On the motions of a liquid-filled rigid body around a fixed point

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Consider a rigid body $B$ with a cavity, $C$, completely filled by a viscous liquid.

We have investigated motions about a fixed point:

- motions under the action of gravity:
  - motions of a liquid-filled physical pendulum;
  - motions of a liquid-filled spherical pendulum;
  - motions of a liquid-filled spinning top.
Examples

$B$ moves while keeping constant the distance between its center of mass and a fixed point $O$.

Figure: Physical Pendulum (left), Spherical Pendulum (center), Spinning Top (right).
Applications

In space engineering:

- study of the motion of fuel within the tank; \(^1\)
- tube dampers filled with a viscous liquid are used to suppress oscillations in spacecraft and artificial satellites. \(^2\)


The liquid has a stabilizing effect on the motion of the solid: after an initial interval of time, whose length depends on the size of initial data and viscosity of the liquid, where the motion has a typically “chaotic” nature, the coupled system eventually reaches a more orderly configuration, due to the combined effect of viscosity and incompressibility.
The liquid has a **stabilizing effect** on the motion of the solid: after an initial interval of time, whose length depends on the size of initial data and viscosity of the liquid, where the motion has a typically “chaotic” nature, the coupled system eventually reaches a more orderly configuration, due to the combined effect of viscosity and incompressibility.

In the case of **inertial motions**:

**Reference:**

Dynamics of the liquid-solid system for values of viscosity

\( \nu = 0.1 \)

\[ \begin{align*}
\omega_1 & = 10 \\
\omega_2 & = 0 \\
\omega_3 & = 10 
\end{align*} \]

\( \nu = 0.001 \)

\[ \begin{align*}
\omega_1 & = 10 \\
\omega_2 & = 5 \\
\omega_3 & = 5 
\end{align*} \]
About the steady states

We shall make the following two assumptions:

1. The coupled system is constrained to move (without friction) about a fixed point $O \in \mathcal{B}$.

2. $G$ belongs to one of the principal axes of inertia of $S$ with respect to $O$.

Steady states with respect to the moving frame $F := \{O, e_1, e_2, e_3\}$, $e_1 \equiv \overrightarrow{OG}/|\overrightarrow{OG}|$ are solutions to the following boundary value problem

$$\begin{align*}
\nu \cdot \nabla \nu + 2(\omega_\infty + a) \times \nu &= \nu \Delta \nu - \nabla \tilde{p} \\
\text{div } \nu &= 0 \\
(\omega_\infty + a) \times \mathbf{I} \cdot \omega_\infty &= \beta^2 e_1 \times \gamma, \\
(\omega_\infty + a) \times \gamma &= 0, \\
\nu &= 0 \quad \text{on } \partial C.
\end{align*}$$

Here, we have set $g := g\gamma$, $\tilde{p} := p/\rho - g\gamma \cdot x$,

$$\beta^2 = Mg|\overrightarrow{OG}|, \quad a := -\rho I^{-1} \cdot \int_C \mathbf{y} \times \nu, \quad \omega_\infty := \omega - a \quad (\Leftrightarrow A_O = \mathbf{I} \cdot \omega_\infty).$$
Lemma

\((v, \omega_\infty, \gamma)\) is a steady solution iff it satisfies

\[
\begin{aligned}
    v &\equiv 0 \quad \text{on } C, \\
    \omega_\infty \times I \cdot \omega_\infty &= \beta^2 e_1 \times \gamma, \\
    \omega_\infty \times \gamma &= 0.
\end{aligned}
\]

Figure: Permanent rotation (left), regular precession (center), steady precession (right).
Equations of motion in a moving frame

Assumptions:
1. The coupled system is constrained to move (without friction) about a fixed point $O \in B$.
2. $G$ belongs to one of the principal axes of inertia of $S$ with respect to $O$.

The equations of motion for the coupled system $S$ with respect to a moving frame $F := \{O, e_1, e_2, e_3\}$, $e_1 \equiv \overrightarrow{OG}/|\overrightarrow{OG}|$, are given by

$$\begin{align*}
\rho \left( \frac{\partial v}{\partial t} + v \cdot \nabla v + (\dot{\omega}_\infty + \dot{a}) \times y + 2(\omega_\infty + a) \times v \right) \\
= \mu \Delta v - \nabla p + \rho g_0 \gamma \\
\text{div} \ v = 0 \\
I \cdot \dot{\omega}_\infty + (\omega_\infty + a) \times I \cdot \omega_\infty = \beta^2 e_1 \times \gamma \\
\dot{\gamma} + (\omega_\infty + a) \times \gamma = 0, \\
v = 0 \quad \text{on } \partial C.
\end{align*}$$

(1)

