.`q.e.d.`

Our aim now is to prove the following.

Theorem 2.5 $\chi_{\aleph_0}(G(\omega, \{\beth_\omega\})) > \beth_\omega$.

To prove the theorem, we shall define first a graph $G^*(\omega, \{\beth_\omega\})$ on the set of vertices $\text{typ}(\omega, \{\beth_\omega)\}$ but with fewer edges than $G(\omega, \{\beth_\omega\})$. We let (a, b) form an edge in G^* iff there are infinitely many $x \in a \cap b$ such that the order-type of $x \cap a$ is different from that of $x \cap b$. In case no G^* edge connects a and b we say that a and b are "eventually consistent". So, $a, b \in \text{typ}(\omega, \{\beth_\omega)\}$ are eventually consistent iff the isomorphism $f : a \to b$ is the identity on "almost all" members of $a \cap b$.

Define the "almost inclusion" relation $X \subseteq^* Y$ iff $Y \setminus X$ is finite, and then define the "almost equal" relation $X =^* Y$ iff $X \subseteq^* Y$ and $Y \subseteq^* X$. If $X \subseteq \omega$, then $[X]_*$ denotes the equivalence class of X. That is, the collection of all subsets of ω that are $=^*$ equivalent to X. In case f and g are functions, $f =^* g$ iff the domain of f is almost equal to the domain of g and $f(x) = g(x)$ for almost all x's in intersection of the domains of f and g.

In these notations, $a, b \in \text{typ}(\omega, \{\beth_\omega)\}$ are eventually consistent iff $\pi_{a\cap b,a} =^* \pi_{a\cap b,b}$.

We first note the following.

Theorem 2.6 *The chromatic number of $G^* = G^*(\omega, \{\beth_\omega\})$ is bigger than $\beth_\omega$.*

The proof is similar to that of Theorem 2.2. Assume that $\tau : G^* \to \beth_\omega$ is separating in the sense that $\tau(M_1) = \tau(M_2)$ implies that M_1 and M_2 are eventually consistent. Then $t_M(\alpha)$ is defined, when $M \subseteq N$ for some N with $\tau(N) = \alpha$, as [range $\pi_{M,N}]_*$, the $=^*$ equivalence class of the set in (1). Then it follows again that $t_M(\alpha)$ does not depend on the set N chosen: any two such supersets will give equivalent sets of positions. At stage n choose two sets M_n and M'_n that are disjoint and such that $t_{M_n} \restriction \beth_n = t_{M'_n} \restriction \beth_n$. The contradiction is obtained as before.

Theorem 2.7 *For $G^* = G^*(\omega, \{\beth_\omega\})$, we have $\chi_{\aleph_0}(G^*) > \beth_\omega$: the $\aleph_0$-chromatic number of G^* is bigger than $\beth_\omega$.*

Proof. Suppose on the contrary that $\tau : \mathrm{typ}(\omega, \{\beth_\omega\}) \to \beth_\omega$ is a $\aleph_0$- separating coloring of the graph $G^*(\omega, \{\beth_\omega\})$. Given $M \in \mathrm{typ}(\omega, \{\beth_\omega\})$ we define t_M on $\beth_\omega$ as follows. For any $\gamma \in \beth_\omega$ define

$$t_M(\gamma) = \{\ [\text{range } \pi_{M\cap B,B}]_* \ \mid\ \tau(B) = \gamma \text{ and } M \subseteq^* B\}$$

In words, $t_M(\gamma)$ is the collection of the almost equality equivalence classes of subsets of ω induced by sets of the form $\{|m \cap B| \mid m \in M \cap B\}$ where $B \in \mathrm{typ}(\omega, \{\beth_\omega\})$ is such that $\tau(B) = \gamma$ and $M \setminus B$ is finite.

We claim that $t_M(\gamma)$ is a finite set (of equivalence classes); this is the content of the following lemma.

Lemma 2.8 *For every $M \in \mathrm{typ}(\omega, \{\beth_\omega\})$ and $\gamma \in \beth_\omega$,*

$$\{\ [\text{range } \pi_{M\cap B,B}]_* \ \mid\ \tau(B) = \gamma \textit{ and } M \subseteq^* B\}$$

is finite.

