

Solutions to the Exercises in Chapter 1,  
Introduction to Commutative Algebra,  
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1. Since  $x$  is nilpotent, there is an  $n$  such that  $x^n = 0$ . so

$$(1+x)(1-x+x^2-\dots+(-x)^{n-1})=1.$$

Now if  $a$  is a unit, then  $1+a^{-1}x$  is a unit, hence  $a+x$  is a unit.

2. (i) Suppose  $f$  is a unit in  $A[x]$ . Let  $b_0+b_1x+\dots+b_mx^m$  be its inverse. Clearly  $a_0b_0=1$  and  $a_nb_m=0$ . We show by induction that  $a_n^{r+1}b_{m-r}=0$  for each  $0\leq r\leq m$ . Since for each  $0\leq r\leq m$  we have

$$\sum_{i+j=n+m-r-1} a_ib_j=0,$$

multiplying  $a_n^{r+1}$  on both sides we obtain  $a_n^{r+2}b_{m-r-1}=0$ . So  $a_n^{m+1}b_0=0$ , that is  $a_n^{m+1}=0$ . Now by Ex. 1  $a_0+a_nx^n$  is a unit, so we may repeat the above argument for  $a_{n-1}$  and so on, so  $a_1,\dots,a_n$  are all nilpotent. The other direction is immediate by Ex. 1.

(ii) Suppose  $f^m=0$ . So  $a_0^m=0$ . Since  $1+f$  is a unit, by (i) we deduce that  $a_1,\dots,a_n$  are all nilpotent. Conversely, since all coefficients are nilpotent, it is easy to see that for sufficiently large  $N$  we have  $f^N=0$ .

(iii) Choose a  $g=b_0+b_1x+\dots+b_mx^m$  of least degree such that  $fg=0$ . Since  $a_nb_m=0$ , we deduce that  $a_ng=0$ . Then for the same reason we see that  $a_i g=0$  for any  $0\leq i\leq n$ . So every nonzero coefficient of  $g$  annihilates  $f$ .

(iv) Suppose for contradiction that  $fg$  is primitive and, say,  $f$  is not primitive. Let  $I=(a_0,a_1,\dots,a_n)_A$ . Since  $I$  is a proper ideal, the ring  $A^*=A/I$

is nontrivial. From this it is readily seen that the ideal generated by the coefficients of  $fg$  is not  $(1)_A$ . So  $fg$  is not primitive, contradiction.

Conversely, if  $fg$  is not primitive then as above we consider the ideal  $I$  generated by the coefficients of  $fg$ . So again  $A/I$  is not trivial. Consider the ring  $A/I[x]$ . Let  $f^*$  and  $g^*$  be the images of  $f$  and  $g$  in  $A/I[x]$  respectively. So  $f^*$  and  $g^*$  are zero divisors. So by (iii) there is a nonzero  $a^* \in A/I$  such that  $a^*f^* = 0$ . This means that there is an  $a \in A$  such that  $a \notin I$  and  $aa_i \in I$  for every  $0 \leq i \leq n$ . Since  $f$  is primitive we deduce that  $a \in I$ , contradiction.

3. Do induction and consider the ring  $A[x_1, \dots, x_r]/(x_1, \dots, x_{r-1})$ .

4. It is clear that the nilradical is contained in the Jacobson radical. Suppose  $f \in \mathfrak{J}$ , then  $1 + f$  is a unit, so by 2(i) above  $1 + a_0$  is a unit, so  $a_1, \dots, a_n$  are all nilpotent. Since, for any  $b \in A$ ,  $bf \in \mathfrak{J}$ , we deduce  $1 + ba_0$  is a unit. So  $a_0$  is also nilpotent. So by 2(ii)  $f$  is nilpotent.

5. (i) Suppose  $a_0b_0 = 1$ . Then working forwards for each power of  $x$  one easily sees that each  $b_n$  can be solved.

(ii) Suppose  $f^n = 0$ . Let  $\mathfrak{R}$  be the nilradical of  $A$ . There is an obvious ring homomorphism  $\phi : A[[x]] \rightarrow A/\mathfrak{R}[[x]]$ . So  $\phi^n(f) = 0$ , which readily shows that each  $a_n$  is nilpotent. The converse is not true. Simply consider  $A = \bigoplus_{n \in \mathbb{N}} \mathbb{Z}/2^n\mathbb{Z}$ .

(iii) This is immediate from (i).

