On the motions of a liquid-filled rigid body around a fixed point

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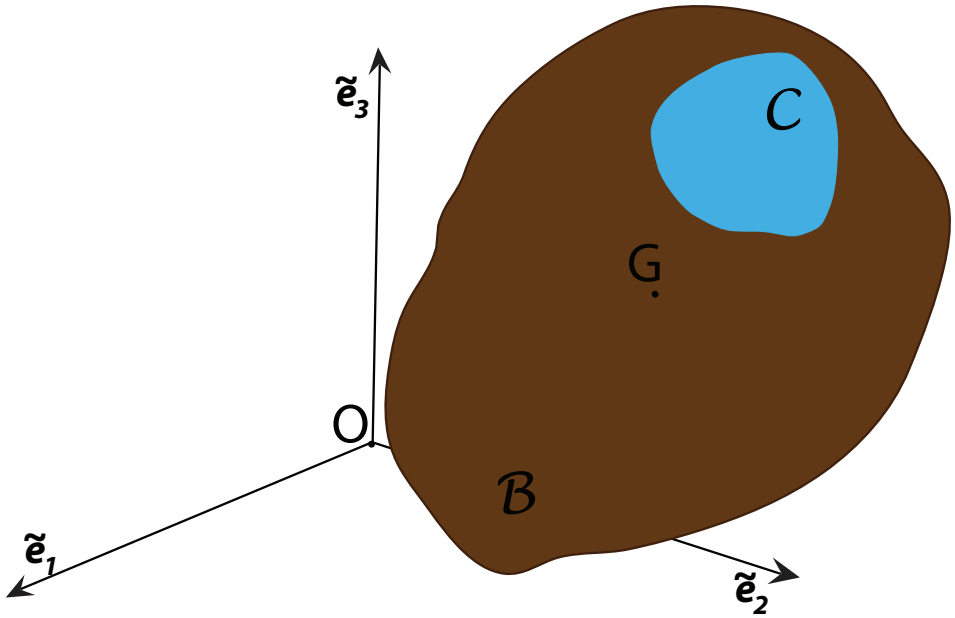
On the motions of a liquid-filled rigid body

Consider a rigid body $\mathcal{B}$ with a cavity, $\mathcal{C}$, completely filled by a viscous liquid

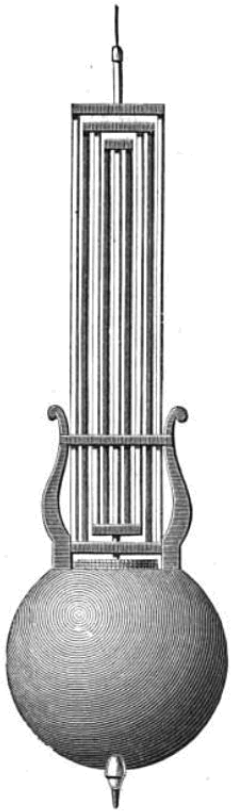


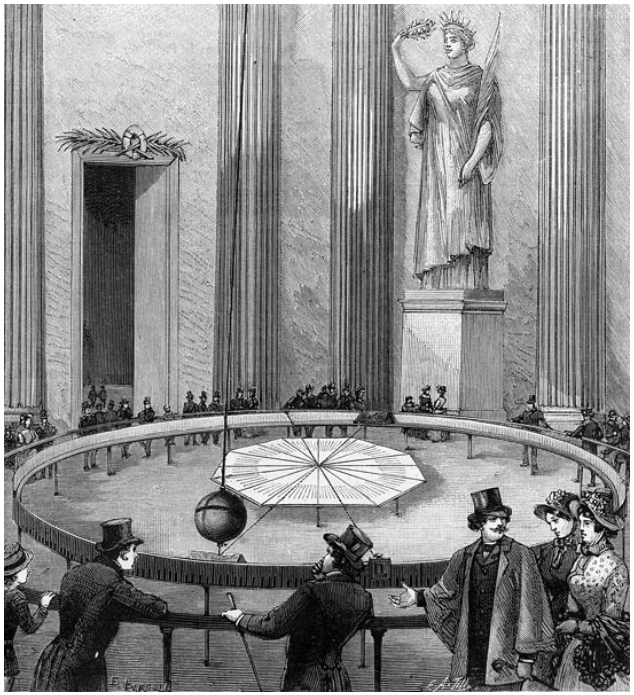
We have investigated motions about a fixed point:

- *motions under the action of gravity*:
  - *motions of a liquid-filled physical pendulum*;
  - *motions of a liquid-filled spherical pendulum*;
  - *motions of a liquid-filled spinning top*.

Examples

$\mathcal{B}$ moves while keeping constant the distance between its center of mass and a fixed point O.