Proof. If not, then there are $B_i \in \mathrm{typ}(\omega, \{\beth_\omega\})$ for $i \in \omega$ such that $\tau(B_i) = \gamma$, $M \subseteq^* B_i$ and

$$\text{range } \pi_{M\cap B_i,B_i} \ \neq^* \ \text{range } \pi_{M\cap B_j,B_j} \tag{3}$$

for all $i \neq j$ (here $\neq^*$ is the negation of $=^*$). We claim that $\{B_i \mid i \in \omega\}$ is a clique, which contradicts the assumption that τ is $\aleph_0$-separating. To prove that (B_i, B_j) is an edge in G^* for $i \neq j$, we must find an infinite number of $m \in B_i \cap B_j$ for which $|m \cap B_i| \neq |m \cap B_j|$. But since $M \subseteq^* B_i$ and $M \subseteq^* B_j$, this follows immediately from (3). `q.e.d.`-lemma

Thus (continuing the proof of the theorem) t_M takes values essentially in $[\mathcal{P}(\omega)]^{<\omega}$, and hence for every $n \in \omega$ there are not more than $2^{\beth_n}$ possible functions of the form $t_M \restriction \beth_n$. It follows, for every fixed $n \in \omega$, that we can find M_n^i for $i \in \omega$ such that $M_n^i \neq^* M_n^j$ and $t_{M_n^i} \restriction \beth_n = t_{M_n^j} \restriction \beth_n$ for all i and j. Now find $K \in \mathrm{typ}(\omega, \{\beth_\omega\})$ such that $M_n^i \subseteq^* K$ for all indices, and consider $\gamma = \tau(K)$. Pick some n such that $\gamma < \beth_n$. Since $M_n^i \neq^* M_n^j$ and these sets are almost included in K, $M_n^i \cap K \neq^* M_n^j \cap K$, and so

$$\text{range } \pi_{M_n^i \cap K, K} \neq^* \text{range } \pi_{M_n^j \cap K, K} \text{ for } i \neq j.$$

On one hand we have $[\text{range } \pi_{M_n^i \cap K, K}]_* \in t_{M_n^i}(\gamma)$ by the definition of $t_{M_n^i}(\gamma)$, but on the other hand there is a fixed F such that $t_{M_n^i}(\gamma) = F$ for all $i \in \omega$ (by definition of $\{M_n^i \mid i \in \omega\}$). Hence F is infinite, in contradiction to the lemma. q.e.d.

Since any edge of $G^* = G^*(\omega, \{\beth_\omega\})$ is also an edge of $G = G(\omega, \{\beth_\omega\})$, we have that $\chi_{\aleph_0}(G) \geq \chi_{\aleph_0}(G^*)$. That is, Theorem 2.5 is proven.

The following remain unresolved.

1. Improve the theorems by finding the exact value of the chromatic number, rather than just saying it is above $\beth_\omega$. For example, is it always $2^{\beth_\omega}$?

2. Can we replace $\aleph_0$ with $\aleph_1$ in Theorem 2.5? I. e, what is the $\aleph_1$ chromatic number of the graph? Observe that there are no cliques of size $(2^{\aleph_0})^+$ (by Erdős Rado).

3 Ladder graphs

The graph $G(\omega, \{\beth_\omega\})$ considered in the previous section has all its sets with the same supremum, namely $\beth_\omega$. Now we consider the other extreme, when all sets have different suprema. These are the ladder graphs.

Let λ be some ordinal and suppose that a "ladder system" $\overline{X} = \langle X_\alpha \mid \alpha \in S_\omega^\lambda \rangle$ is given where $S_\omega^\lambda \subset \lambda$ is the subset of λ of limit ordinals with countable cofinality, and $X_\alpha \subset \alpha$ is unbounded in α and of order-type ω. A ladder graph induced by $\overline{X}$ is a subgraph of $G(\omega, \lambda)$ having the X_α's as vertices, and edges all inconsistent pairs. It is easy to have such a graph with no edges at all: just assume that each X_α has the form $\{x_i \mid i \in \omega\}$ an increasing enumeration where each x_i is the ith successor of some limit ordinal. So the question is about constructing such a graph with large chromatic number. We concentrate on Ladder subgraphs of $G(\omega, \omega_1)$ and prove that assuming $\diamondsuit_{\omega_1}$ there are such graphs of chromatic number $\aleph_1$, but under $\mathrm{MA}_{\aleph_1}$ each such graph has countable chromatic number.