(iv) Let  $\mathfrak{m}$  be a maximal ideal of  $A[[x]]$ . Clearly  $\mathfrak{m}^c$  is an ideal  $A$ . Suppose it is not maximal, then consider any maximal ideal  $\mathfrak{m}^*$  that contains it. The ideal of  $A[[x]]$  generated by  $\mathfrak{m}$  and  $\mathfrak{m}^*$  is proper, contradicting the maximality of  $\mathfrak{m}$ . Now clearly  $\mathfrak{m} \subseteq (\mathfrak{m}^c, x)$ , so by maximality we have  $\mathfrak{m} = (\mathfrak{m}^c, x)$ .

(v) For any prime ideal  $\mathfrak{p}$  of  $A$   $(\mathfrak{p}, x)$  is a prime ideal of  $A[[x]]$ .

6. If  $\mathfrak{R}$  and  $\mathfrak{J}$  are not the same, then there is an idempotent  $e \in \mathfrak{J} \setminus \mathfrak{R}$  (nonzero idempotent elements cannot be nilpotent). So  $1 - e$  is a unit. Since  $e(1 - e) = 0$ , we have  $e = 0$ , contradiction.

7. Let  $\mathfrak{p}$  be a prime. For any  $a \notin \mathfrak{p}$  we have  $(a/\mathfrak{p})^n = a/\mathfrak{p}$  for some  $n > 1$ , so  $a/\mathfrak{p}$  is either a nonzero divisor or a unit. The former is impossible as  $A/\mathfrak{p}$  is an ID. This implies that  $A/\mathfrak{p}$  is a field, so  $\mathfrak{p}$  is maximal.

8. The intersection of a descending chain of prime ideals is a prime ideal.

9. For any  $a \notin \mathfrak{a}$  we have  $\{1, a, \dots, a^n, \dots\} \cap \mathfrak{a} = \emptyset$ . So there is a maximal (hence prime) ideal  $\mathfrak{b}_a$  which contains  $\mathfrak{a}$  and is disjoint from  $\{1, a, \dots, a^n, \dots\}$ . So  $\mathfrak{a} = \bigcap_{a \notin \mathfrak{a}} \mathfrak{b}_a$ . The other direction is clear.

10. (i) $\Rightarrow$ (ii): Suppose  $a$  is not a unit. Then there is a maximal ideal which contains  $a$ , which must be the only prime ideal, which must be  $\mathfrak{R}$ .

(ii) $\Rightarrow$ (iii):  $\mathfrak{R}$  is maximal.

(iii) $\Rightarrow$ (i): Since  $\mathfrak{R}$  is maximal, clearly  $\mathfrak{R}$  is the only prime ideal.

11. (i) Since  $(1+x)^2 = 1+x$  and  $x^2 = x$ , naturally  $2x = 0$ .

(ii) That any prime  $\mathfrak{p}$  is maximal follows from 7 above.  $A/\mathfrak{p}$  has only two elements because each nonzero element of it is an idempotent unit, which must be 1.

(iii) ETS any ideal generated by two elements  $a, b$  is principal. Observe that  $(a+b-ab)a = a$  and  $(a+b-ab)b = b$ .

12. Let  $\mathfrak{m}$  be the unique maximal ideal. Let  $e$  be an idempotent element. Since  $e(1-e) = 0$ , either  $e \in \mathfrak{m}$  or  $1-e \in \mathfrak{m}$  but not both. So either  $e$  or  $1-e$  is a unit, i.e.  $e$  is either 0 or 1.

13. See Theorem V.2.5 in Lang [1]. (In the light of modern logical machinery this construction looks a bit old-fashioned.)

14. That maximal elements exist is by Zorn's Lemma. The set  $C$  of non-zero-divisors of  $A$  is a MC set. This means that any maximal ideal that avoids  $C$  is prime.

15. The first three are pretty obvious. For (iv), clearly  $V(\mathfrak{a}) \cup V(\mathfrak{b}) \subseteq V(\mathfrak{a} \cap \mathfrak{b}) \subseteq V(\mathfrak{a}\mathfrak{b})$ . For a prime ideal  $\mathfrak{p}$ , if  $a \in \mathfrak{a} \setminus \mathfrak{p}$  and  $b \in \mathfrak{b} \setminus \mathfrak{p}$  then  $ab \notin \mathfrak{p}$ . This shows that the first two must be equal. Setting  $a = b$  we see that the last two must be equal too.

16. This is hard.

17. (i) Clear. This shows that the sets  $X_f$ 's form a basis.

(ii) If  $f$  is not nilpotent then there is a prime ideal that avoids the set  $\{1, f, \dots, f^n, \dots\}$ . The other direction is clear.

(iii) Clear.