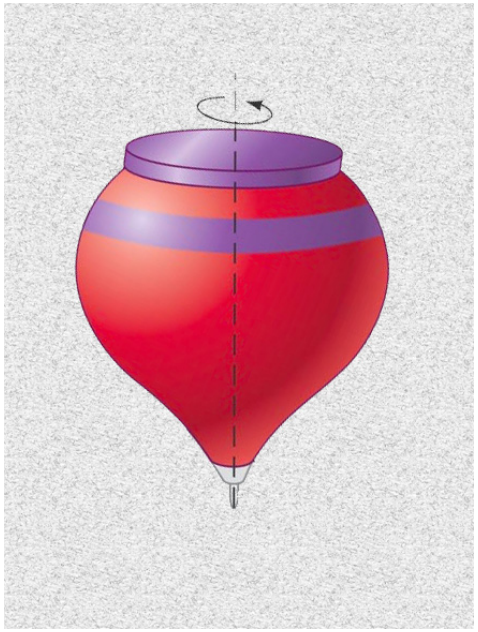


Figure: Physical Pendulum (left), Spherical Pendulum (center), Spinning Top (right).

Applications

In space engineering:

- study of the motion of fuel within the tank; [1]
- tube dampers filled with a viscous liquid are used to suppress oscillations in spacecraft and artificial satellites. [2]

[1] Abramson, H. N. (1966) *Dynamic behavior of liquids in moving containers with applications to propellants in space vehicle fuel tanks*. NASA-SP-106.

[2] Bhuta, P.G. & Koval, L.R. (1966) *A viscous ring damper for a freely precessing satellite*. Intern. J. Mech. Sci. **8** 5.

About the long-time behavior: numerical simulations

The liquid has a stabilizing effect on the motion of the solid: after an initial interval of time, whose length depends on the size of initial data and viscosity of the liquid, where the motion has a typically "chaotic" nature, the coupled system eventually reaches a more orderly configuration, due to the combined effect of viscosity and incompressibility.

About the long-time behavior: numerical simulations

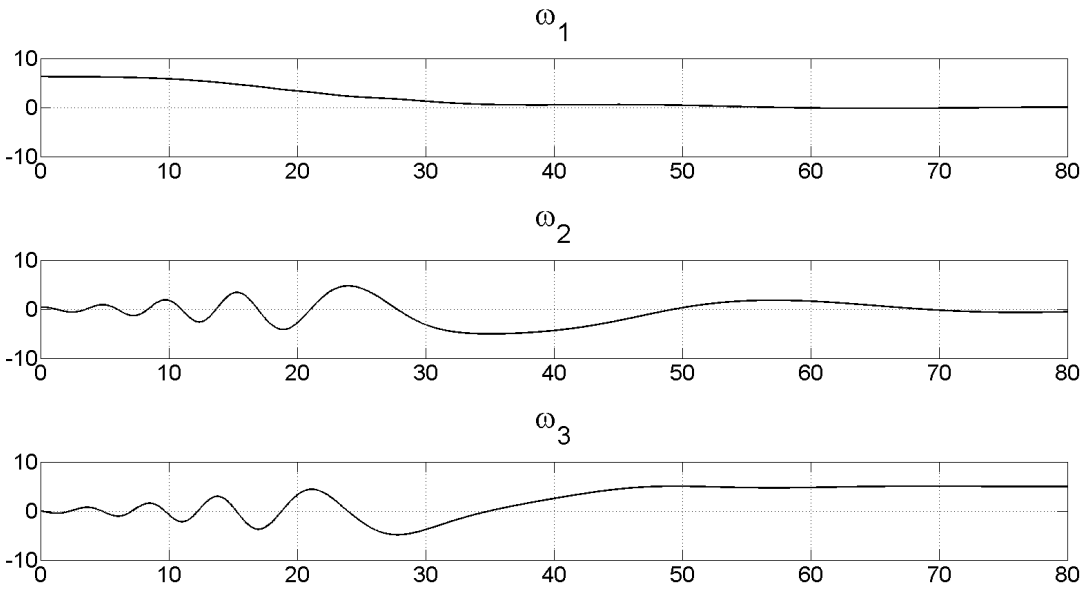
The liquid has a stabilizing effect on the motion of the solid: after an initial interval of time, whose length depends on the size of initial data and viscosity of the liquid, where the motion has a typically "chaotic" nature, the coupled system eventually reaches a more orderly configuration, due to the combined effect of viscosity and incompressibility.
In the case of inertial motions:

Reference:

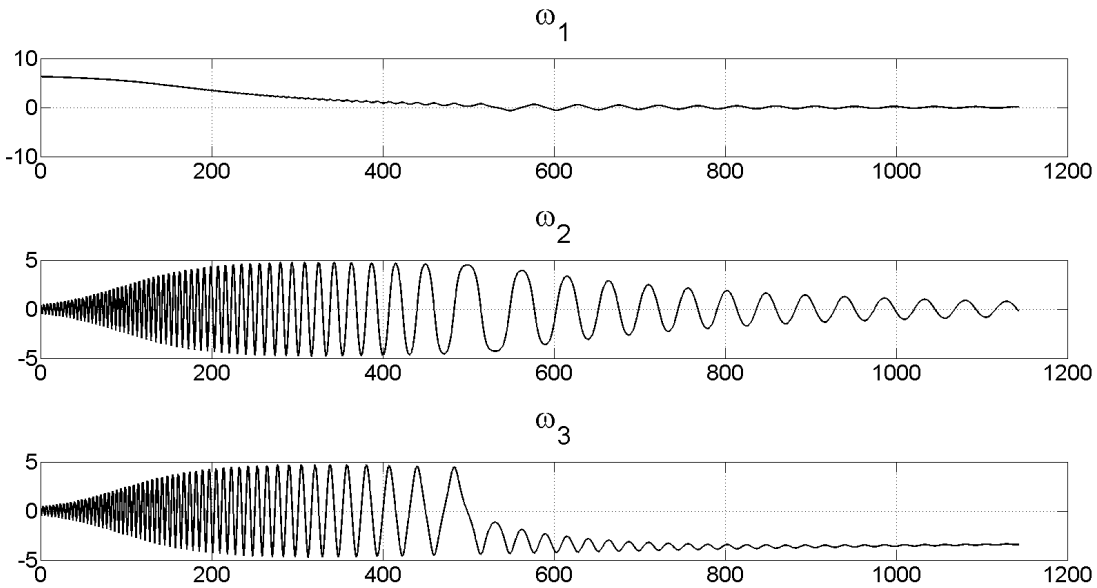
K. Disser, G. P. Galdi, G. Mazzone, and P. Zunino,
Inertial motions of a rigid body with a cavity filled with a viscous liquid,
Arch. Rational Mech. Anal., Volume 221, Issue 1, pp. 487–526 (2016) (online, 04 February 2016).