Our referee noticed that if $S \subset \omega_1$ is non-stationary, then the ladder graph built on $\langle X_\alpha \mid \alpha \in S \rangle$ has countable chromatic number. To see this, take C club disjoint to S and such that every $\alpha \in C$ is sufficiently closed. Then define the coloring on the interval (α, α') for $\alpha \in C$ by induction on α (where $\alpha' > \alpha$ is the next ordinal in C) so that vertices in (α, α') have different colors. The inductive requirement for $\alpha \in C$ ($\alpha > 0$) is that for every $\beta > \alpha$ there is an infinite number of differently colored $\beta' < \alpha$ with $X_\beta \cap \alpha$ an initial segment of $X_{\beta'}$. Now when a color has to be chosen for X_γ where $\gamma \in (\alpha, \alpha')$ while finitely many colors are to be avoided, an example is taken from some already defined $X_{\beta'}$ that extends $X_\gamma \cap \alpha$.

This situation is reminiscent of the one of the Hajnal–Máté graphs defined on ω_1 (in [8]). These graphs are also defined by means of a ladder system $\langle X_\alpha \mid \alpha \in S^\lambda_\omega \rangle$, by joining $\alpha < \beta$ with an edge if $\alpha \in X_\beta$. It is proven in [8] that the diamond $\diamondsuit_{\omega_1}$ implies that there is a Hajnal–Máté graph of chromatic number $\aleph_1$, while $\mathrm{MA}_{\aleph_1}$ implies that all such graphs have chromatic number $\leq \aleph_0$. Yet, the situation with respect to the continuum hypothesis is clearer with the Hajnal–Máté graphs (see [1] and [2]): we know that it is consistent that CH holds and all of these graphs have countable chromatic number, but we do not know the impact of CH on the ladder graphs defined here.

We first note the following.

Theorem 3.1 *If G is any ladder subgraph of $G(\omega, \lambda)$ induced by $\overline{X} = \langle X_\alpha \mid \alpha \in S^\lambda_\omega \rangle$, then $\chi_{\aleph_1}(G) = 1$. That is, there are no uncountable cliques in G.*

Proof. Given $S \subseteq \lambda^\lambda_\omega$ a set of cardinality $\aleph_1$, we shall find $\alpha_1 < \alpha_2$ in S such that X_{α_1} and X_{α_2} are consistent (and hence not connected in the graph). Take M a countable elementary substructure of some H_κ rich enough to contain G and S. Let A be the closure of $M \cap \lambda$ in λ. That is, the set of all ordinals that are in $M \cap \lambda$ or are limits of ordinals in $M \cap \lambda$. Then A is countable and we can pick $\alpha_1 \in S \setminus A$. Since $\alpha_1 \notin A$, $F = X_{\alpha_1} \cap M$ is finite. For every $x \in F$, write $n(x) = |x \cap X_{\alpha_1}|$. Since M is an elementary substructure, there is $\alpha_2 \in M \cap S$ such that for every $x \in F$, we have $|x \cap X_{\alpha_2}| = n(x)$. Now $X_{\alpha_1} \cap X_{\alpha_2} \subseteq F$ and X_{α_1}, X_{α_2} are consistent. `q.e.d.`

Theorem 3.2 *Assume $\diamondsuit_{\omega_1}$. There is a ladder graph G on $S^{\omega_1}_\omega$ such that $\chi_{\aleph_0}(G) = \aleph_1$.*

Proof. Let $\langle S_\alpha \mid \alpha \in \omega_1 \rangle$ be the assumed diamond sequence. We define a ladder system $\langle X_\alpha \mid \alpha \in S^{\omega_1}_\omega \rangle$, where X_α is defined by induction on limit