(iv) If  $h \in r((f)) \setminus r((g))$  then  $\{1, h, \dots, h^n, \dots\} \cap r((g)) = \emptyset$  and there is a prime ideal  $\mathfrak{p}$  that contains  $r((g))$  and avoids the set  $\{1, h, \dots, h^n, \dots\}$ . So by 15(i)  $\mathfrak{p} \in X_f$  but  $\mathfrak{p} \notin X_g$ . The other direction is clear by 15(i).

(v) Following the hint we argue that for some finite  $J \subseteq I$  the equation  $\sum_{j \in J} g_j f_j = 1$  holds for some  $g_j \in A$ , because otherwise  $\{f_i : i \in I\}$  generates a proper ideal, which in turn can be extended to a maximal ideal that is not covered by the  $X_i$ 's. Then ideals that contain  $\{f_j : j \in J\}$  cannot be proper, i.e.  $\{X_j : j \in J\}$  must cover  $X$ .

(vi) Replace "1" with " $f^n$  for some  $n$ " in the equation in (v).

(vii) Clear.

18. (i) If  $\{x\}$  is closed then  $X \setminus \{x\} = \bigcup_{a \in \mathfrak{p}_x} X_a$ , hence the equivalence.

(ii) Since  $X \setminus \overline{\{x\}} = \bigcup_{a \in \mathfrak{p}_x} X_a$ , we have  $\overline{\{x\}} = \{\mathfrak{p} \in X : \mathfrak{p}_x \subseteq \mathfrak{p}\} = V(\mathfrak{p}_x)$ .

(iii) Obvious by (ii).

(iv) Otherwise there are two points  $x \neq y$  such that  $y \in \overline{\{x\}}$  and  $x \in \overline{\{y\}}$ , so by (iii)  $\mathfrak{p}_x = \mathfrak{p}_y$ , contradiction.

19. If the nilradical  $\mathfrak{R}$  is not prime then find  $a, b \notin \mathfrak{R}$  such that  $ab \in \mathfrak{R}$ . So  $X_a \cap X_b = X_{ab} = \emptyset$ . The other direction is clear.

20. (i) Otherwise there are two open sets  $A, B$  of  $\overline{Y}$  such that  $A \cap B = \emptyset$ . Since  $A^* = A \cap Y \neq \emptyset$  and  $B^* = B \cap Y \neq \emptyset$ , we have a contradiction.

(ii) Use Zorn's Lemma in the usual way.

(iii) By (ii) the maximal subspaces are closed. They cover  $X$  because  $\{x\}$  is an irreducible subspace for any  $x \in X$ . For a Hausdorff space the irreducible components must be the singletons.

(iv) Given any prime ideal  $\mathfrak{p}$ , if  $X_a \cap V(\mathfrak{p}) \neq \emptyset$  and  $X_b \cap V(\mathfrak{p}) \neq \emptyset$  then  $a, b \notin \mathfrak{p}$ , so  $ab \notin \mathfrak{p}$ , so  $X_{ab} \cap V(\mathfrak{p}) \neq \emptyset$ , so  $V(\mathfrak{p})$  is irreducible. Now suppose  $V(r(\mathfrak{a}))$  is an irreducible subspace for some ideal  $\mathfrak{a}$ . If  $r(\mathfrak{a})$  is not a prime then there are  $a, b \notin r(\mathfrak{a})$  such that  $ab \in r(\mathfrak{a})$ . But clearly  $X_a \cap V(r(\mathfrak{a})) \neq \emptyset$  and  $X_b \cap V(r(\mathfrak{a})) \neq \emptyset$  and  $X_a \cap X_b \cap V(r(\mathfrak{a})) = X_{ab} \cap V(r(\mathfrak{a})) = \emptyset$ , contradiction.

If  $Y$  is an irreducible component of  $X$  then  $Y$  must be closed and hence must be of the form  $V(r(\mathfrak{a}))$  for some ideal  $\mathfrak{a}$ .  $r(\mathfrak{a})$  must be a minimal prime as  $V(r(\mathfrak{a}))$  is maximal. Conversely if  $\mathfrak{p}$  is a minimal prime and  $V(\mathfrak{p})$  is not maximal, then it is contained in an irreducible component  $V(\mathfrak{q})$  where  $\mathfrak{q}$  is a prime. So  $\mathfrak{q} \subsetneq \mathfrak{p}$ , contradiction.

