Mixing Strategies
Intermediate Microeconomics
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Story: Hagar the Horrible is trying to avoid Attila the Hun after having looted the Treasure of the Golden Horde (Attila's gang). He must either flee North or South; Attila can search either North or South.

Hagar would prefer to go South (he can sack Rome even if Attila catches him and forces him to give back the treasure).

Payoff matrix: Note that only the payoffs to Hagar are shown; this is a zero-sum game.

		Attila	
		Search North	**Search South**
Hagar	**Go North**	**- 20**	**0**
	Go South	**+ 5**	**- 10**

Clearly, Hagar would prefer to do the opposite of what Attila does, and Attila would prefer to take the same route Hagar does, so there will be no equilibrium in pure strategies.

There is however a Nash equilibrium in mixed strategies: a best response to the opponent's best response. It can be found by each player adopting a **non-exploitable strategy** -- a strategy which guarantees the player a minimum payoff no matter what the opposing player does.

Consider the problem from Hagar's viewpoint. He would prefer to go South, but he knows that Attila knows he would prefer to go South. Should he go North? If he gives Attila credit for being one smart Hun, he should worry about Attila being able to think that Hagar might deliberately take the less desirable action to throw Attila off the track, and so Hagar might think of going South after all. But won't Attila anticipate this... [continued infinitely].

The answer is that Hagar should make his choice randomly -- but weight the odds in favor of going South, so that Hagar will guarantee himself the maximum expected payoff no matter what strategy Attila adopts.

Language:

Choice variable: the probability with which Hagar goes North is his **choice variable.**
We will denote it by **p.**
The probability with which Attila searches North is Attila's choice variable,
We will denote it by **q**

Expected value: The expected value of the game once Hagar has chosen p is,
if Attila searches North: $EV = -20\,p + 5\,(1 - p)$
if Attila searches South: $EV = 0\,p - 10\,(1 - p)$

Solution technique:
Hagar will choose the probability of going North (which we will denote by p) so that the expected value of the game is the same no matter what Attila does.
Choose p so that

EV if Attila goes North = EV if Attila goes South
If Attila goes North, EV = p (- 20) + (1 - p) 5 = -20 p + 5 - 5 p = - 25 p + 5
If Attila goes South, EV = p (0) + (1 - p) (- 10) = - 10 + 10 p

Hence is found by setting:

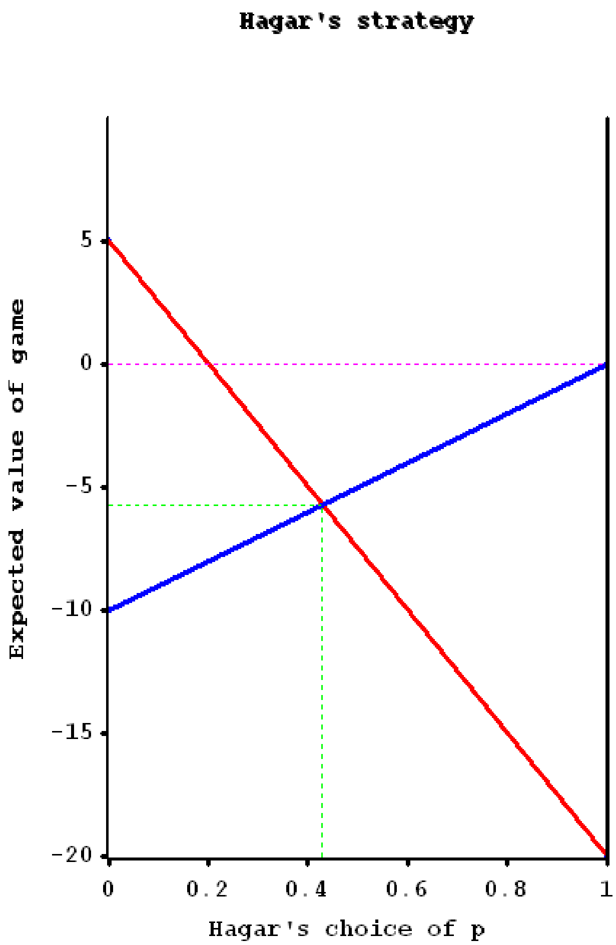
- 25 p + 5 = - 10 + 10 p
15 = 35 p
p = 15 / 35 = 3 / 7   Life will be easier if you leave 3 / 7 as a fraction.

Having found the optimal p for Hagar, we can find the value of the game:

If Attila goes North, EV = - 25 * (3 / 7) + 5 = - 75 / 7 + 35 / 7 = - 40 / 7
If Attila goes South, EV = - 10 + 10 * (3 / 7) = - 70 / 7 + 30 / 7 = - 40 / 7

This may be visualized graphically

the horizontal axis shows Hagar's choice of p.
the downward sloping red line is the EV for any choice of p if Attila goes North
the upward sloping blue line shows the EV for any choice of p if Attila goes South



Can Attila exploit this strategy?

Answer: no, since any mixture of North and South on his part will still lead to an expected value of minus 40 / 7.

Consider what will happen if Attila plays North with probability q.

P (H=N and A=N) = 3/7 * q
P (H=N and A=S) = 3/7 * (1 – q)
P (H=S and A=N) = 4/7 * q
P (H=S and A=S) = 4/7 (1 – q)

Multiply the probabilities times the payoffs and add the to find the expected value of the game:

EV = 3/7 * q * (- 20) + 3/7 * (1 – q) 0 + 4/7 * q * (5) + 4/7 * (1 – q) * (- 10)
EV = - 60/7 * q + 0 + 20/7 * q - 40/7 + 40/7 * q = - 40/7

Whatever strategy Attila adopts, Hagar has guaranteed himself an expected payoff of -40/7.
(Note that an expected payoff is not the actual payoff that will result:
the payoff will actually turn out to be -20 or -10 or 0 or + 5).

So, does it matter what strategy Attila adopts? Yes, because if Attila adopted the wrong strategy, it would be exploitable by Hagar if Hagar knew of it: if Attila played North all the time (or even North more often than optimally), Hagar could do better by going South all the time.

Attila must go through the same process as Hagar to develop a **non-exploitable strategy** by choosing the probability (call it q) with which he will go North.

(Note: Try to develop Attila's strategy without reading any further):

Calculate the expected value of the game with Attila choosing q and
Hagar choosing North (the notation means EV **given that** Hagar chooses North):
EV | HN = - 20 * q + 0 (1 – q) = - 20 q
Hagar choosing South:
EV | HS = 5 * q - 10 * (1 – q) = 15 * q – 10

Solve for a guaranteed expected payoff:
-20 * q = 15 * q – 10
35 * q = 10
q = 10 / 35 = 2 / 7

Note that Attila's probability of going North is NOT the same as Hagar's probability of going North. He is taking into account the fact that Hagar would really prefer to go South.

The expected value of the game, if Hagar goes North, is - 20 (2 / 7) = - 40 / 7
and if Hagar goes South, it will be 15 * (2 / 7) - 10 = 30 / 7 – 70 / 7 = - 40 / 7

The horizontal axis shows Attila's choice of q.

The downward sloping red line shows the expected value of the game if Hagar goes North.

The upward sloping blue line shows the EV of the game if Hagar goes South.

The difference in the intercepts from the previous graph is worth noting to get a visual sense of why Attila's optimal probability of going North differs from Hagar's.