$\alpha \in \omega_1$, an ω set cofinal in α. At stage α consider S_α and suppose that it encodes a function $f_\alpha : \alpha \to \omega$. Let $\langle \gamma_i \mid i \in \omega \rangle$ be an ω sequence increasing and cofinal in α. We define $x_i^\alpha \in \alpha$ for $i \in \omega$ by induction with the aim of defining $X_\alpha = \{x_i^\alpha \mid i \in \omega\}$. At stage i of the construction we have defined $k(i) \in \omega$ and the first $k(i)$ members of the sequence, denoted $\langle x_j^\alpha \mid j < k(i) \rangle$. We will choose as follows a finite increasing sequence of the form $x_{k(i)}^\alpha, \ldots, x_{k(i+1)-1}^\alpha$ in $\alpha \setminus (\gamma_i \cup \max\{x_0^\alpha, \ldots, x_{k(i)-1}^\alpha\}) + 1$ (to ensure that the resulting sequence is increasing and cofinal in α). Let B_i be the collection of all limit $\alpha' \in \alpha \setminus (\gamma_i \cup \max\{x_0^\alpha, \ldots, x_{k(i)-1}^\alpha\}) + 1$ such that $x_0^\alpha, \ldots, x_{k(i)-1}^\alpha$ is an initial segment of $X_{\alpha'}$ and $f_\alpha(\alpha') = i$. Suppose that $\{X_{\alpha'} \mid \alpha' \in B_i\}$, being a subgraph of the graph constructed so far, contains a finite maximal clique. In this case let $\alpha_0, \ldots, \alpha_{k-1} \in B_i$ be the set of indices of a maximal clique enumerated in increasing order. In fact, we take $\{\alpha_0, \ldots, \alpha_{k-1}\}$ to be minimal in some well ordering of the finite sets of ordinals. Define $x_{k(i)}^\alpha, \ldots, x_{k(i)+k-1}^\alpha$ so that X_α (no matter how it is going to be completed) and X_{α_j} are inconsistent for every $j < k$.

We must prove that the resulting graph has $\aleph_0$-chromatic number $\aleph_1$. Suppose $f : \omega_1 \to \omega$ is a coloring (that is, the function taking X_α to $f(\alpha)$ defined on the vertices of the graph is the coloring). We have to prove that for some $i \in \omega$, $f^{-1}\{i\}$ contains an infinite clique. Suppose on the contrary that for every $i \in \omega$ all cliques of $f^{-1}\{i\}$ are finite (actually, the contradiction is derived from the assumption that every $f^{-1}\{i\}$ contains a maximal finite clique).

Let $\langle M_\alpha \mid \alpha \in \omega_1 \rangle$ be an increasing and continuous sequence of countable elementary substructures of H_κ which is large enough to contain the graph and the function f. Find $\alpha \in \omega_1$ so that $f \restriction \alpha$ is encoded by S_α and $\alpha = M_\alpha \cap \omega_1$. Suppose $f(\alpha) = i_0$ and consider stage i_0 in the definition of the sequence x_i^α. The definition of B_{i_0} can be done in M_α and it contains a finite maximal clique. So the maximal clique, subset of B_{i_0} used at stage i_0 is in M_α, and by the construction it turns out that it is not maximal since X_α is inconsistent with each X_{α_i}. This contradiction proves the theorem. q.e.d.

Assuming MA_{ω_1}, any $G = \langle X_\alpha \mid \alpha \in \lim \omega_1 \rangle$, a ladder subgraph of $G(\omega, \omega_1)$, has countable chromatic number.

Theorem 3.3 *Assume Martin's Axiom and $2^{\aleph_0} > \kappa$. If $G = \langle X_\alpha \mid \alpha \in S^\kappa_\omega$ is a ladder subgraph of $G(\omega, \kappa)$, then $\chi(G) \leq \omega$.*

Proof. Consider the poset P of all finite approximation to a separating function. That is, $p \in P$ iff $\mathrm{dom}(p) \subset S^\kappa_\omega$, $p : \mathrm{dom}(p) \to \omega$, and for every $\alpha, \beta \in \mathrm{dom}(p)$, if $p(\alpha) = p(\beta)$ then X_α and X_β are consistent. The ordering of P is plain extension. Clearly, any condition can be extended to include any given limit ordinal in its domain, since the range of p is finite. The countable chain condition of P is proved below and so Martin's Axiom applies to yield that the chromatic number of G is countable.