Dynamics of the liquid-solid system for values of viscosity

$\nu = 0.1$



$\nu = 0.001$



About the steady states

We shall make the following two assumptions:

1. The coupled system is constrained to move (without friction) about a fixed point $O \in \mathcal{B}$.
2. G belongs to one of the principal axes of inertia of $\mathcal{S}$ with respect to O.

Steady states with respect to the *moving frame* $\mathsf{F} := \{O, \boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$, $\boldsymbol{e}_1 \equiv \overrightarrow{OG}/|\overrightarrow{OG}|$ are solutions to the following boundary value problem

$$\left.\begin{aligned} &\boldsymbol{v}\cdot\nabla\boldsymbol{v} + 2(\boldsymbol{\omega}_\infty + \boldsymbol{a})\times\boldsymbol{v} = \nu\Delta\boldsymbol{v} - \nabla\tilde{p} \\ &\operatorname{div}\boldsymbol{v} = 0 \end{aligned}\right\} \text{ on } \mathcal{C},$$

$$(\boldsymbol{\omega}_\infty + \boldsymbol{a})\times\boldsymbol{I}\cdot\boldsymbol{\omega}_\infty = \beta^2\boldsymbol{e}_1\times\boldsymbol{\gamma},$$

$$(\boldsymbol{\omega}_\infty + \boldsymbol{a})\times\boldsymbol{\gamma} = \boldsymbol{0},$$

$$\boldsymbol{v} = \boldsymbol{0} \qquad \text{on } \partial C.$$

Here, we have set $\boldsymbol{g} := g\boldsymbol{\gamma}$, $\tilde{p} := p/\rho - g\boldsymbol{\gamma}\cdot\boldsymbol{x}$,

$$\beta^2 = Mg|\overrightarrow{OG}|, \quad \boldsymbol{a} := -\rho\boldsymbol{I}^{-1}\cdot\int_{\mathcal{C}}\boldsymbol{y}\times\boldsymbol{v}, \quad \boldsymbol{\omega}_\infty := \boldsymbol{\omega} - \boldsymbol{a} \quad (\Leftrightarrow \boldsymbol{A}_O = \boldsymbol{I}\cdot\boldsymbol{\omega}_\infty).$$

About the steady states (continued)

Lemma

$(\boldsymbol{v}, \boldsymbol{\omega}_\infty, \boldsymbol{\gamma})$ is a steady solution iff it satisfies

$$\begin{cases} \boldsymbol{v} \equiv \boldsymbol{0} & \text{on } \mathcal{C}, \\ \boldsymbol{\omega}_\infty \times \boldsymbol{I} \cdot \boldsymbol{\omega}_\infty = \beta^2 \boldsymbol{e}_1 \times \boldsymbol{\gamma}, \\ \boldsymbol{\omega}_\infty \times \boldsymbol{\gamma} = \boldsymbol{0}. \end{cases}$$