21. (i) For any  $\mathfrak{q} \in Y_{\phi(f)}$ , we have  $\phi^{-1}(\mathfrak{q}) \cap \phi^{-1}(\phi(f)) = \emptyset$ , so  $\phi^{-1}(\mathfrak{q}) \in X_f$ , so  $\mathfrak{q} \in \phi^{*-1}(X_f)$ . Conversely if  $\mathfrak{q} \in \phi^{*-1}(X_f)$  then  $\mathfrak{q} \in \phi^{*-1}(\mathfrak{p})$  for some  $\mathfrak{p} \in X_f$ , so  $\phi^{-1}(\mathfrak{q}) = \mathfrak{p}$ , so  $\phi(f) \notin \mathfrak{q}$ .

(ii) We have:  $\phi^{*-1}(V(\mathfrak{a})) = Y \setminus \phi^{*-1}(X \setminus V(\mathfrak{a})) = Y \setminus \phi^{*-1}(\bigcup_{a \in \mathfrak{a}} X_a) = Y \setminus \bigcup_{a \in \mathfrak{a}} \phi^{*-1}(X_a) = Y \setminus \bigcup_{a \in \mathfrak{a}} Y_{\phi(a)} = \bigcap_{a \in \mathfrak{a}} (Y \setminus Y_{\phi(a)}) = \bigcap_{a \in \mathfrak{a}} V(\overline{\phi(a)}) = V(\bigcup_{a \in \mathfrak{a}} \overline{\phi(a)}) = V(\mathfrak{a}^e)$ .

(iii) Since  $\phi^*(V(\mathfrak{b})) \subseteq V(\mathfrak{b}^c)$  and  $V(\mathfrak{b}^c)$  is closed, clearly  $\overline{\phi^*(V(\mathfrak{b}))} \subseteq V(\mathfrak{b}^c)$ . Now if  $\mathfrak{p} \in V(\mathfrak{b}^c)$  is a prime ideal such that  $\mathfrak{p} \notin \overline{\phi^*(V(\mathfrak{b}))}$ , then there is an  $f \in A$  such that  $f \notin r(\mathfrak{b}^c)$  and  $\mathfrak{p} \in X_f$  and  $X_f \cap \overline{\phi^*(V(\mathfrak{b}))} = \emptyset$ . By (i)  $\phi^{*-1}(\mathfrak{p}) \subseteq \phi^{*-1}(X_f) = Y_{\phi(f)}$ , hence  $Y_{\phi(f)} \cap V(\mathfrak{b}) = \emptyset$ . But since  $\phi(f) \notin r(\mathfrak{b})$ , we see that this is impossible.

(iv) Only observe that if  $\phi$  is surjective then  $B \cong A/\ker(\phi)$ .

(v) Let  $\mathfrak{R}_A$  and  $\mathfrak{R}_B$  be the nilradicals of  $A$  and  $B$  respectively. If  $\phi^*(Y)$  is dense in  $X$  then  $X_a \cap \phi^*(Y) \neq \emptyset$  for every  $a \in X$  such that  $a \notin \mathfrak{R}_A$ . Hence  $\ker(\phi) \subseteq \mathfrak{R}_A$ . Conversely, we know that  $\phi^{-1}(\mathfrak{R}_B) \subseteq r(\ker(\phi)) = \mathfrak{R}_A$  (in fact also  $\phi(\mathfrak{R}_A) \subseteq \mathfrak{R}_B$  and hence  $\phi(\mathfrak{R}_A) = \mathfrak{R}_B$ ), so  $Y_{\phi(a)}$  is nonempty for every

$a \in X$  such that  $a \notin \mathfrak{R}_A$ .

(vi) Clear.

(vii)  $A$  is a local ring, so  $A/\mathfrak{p}$  is a field. It is not hard to see that there are only two prime ideals of  $B$ :  $(A/\mathfrak{p}) \times \{0\}$  and  $\{0\} \times K$ . Naturally  $\phi^*((A/\mathfrak{p}) \times \{0\}) = 0$  and  $\phi^*(\{0\} \times K) = \mathfrak{p}$ . But  $\{0\} \times K$  is open while  $\mathfrak{p}$  is not.

22. Let  $e_i = (0, \dots, 0, 1_i, 0, \dots, 0)$ . For any prime ideal  $\mathfrak{p}$  of  $A$  clearly not all of the  $e_i$ 's are in  $\mathfrak{p}$ , but there can only be exactly one  $e_i$  that is not in  $\mathfrak{p}$  because  $e_i e_j = 0$  whenever  $i \neq j$ . Hence  $\mathfrak{p}$  is of the form  $\mathfrak{p}_i \times \prod_{j \neq i} A_j$  for some prime ideal  $\mathfrak{p}_i$  of  $A_i$ . From this the first claim readily follows.