Lemma 3.4 *P satisfies the c.c.c.*

Proof. Let $P_0 \subseteq P$ be uncountable. We may assume that the sets $\{\mathrm{dom}(p) \mid p \in P_0\}$ form a Δ-system with core D_0. We may even assume that the sets $\mathrm{dom}(p)$ for $p \in P_0$ are pairwise disjoint and $D_0 = \emptyset$ (just replace p with $p \restriction (\mathrm{dom}(p) \setminus D_0)$). Pick a countable $M \prec H_\kappa$ with $G, P, P_0 \in M$. Let $\overline{M}$ be the union of $M \cap \lambda$ with its set of accumulation points in λ. As M is countable $\overline{M}$ is countable. As P_0 is uncountable, we can find $p \in P_0$ such that $\mathrm{dom}(p) \cap \overline{M} = \emptyset$. Thus for every $x \in \mathrm{dom}(p)$, $C_x \cap M$ is finite. We think of p as a structure with universe $\omega \cup \mathrm{dom}(p) \cup \bigcup_{x \in \mathrm{dom}(p)} C_x$, with predicates the ordering relation and the two binary relations $a \in C_x$ and $\alpha \in \mathrm{dom}(p)$, and with constants all members of $C_x \cap M$ for $x \in \mathrm{dom}(p)$. The function p itself is also part of that structure. Since M is an elementary substructure there is $q \in P_0 \cap M$ such that the structures of p and q are isomorphic with an isomorphism f that does not move the constants.

We claim that p and q are compatible. Suppose not, and $\alpha \in \mathrm{dom}(p)$ and $\beta \in \mathrm{dom}(q)$ are such that $p(\alpha) = q(\beta)$ but X_α and X_β are inconsistent. Recall that $\alpha \notin M$. Say $\alpha' = f(\alpha)$. Then $\alpha' \in \mathrm{dom}(q)$ and $C_\alpha \cap M = C_\alpha \cap C_{\alpha'}$. Moreover, each $t \in C_\alpha \cap C_{\alpha'}$ has the same position in C_α as it has in $C_{\alpha'}$. It also follows that $q(\alpha') = q(\beta)$. Supposedly there is $x \in C_\alpha \cap C_\beta$ that has different positions in C_α and C_β. But then $x \in C_\alpha \cap M$ and so $x \in C_{\alpha'}$ and has the same position there as it has in C_α. Which is impossible since α' and β are both in $\mathrm{dom}(q)$. `q.e.d.`

In view of the last three theorems, we ask: is there (in ZFC) a graph G with $\chi_{\aleph_0}(G) = \aleph_1$ and $\chi_{\aleph_1}(G) = 1$?

Moving one cardinal higher we look at ladder subgraphs of $G(\omega, \omega_2)$. By the previous theorem, under MA$+2^{\aleph_0} > \aleph_2$ they all have countable chromatic number. If CH holds then they have $\aleph_1$ chromatic number (by Corollary 1.2, take $\mu = \theta = \aleph_1$). We prove next that for the case that $2^{\aleph_0} = \aleph_2$ it consistent to have a ladder subgraph of $G(\omega, \omega_2)$ with chromatic number $\aleph_2$: just add Cohen reals.

Theorem 3.5 *In a model obtained by adding $\aleph_2$ many Cohen reals there is a ladder subgraph of $G(\omega, \omega_2)$ with chromatic number $\aleph_2$.*

Proof. Pick for any $\alpha \in S^{\omega_2}_{\omega}$ an unbounded ω set with an increasing enumeration $C_\alpha = \{C_\alpha(n) \mid n \in \omega\}$. Suppose G is a V-generic filter over the Cohen forcing poset (of finite functions from ω_2 to 2). Let $g = \bigcup G$ be the resulting generic function from ω_2 to 2, and denote for any limit ordinal $\alpha \in \omega_2$ $g_\alpha = g \restriction [\alpha, \alpha + \omega)$. In $V[G]$ define $D_\alpha \subset C_\alpha$ as the subset of C_α obtained by picking only those member of the C_α sequence in positions that are in g_α. That is, $C_\alpha(n) \in D_\alpha$ iff $g_\alpha(\alpha + n) = 1$.