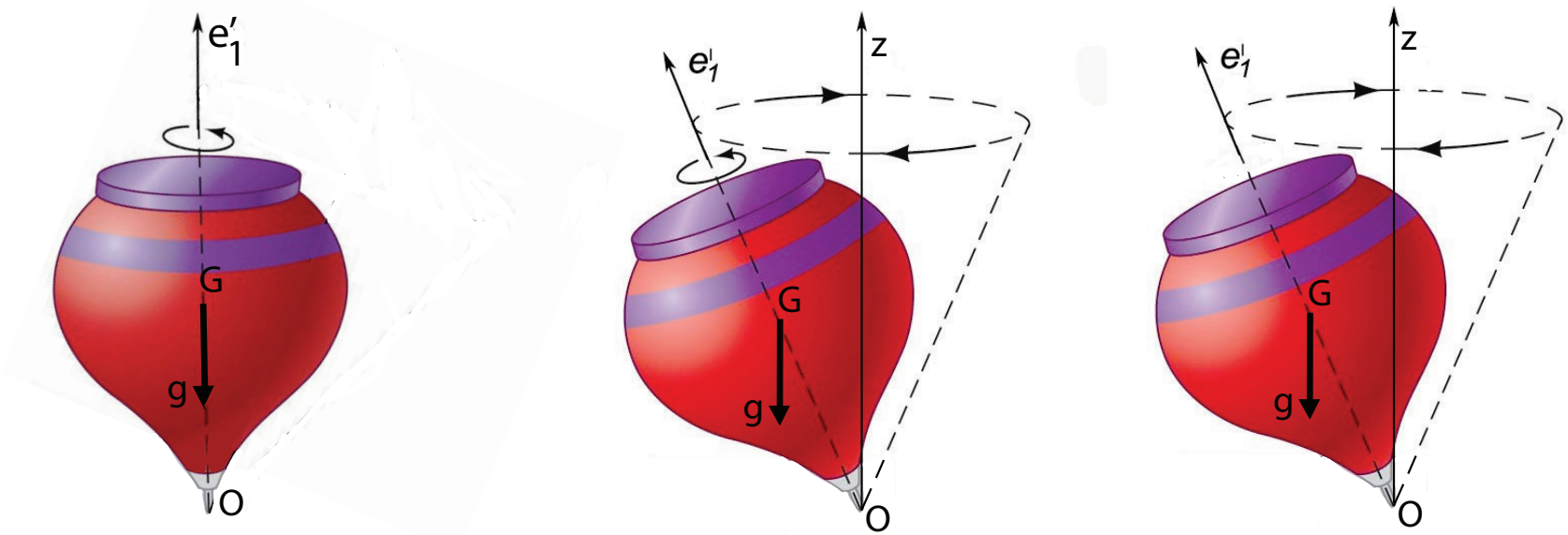


Figure: Permanent rotation (left), regular precession (center), steady precession (right).

Equations of motion in a moving frame

Assumptions:

1. The coupled system is constrained to move (without friction) about a fixed point $O \in \mathcal{B}$.
2. G belongs to one of the principal axes of inertia of $\mathcal{S}$ with respect to O.

The equations of motion for the coupled system $\mathcal{S}$ with respect to a *moving frame* $\mathsf{F} := \{O, \boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$, $\boldsymbol{e}_1 \equiv \overrightarrow{OG}/|\overrightarrow{OG}|$, are given by

$$\left.\begin{aligned} &\rho\left(\frac{\partial \boldsymbol{v}}{\partial t} + \boldsymbol{v}\cdot\nabla\boldsymbol{v} + (\dot{\boldsymbol{\omega}}_\infty + \dot{\boldsymbol{a}})\times\boldsymbol{y} + 2(\boldsymbol{\omega}_\infty + \boldsymbol{a})\times\boldsymbol{v}\right) \\ &\qquad\qquad = \mu\Delta\boldsymbol{v} - \nabla p + \rho g_0 \boldsymbol{\gamma} \\ &\operatorname{div}\boldsymbol{v} = 0 \end{aligned}\right\} \text{ on } \mathcal{C}\times(0,\infty),$$

$$\boldsymbol{I}\cdot\dot{\boldsymbol{\omega}}_\infty + (\boldsymbol{\omega}_\infty + \boldsymbol{a})\times\boldsymbol{I}\cdot\boldsymbol{\omega}_\infty = \beta^2\boldsymbol{e}_1\times\boldsymbol{\gamma} \qquad \text{in } (0,\infty),$$

$$\dot{\boldsymbol{\gamma}} + (\boldsymbol{\omega}_\infty + \boldsymbol{a})\times\boldsymbol{\gamma} = \boldsymbol{0},$$

$$\boldsymbol{v} = \boldsymbol{0} \qquad \text{on } \partial\mathcal{C}. \tag{1}$$

Here, we recall that

$$\beta^2 = Mg|\overrightarrow{OG}|, \quad \boldsymbol{a} = -\rho\boldsymbol{I}^{-1}\cdot\int_{\mathcal{C}}\boldsymbol{y}\times\boldsymbol{v}, \quad \boldsymbol{\omega}_\infty = \boldsymbol{\omega} - \boldsymbol{a} \quad (\Leftrightarrow \boldsymbol{A}_O = \boldsymbol{I}\cdot\boldsymbol{\omega}_\infty). \tag{2}$$

Energy balance

The *energy balance* is given by

$$\frac{d}{dt}(\mathcal{E}+\mathcal{U})+2\mu\|\nabla\boldsymbol{v}\|_2^2=0 \tag{3}$$

where we have denoted by

$$\mathcal{E}(t):=\mathcal{E}_F(t)+\boldsymbol{\omega}_\infty\cdot\boldsymbol{I}\cdot\boldsymbol{\omega}_\infty \quad \text{and} \quad \mathcal{U}(t):=-2\beta^2\boldsymbol{\gamma}\cdot\boldsymbol{e}_1,$$

the *kinetic* and *potential energy* of $\mathcal{S}$, respectively.

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the *kinetic* and *potential energy* of $\mathcal{S}$, respectively. Specifically,

$$\mathcal{E}_F(t):=\rho\|\boldsymbol{v}\|_2^2-\boldsymbol{a}\cdot\boldsymbol{I}\cdot\boldsymbol{a}, \tag{4}$$

and it satisfies

$$c_1\|\boldsymbol{v}\|_2^2\le\mathcal{E}_F(t)\le c_2\|\boldsymbol{v}\|_2^2, \tag{5}$$

for some positive constants c_1 and c_2.