For an explanation of the topological concepts used in this book see Munkres [2].

(i) $\Rightarrow$ (ii): This is trickier than you think. Let  $S_1$  and  $S_2$  witness  $X = \text{Spec}(A)$ 's disconnectedness. So  $S_1 = V(r(\mathfrak{a}))$  and  $S_2 = V(r(\mathfrak{b}))$  for some ideals  $\mathfrak{a}, \mathfrak{b}$ . Since  $S_1 \cap S_2 = \emptyset$ , it follows that  $r(\mathfrak{a}) + r(\mathfrak{b}) = (1)$ . Since  $X = V(r(\mathfrak{a})) \cup V(r(\mathfrak{b})) = V(r(\mathfrak{a}) \cap r(\mathfrak{b}))$ , it follows that  $r(\mathfrak{a}) \cap r(\mathfrak{b}) = \mathfrak{R}$ . Fix an  $a \in \mathfrak{a}$  and a  $b \in \mathfrak{b}$  such that  $a + b = 1$ . So  $a, b \notin \mathfrak{R}$ .

Now, since  $X_a \subseteq S_2$  and  $X_b \subseteq S_1$ , we have  $X_a \cap X_b = \emptyset$  and hence  $ab \in \mathfrak{R}$ . Suppose  $a^n b^n = 0$ . So  $a^n \neq 0$  and  $b^n \neq 0$ . Since  $(a + b)^n = 1$ , we see that  $a^n + b^n = 1 - abf$  for some element  $f \in A$ . Clearly  $abf$  is nilpotent, and since  $\mathfrak{R} \subseteq \mathfrak{J}$ , we see that  $1 - abf$  is a unit. So for some unit  $g \in A$  we have  $a^n g + b^n g = 1$  and  $a^n g b^n g = 0$ . Clearly  $a^n g, b^n g \neq 0, 1$ . Since  $a^n g(1 - a^n g) = 0$ , we conclude that  $a^n g$  is a nontrivial (i.e.  $\neq 0, 1$ ) idempotent element. Similarly  $b^n g$  is a nontrivial idempotent element.

In general, if  $a, b$  are two nontrivial idempotent elements such that  $a + b = 1$  and  $ab = 0$ , then  $(a) \cap (b) = \{0\}$ : if  $ac = bd$ , then  $ac = (1 - a)d$ , so  $a(c + d) = d$ , so  $0 = ba(c + d) = bd$ . Now let  $A_1 = A/(a)$  and  $A_2 = A/(b)$ , we have  $A \cong A_1 \times A_2$ .

(ii) $\Rightarrow$ (iii):  $(0, 1)^2 = (0, 1)$ .

(iii) $\Rightarrow$ (i): Let  $e$  be such an idempotent element. Since  $e(1 - e) = 0$ , clearly  $X_e$  and  $X_{1-e}$  witness that  $X$  is disconnected.

23. (i)  $X_e$  and  $X_{1-e}$  are both open and complementary to each other.

(ii)  $\bigcup_i X_{f_i} = X \setminus \bigcap_i V(f_i) = X \setminus V(\{f_1, \dots, f_n\})$ . Now use 11(iii).

(iii) The hint reveals all!

(iv) It is not clear what "compact Hausdorff" means in the book's terminology. We assume it means "quasi-compact and Hausdorff". Since all Zariski topologies are quasi-compact, we just need to show that for a boolean ring  $A$   $\text{Spec}(A)$  is Hausdorff. If two different prime ideals  $\mathfrak{a}$  and  $\mathfrak{b}$  cannot be contained in two disjoint open sets respectively, then either  $\mathfrak{a} \subseteq \mathfrak{b} \vee \mathfrak{b} \subseteq \mathfrak{a}$

or  $ab \notin \mathfrak{A}$  for every  $a \in \mathfrak{a} \setminus \mathfrak{b}$  and every  $b \in \mathfrak{b} \setminus \mathfrak{a}$ . Check that this does not happen for a boolean ring (use 11(ii)).

24. This is tedious.

25. This is immediate from the last two exercises.

26. The question we are asked to do is trivial (the overall result is not, of course).

27. This is by Hilbert's Nullstellensatz.

28. This is essentially just definition checking. (Ask: Where the coordinate functions on  $Y$  are mapped to by a  $k$ -algebra homomorphism?)

## References

- [1] S. Lang, *Algebra*, revised third ed., Springer-Verlag, New York, 2002.
- [2] J. R. Munkres, *Topology*, second ed., Prentice Hall, New Jersey, 1999.