We claim that the resulting graph has chromatic number $\aleph_2$. Suppose for a contradiction that $\underset{\sim}{f}$ is a name forced by every condition to be a function from ω_2 in ω_1. We shall find a condition (extending a given condition) that forces two vertices to be connected and have the same color under $\underset{\sim}{f}$.

So let r_0 be an arbitrary condition. It is a finite function from a subset of ω_2 to 2. Let M be an elementary substructure of some large enough H_κ and with cardinality $\aleph_1$ such that $r_0, \underset{\sim}{f} \in M$ and $\delta = M \cap \omega_2 > \omega_1$ is of countable cofinality. Let r_1 be an extension of r_0 that forces that $\underset{\sim}{f}(\delta) = \xi$ for some $\xi \in \omega_1$. Let n be the cardinality of $\mathrm{dom}(r_1) \cap [\delta, \delta + \omega)$. So r_1 determines which of the first members of C_δ are in D_δ. Let $x_0, \ldots, x_{n+1}$ be the first $n+2$ members of C_δ. It will be soon evident why we want those two additional members of the sequence, x_n and x_{n+1}.

Let $s = r_1 \restriction M$. Then $s \in M$ and s is also an extension of r_0. Since M is an elementary substructure, there is in M a condition s_1 that extends s and "reflects" r_1. That is, there is an isomorphism $i : \mathrm{dom}(r_1) \to \mathrm{dom}(s_1)$ such that for $\delta' = i(\delta)$ we have that:

1. $x_0, \ldots, x_{n+1}$ are also the first $n + 2$ members of $C_{\delta'}$.
2. s_1 forces that $\underset{\sim}{f}(\delta') = \xi$.

Now extend s_1 to force that $D_{\delta'}$ includes both x_n and x_{n+1}, and extend r_1 to force that D_δ includes x_{n+1} but not x_n. Then these two extensions are compatible in the Cohen poset and they force that $D_{\delta'}$ and D_δ are inconsistent. q.e.d.

Suppose the GCH. What is the chromatic number of ladder subgraphs of $G(\omega, \omega_3)$? Certainly $\leq \aleph_2$ (by Corollary 1.2). Can we define a ladder graph (in L? with forcing?) so that its chromatic number is $\aleph_2$?

What are the chromatic numbers of ladder subgraphs of $G(\omega, \aleph_{\omega+1})$?

4 Graphs of the form $G(\omega, \mu)$

In the previous sections we considered graphs of ω sequences that had all the same supremum or all different suprema. Now we consider graphs of the form $G(\omega, \mu)$ where μ is a cardinal. That is, graphs of all ω sequences in μ with no restriction on their suprema.

One can consider the more general case $G(\alpha, \mu)$ of all subsets of μ of order-type α (edges defined as subgraphs of $\mathcal{P}(\mu)$). The case $G(2, \mu)$ was considered in [4], and here we extend this discussion. They proved that $G(2, (2^\kappa)^+)$ (called there a shift graph) is a triangle free graph with chromatic number $\geq \kappa^+$ such that all its subgraphs of cardinality $\leq 2^\kappa$ have chromatic number $\leq \kappa$. Our example below is different: not only the chromatic number of the graph is greater than κ, but its $\aleph_0$-chromatic number is also above κ.

Theorem 4.1 *For $G = G(\omega, (2^\kappa)^+)$, $\chi_{\aleph_0}(G) > \kappa$. If $\kappa^{\aleph_0} = \kappa$ then any subgraph of G of smaller cardinality has chromatic number $\leq 2^\kappa$.*

Proof. In the following, G denotes the set of vertices of the graph (all ω-subsets of $(2^\kappa)^+$). Suppose that $\chi_{\aleph_0}(G) \leq \kappa$ and $\tau : G \to \kappa$ is an $\aleph_0$-separating function. That is, $\tau(X_i) = \tau(X_j)$ for all $i, j \in \omega$ implies that for some $i \neq j$ X_i and X_j are consistent sequences.