Weak solutions

Definition

We will say that the triple $(\boldsymbol{v}, \boldsymbol{\omega}_\infty, \boldsymbol{\gamma})$ is a *weak solution* if it satisfies the following conditions:

(a) $\boldsymbol{v} \in C_w([0,\infty); H(\mathcal{C})) \cap L^\infty(0,\infty; H(\mathcal{C})) \cap L^2(0,\infty; W_0^{1,2}(\mathcal{C}))$;

(b) $\boldsymbol{\omega}_\infty \in C([0,\infty)) \cap C^1(0,\infty)$, $\boldsymbol{\gamma} \in C^1([0,\infty); \mathsf{S}^2)$;

(c) $(\boldsymbol{v}, \boldsymbol{\omega}_\infty, \boldsymbol{\gamma})$ satisfies the equations of motions in the distributional sense and the boundary condition in the trace sense;

(d) the Strong Energy Inequality:

$$\mathcal{E}(t) + \mathcal{U}(t) + 2\mu \int_s^t \|\nabla \boldsymbol{v}(\tau)\|_2^2 \, d\tau \leq \mathcal{E}(s) + \mathcal{U}(s) \tag{6}$$

holds for all $t \geq s$ and a.a. $s \geq 0$ including $s = 0$.

Large-time properties of weak solutions

Lemma

For every $\boldsymbol{v}_0 \in H(\mathcal{C}), \boldsymbol{\omega}_{\infty 0} \in \mathbb{R}^3$ *and* $\boldsymbol{\gamma}_0 \in \mathsf{S}^2$, *there exists at least one weak solution* $(\boldsymbol{v}, \boldsymbol{\omega}_\infty, \boldsymbol{\gamma})$. *Moreover,*

$$\lim_{t \to \infty} \|\boldsymbol{v}(t)\|_2 = 0.$$

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Let $\mathsf{s} = (\boldsymbol{v}, \boldsymbol{\omega}_\infty, \boldsymbol{\gamma})$ be a weak solution, and set $\mathcal{H} := H(\mathcal{C}) \times \mathbb{R}^3 \times \mathsf{S}^2$. The *$\Omega$-limit set* of s is defined as

$$\Omega(\mathsf{s}) := \{(\boldsymbol{u}, \boldsymbol{\omega}, \boldsymbol{q}) \in \mathcal{H} : \text{ there exists } t_k \geq 0,\ t_k \nearrow \infty \text{ s.t.}$$
$$\lim_{k\to\infty} \|\boldsymbol{v}(t_k) - \boldsymbol{u}\|_2 = \lim_{k\to\infty} |\boldsymbol{\omega}_\infty(t_k) - \boldsymbol{\omega}| = \lim_{k\to\infty} |\boldsymbol{\gamma}(t_k) - \boldsymbol{q}| = 0\}.$$

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Definition

$\Omega(\mathsf{s})$ is positively invariant if the following implication holds:

$$\mathsf{y} \in \Omega(\mathsf{s}) \quad \Rightarrow \quad \mathsf{w}(t;\mathsf{y}) \in \Omega(\mathsf{s}), \qquad \text{all } t \geq 0,$$

and for all weak solutions $\mathsf{w}(t;\mathsf{y})$.

Large-time properties of weak solutions (continued)

Proposition. Let $\mathsf{s} = \mathsf{s}(t; \mathsf{s}_0) := (\boldsymbol{v}, \boldsymbol{\omega}_\infty, \boldsymbol{\gamma})$ be a weak solution corresponding to initial data, $\mathsf{s}_0 \in \mathcal{H}$. Then,

a) $\Omega(\mathsf{s})$ is non-empty, compact, and connected,

b) $\Omega(\mathsf{s})$ is positively invariant in the class of weak solutions. Moreover,

c) $\Omega(\mathsf{s}) \subset \{(\bar{\boldsymbol{v}}, \bar{\boldsymbol{\omega}}, \bar{\boldsymbol{\gamma}}) \in \mathcal{H} : \ \bar{\boldsymbol{v}} \equiv \mathbf{0}, \ \ \bar{\boldsymbol{\omega}} \times \boldsymbol{I} \cdot \bar{\boldsymbol{\omega}} = \beta^2 \boldsymbol{e}_1 \times \bar{\boldsymbol{\gamma}}, \ \ \bar{\boldsymbol{\omega}} \times \bar{\boldsymbol{\gamma}} = \mathbf{0}\}$.

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Steps of the proof.

Step 1.

The non-emptiness, the compactness and the connectedness of $\Omega(\mathsf{s})$ follows from the following facts:

- weak solutions corresponding to initial data of finite kinetic energy exist;
- weak solutions satisfy the strong energy inequality;
- $\boldsymbol{\omega}_\infty, \boldsymbol{\gamma} \in C([0,\infty))$ and $\boldsymbol{v} \to 0$ as $t \to \infty$.

Large-time properties of weak solutions (continued)

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Steps of the proof.

Lemma (Step 2.)

1. s *is unique in the class of weak solutions in* $[t_0, \infty)$*;*
2. s *depends continuously upon the data in* $[t_0, \infty)$*, in the class of weak solutions.*

Moreover,

$$\lim_{t\to\infty} \|\nabla \boldsymbol{v}\|_2 = 0.$$

Large-time properties of weak solutions (continued)

Proposition. Let $\mathsf{s} = \mathsf{s}(t;\mathsf{s}_0) := (\boldsymbol{v}, \boldsymbol{\omega}_\infty, \boldsymbol{\gamma})$ be a weak solution corresponding to initial data, $\mathsf{s}_0 \in \mathcal{H}$. Then,

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Steps of the proof.

Step 3.

The dynamics on $\Omega(\mathsf{s})$ is governed by

$$\boldsymbol{v} \equiv \boldsymbol{0}, \quad \rho \int_{\mathcal{C}} (\dot{\boldsymbol{\omega}}_\infty \times \boldsymbol{y}) \cdot \boldsymbol{\psi} = 0 \text{ for all } \boldsymbol{\psi} \in \mathcal{D}^{1,2}_0(\mathcal{C}),$$

$$\boldsymbol{I} \cdot \dot{\boldsymbol{\omega}}_\infty + \boldsymbol{\omega}_\infty \times \boldsymbol{I} \cdot \boldsymbol{\omega}_\infty = \beta^2 \boldsymbol{e}_1 \times \boldsymbol{\gamma},$$

$$\dot{\boldsymbol{\gamma}} + \boldsymbol{\omega}_\infty \times \boldsymbol{\gamma} = \boldsymbol{0}.$$

Large-time properties of weak solutions (continued)

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Steps of the proof.

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The dynamics on $\Omega(\mathsf{s})$ is governed by

$$\begin{aligned} &\boldsymbol{v} \equiv \boldsymbol{0}, \quad \dot{\boldsymbol{\omega}}_\infty \times \boldsymbol{x} = \nabla \varphi, \\ &\boldsymbol{I} \cdot \dot{\boldsymbol{\omega}}_\infty + \boldsymbol{\omega}_\infty \times \boldsymbol{I} \cdot \boldsymbol{\omega}_\infty = \beta^2 \boldsymbol{e}_1 \times \boldsymbol{\gamma}, \\ &\dot{\boldsymbol{\gamma}} + \boldsymbol{\omega}_\infty \times \boldsymbol{\gamma} = \boldsymbol{0}. \end{aligned}$$

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Large-time properties of weak solutions (continued)

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Large-time properties of weak solutions (continued)

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The dynamics on $\Omega(\mathsf{s})$ is governed by

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$$\dot{\boldsymbol{\gamma}} + \boldsymbol{\omega}_\infty \times \boldsymbol{\gamma} = \boldsymbol{0} \quad \Rightarrow \boldsymbol{e}_1 \cdot \dot{\boldsymbol{\gamma}} = 0.$$

Large-time properties of weak solutions (continued)

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Motion of a liquid-filled pendulum

Reference:

G. P. Galdi and G. Mazzone,
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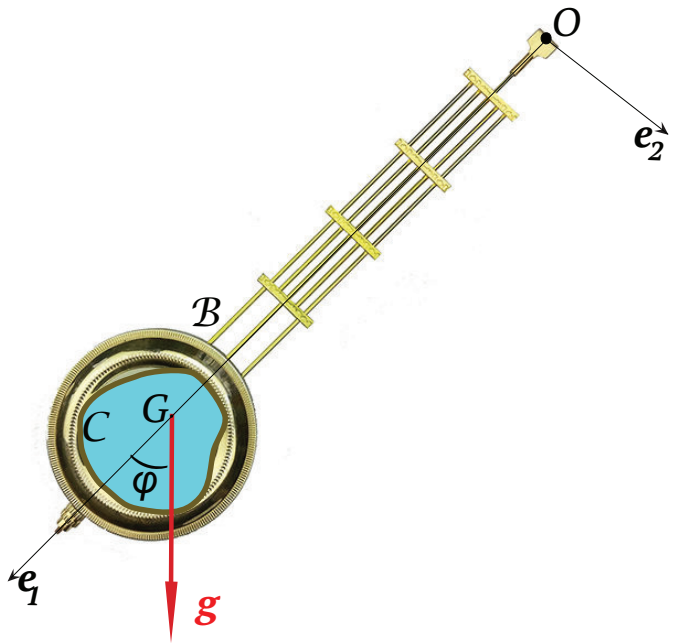
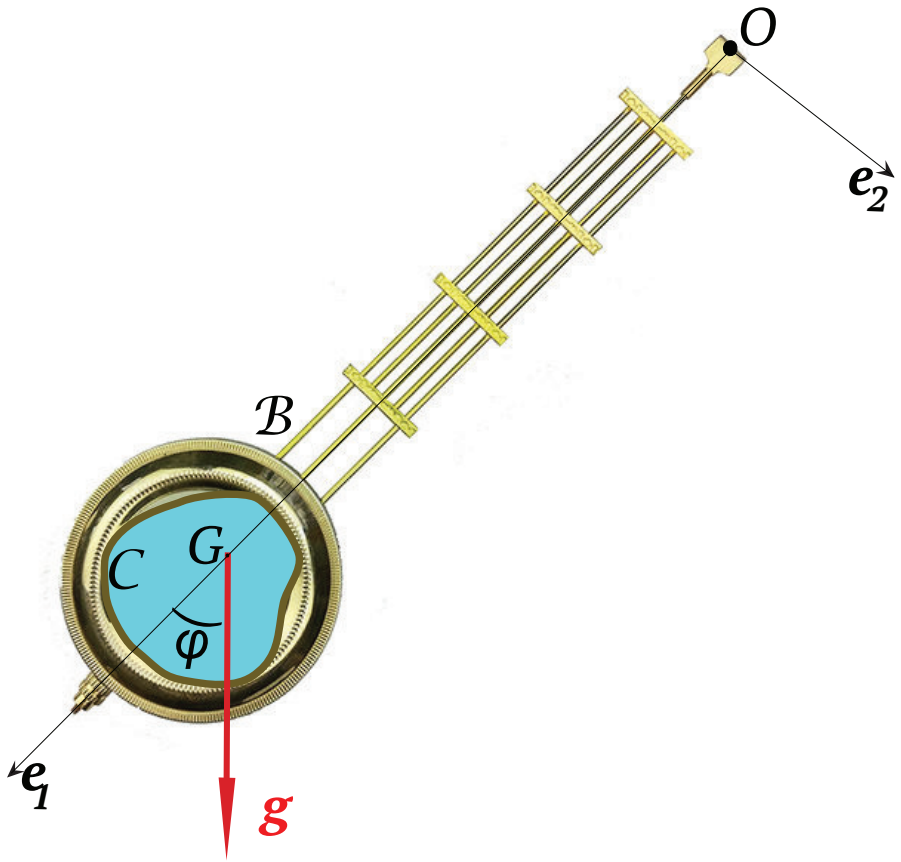
in Recent progress in the theory of the Euler and Navier-Stokes Equations, London Mathematical Society Lecture Note Series: 430, Cambridge University Press, 2016, pp. 37–56.