For every $\alpha \in (2^\kappa)^+$ define a function $g_\alpha : \kappa \to \omega$ as follows. Given $\xi \in \kappa$, define $g_\alpha(\xi) = \max\{|X \cap \alpha| \mid \alpha \in X \text{ and } \tau(X) = \xi\}$. We claim that $g_\alpha(\xi) \in \omega$. Otherwise there are $X_i \in G$ (for $i \in \omega$) such that $|X_i \cap \alpha| \neq |X_j \cap \alpha|$ for $i \neq j$ and yet $\tau(X_i) = \xi$ for all i. But this contradicts the property of τ since any two X_i's are inconsistent.

Since each g_α can be encoded as a subset of κ, there is a set $A \subset (2^\kappa)^+$ of cardinality $(2^\kappa)^+$ and such that $g_\alpha = g_\beta$ for every $\alpha, \beta \in A$. Let $X = \{x_i \mid i \in \omega\}$ be an increasing ω enumeration of ordinals from A. Say $\xi = \tau(X)$. We claim that $m < g_{x_0}(\xi)$ for every $m \in \omega$, and this is a contradiction. Clearly $m \leq g_{x_m}(\xi)$ since $X \cap x_m = \{x_0, \ldots, x_{m-1}\}$. But $g_{x_0} = g_{x_m}$ and hence $m \leq g_{x_0}(\xi)$.

The second statement of the theorem is that if G_0 is a subgraph of $G(\omega, (2^\kappa)^+)$ generated by $\leq 2^\kappa$ vertices, then the chromatic number of G_0 is $\leq \kappa$. This follows if we prove for every $\lambda < (2^\kappa)^+$ that the chromatic number of $G(\omega, \lambda)$ is $\leq \kappa$. We use here Corollary 1.2 to the Engelking and Karłowicz theorem with $\mu = \kappa$ and $\theta = \aleph_0$. `q.e.d.`

Theorem 4.2 *Assume that $\kappa^{\aleph_0} = \kappa$. Then $\chi_{\aleph_1}(G(\omega, (2^\kappa)^+)) \leq \kappa$.*

Proof. Fix for every $\beta < (2^\kappa)^+$ a function $\tau_\beta : \mathrm{typ}(\omega, \{\beta\}) \to \kappa$ such that if $\tau(M_1) = \tau(M_2)$ then M_1 and M_2 are consistent. This is possible by Corollary 1.2 since β has cardinality $\leq 2^\kappa$. Now we define the $\aleph_1$-separating function $\tau : \mathrm{typ}(\omega, (2^\kappa)^+) \to \kappa$ as follows. For any ω-set $X \subset (2^\kappa)^+$, let $\beta = \sup X$, and define $\tau(X) = \tau_\beta(X)$. We prove that τ is $\aleph_1$-separating. Suppose that $\{X_i \mid i \in \omega_1\}$ is a collection of $\aleph_1$ vertices and that for some fixed $\alpha \in \kappa$ we have $\tau(X_i) = \alpha$ for all i. We must prove that this collection is not a clique. Denote $\beta_i = \sup X_i$ for all i. In case, for some $i \neq j$, we have $\beta_i = \beta_j = \beta$, then X_i and X_j are consistent by the property of τ_β. Otherwise, $\{X_i \mid i \in \omega_1\}$ forms a ladder system and is hence not a clique (by Theorem 3.1). `q.e.d.`

For example, for $\kappa = 2^{\aleph_0}$ and $G = G(\omega, (2^{2^{\aleph_0}})^+)$ we get by the last two theorems that $\chi_{\aleph_0}(G) > \kappa \geq \chi_{\aleph_1}(G)$. When $\kappa = 2^{\aleph_0}$ is regular, $\chi_{\aleph_1}(G) = \kappa$ because $G(\omega, \omega)$ has a clique of size $2^{\aleph_0}$.

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