Motion of a liquid-filled pendulum

Consider the coupled system constituted by a rigid body, $\mathcal{B}$, with an interior cavity, $\mathcal{C}$, entirely filled with a Navier-Stokes liquid, satisfying the following properties:

- $\mathcal{B}$ is constrained to move without friction about a fixed axis, **a**,



Motion of a liquid-filled pendulum

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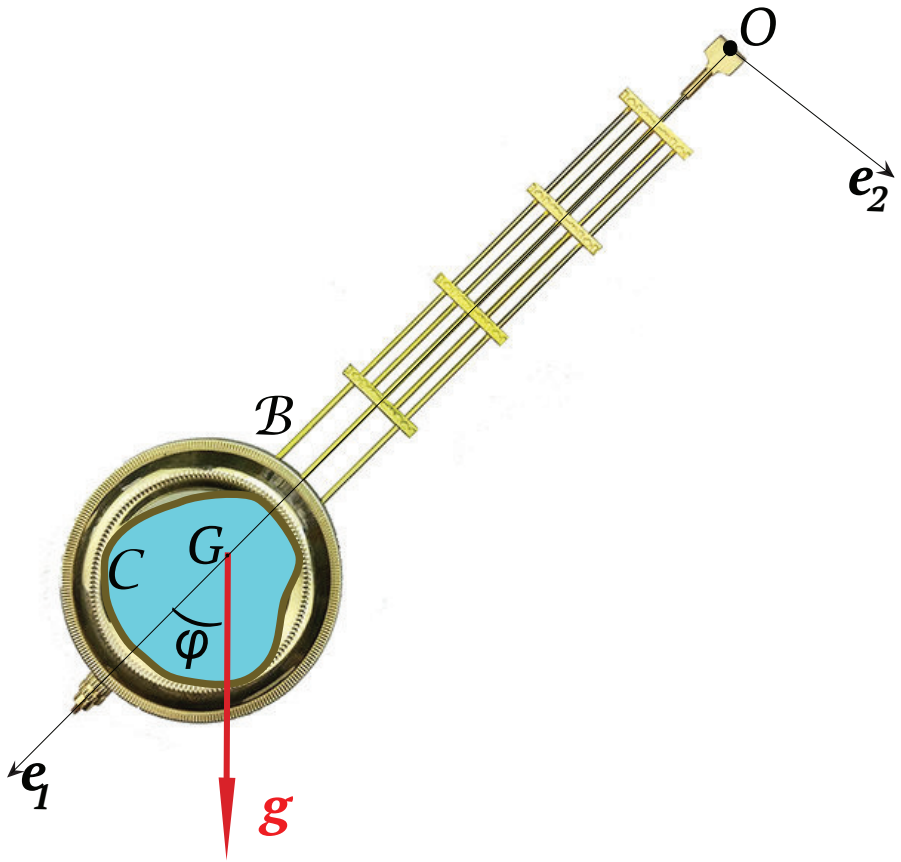

- the angular velocity is: $\omega(t)\boldsymbol{e}_3$;
- $\boldsymbol{\gamma} = (\cos\varphi, -\sin\varphi, 0)$.

Asymptotic Behavior

Theorem

Let $(\boldsymbol{v}, \omega, \boldsymbol{\gamma})$ be a weak solution corresponding to initial data of finite energy. Then,

$$\lim_{t\to\infty}\left(\|\boldsymbol{v}(t)\|_{2,2}+\left\|\frac{\partial \boldsymbol{v}(t)}{\partial t}\right\|_2\right)=0\,,\quad \textit{and}\quad \lim_{t\to\infty}\left(\max_{x\in\mathcal{C}}|\boldsymbol{v}(x,t)|\right)=0\,,$$

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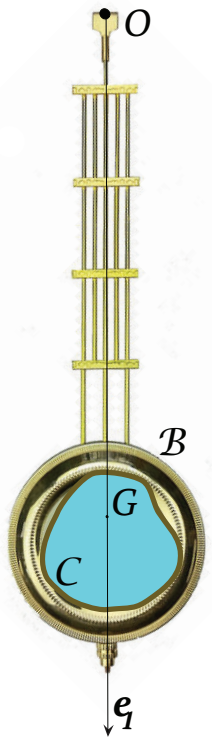

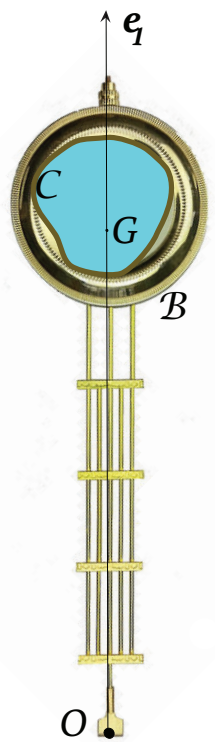

Attainability & Stability

Theorem (Attainability)

Let $\mathcal{C}$ be of class C^2, and let $(\boldsymbol{v}_0, \omega_0, \boldsymbol{\gamma}_0) \in H(\mathcal{C}) \times \mathbb{R} \times \mathsf{S}^1$ be given with

$$\rho \, \|\boldsymbol{v}_0\|_2^2 + C \, (\omega_0 - a(0))^2 < 2 \, C \, \beta^2 \, (1 + \gamma_{1,0}) \, .$$

Then all weak solutions corresponding to $(\boldsymbol{v}_0, \omega_0, \boldsymbol{\gamma}_0)$ tend $(\boldsymbol{v} \equiv \boldsymbol{0}, \omega \equiv 0, \boldsymbol{\gamma} \equiv \boldsymbol{e}_1)$.

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Theorem (Stability)

- $(\boldsymbol{v} \equiv \boldsymbol{0}, \omega \equiv 0, \boldsymbol{\gamma} \equiv \boldsymbol{e}_1))$ *is stable in the sense of Lyapunov. .*
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Attainability & Stability

Theorem (Attainability)

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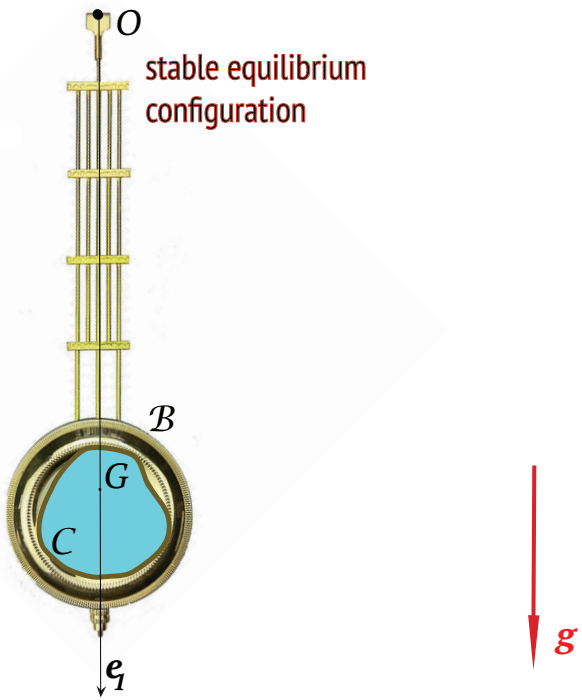
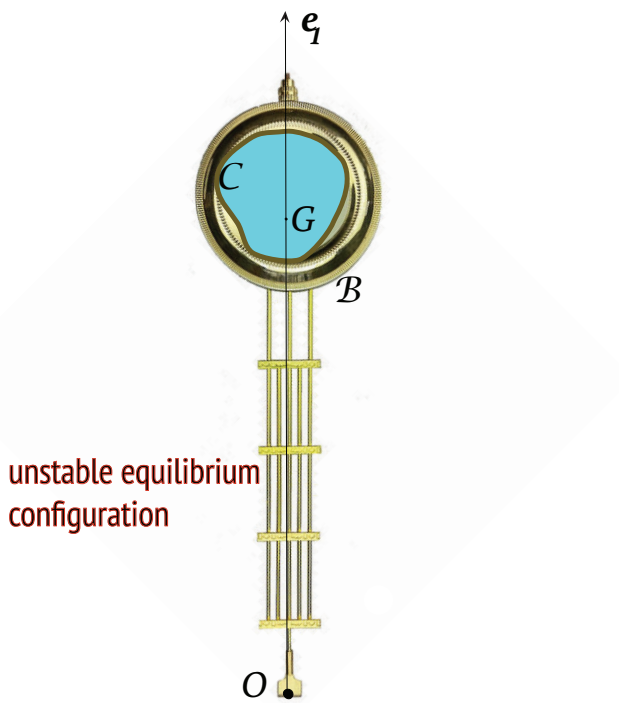
Stability results:





THANK YOU!