Here, we recall that

$$\beta^2 = M g |\overrightarrow{OG}|, \quad a = -\rho I^{-1} \cdot \int_C y \times v, \quad \omega_\infty = \omega - a \quad (\Leftrightarrow A_O = I \cdot \omega_\infty).$$

(2)
The **energy balance** is given by

\[
\frac{d}{dt}(E + U) + 2\mu \|\nabla v\|_2^2 = 0 \tag{3}
\]

where we have denoted by

\[
E(t) := E_F(t) + \omega_\infty \cdot I \cdot \omega_\infty \quad \text{and} \quad U(t) := -2\beta^2 \gamma \cdot e_1,
\]

the **kinetic** and **potential energy** of $S$, respectively.
Energy balance

The *energy balance* is given by

$$\frac{d}{dt}(\mathcal{E} + \mathcal{U}) + 2\mu \|\nabla v\|_2^2 = 0$$

(3)

where we have denoted by

$$\mathcal{E}(t) := \mathcal{E}_F(t) + \omega_\infty \cdot I \cdot \omega_\infty \quad \text{and} \quad \mathcal{U}(t) := -2\beta^2 \gamma \cdot e_1,$$

the *kinetic* and *potential energy* of $S$, respectively. Specifically,

$$\mathcal{E}_F(t) := \rho \|v\|_2^2 - a \cdot I \cdot a,$$

(4)

and it satisfies

$$c_1 \|v\|_2^2 \leq \mathcal{E}_F(t) \leq c_2 \|v\|_2^2,$$

(5)

for some positive constants $c_1$ and $c_2$. 
We will say that the triple \((v, \omega_\infty, \gamma)\) is a \textit{weak solution} if it satisfies the following conditions:

(a) \(v \in C_w([0, \infty); H(C)) \cap L^\infty(0, \infty; H(C)) \cap L^2(0, \infty; W^{1,2}_0(C))\);

(b) \(\omega_\infty \in C([0, \infty)) \cap C^1(0, \infty), \gamma \in C^1([0, \infty); S^2)\);

(c) \((v, \omega_\infty, \gamma)\) satisfies the equations of motions in the distributional sense and the boundary condition in the trace sense;

(d) the Strong Energy Inequality:

\[
E(t) + U(t) + 2\mu \int_s^t \| \nabla v(\tau) \|_2^2 \, d\tau \leq E(s) + U(s) \tag{6}
\]

holds for all \(t \geq s\) and a.a. \(s \geq 0\) including \(s = 0\).
Large-time properties of weak solutions

**Lemma**

For every \( v_0 \in H(C), \omega_\infty \in \mathbb{R}^3 \) and \( \gamma_0 \in S^2 \), there exists at least one weak solution \((v, \omega_\infty, \gamma)\). Moreover,

\[
\lim_{t \to \infty} \|v(t)\|_2 = 0.
\]
Large-time properties of weak solutions

**Lemma**

For every $v_0 \in H(C), \omega_{\infty 0} \in \mathbb{R}^3$ and $\gamma_0 \in S^2$, there exists at least one weak solution $(v, \omega_{\infty}, \gamma)$. Moreover,

$$\lim_{t \to \infty} \|v(t)\|_2 = 0.$$

Let $s = (v, \omega_{\infty}, \gamma)$ be a weak solution, and set $\mathcal{H} := H(C) \times \mathbb{R}^3 \times S^2$. The **$\Omega$-limit set** of $s$ is defined as

$$\Omega(s) := \{ (u, \omega, q) \in \mathcal{H} : \text{there exists } t_k \geq 0, t_k \uparrow \infty \text{ s.t.}$$

$$\lim_{k \to \infty} \|v(t_k) - u\|_2 = \lim_{k \to \infty} |\omega_{\infty}(t_k) - \omega| = \lim_{k \to \infty} |\gamma(t_k) - q| = 0 \}. $$
Lemma

For every \( v_0 \in H(C), \omega_{\infty 0} \in \mathbb{R}^3 \) and \( \gamma_0 \in S^2 \), there exists at least one weak solution \((v, \omega_\infty, \gamma)\). Moreover,

\[
\lim_{t \to \infty} \|v(t)\|_2 = 0.
\]

Let \( s = (v, \omega_\infty, \gamma) \) be a weak solution, and set \( \mathcal{H} := H(C) \times \mathbb{R}^3 \times S^2 \). The \( \Omega \)-limit set of \( s \) is defined as

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\Omega(s) := \{(u, \omega, q) \in \mathcal{H} : \text{there exists } t_k \geq 0, \ t_k \to \infty \text{ s.t. } \\
\lim_{k \to \infty} \|v(t_k) - u\|_2 = \lim_{k \to \infty} |\omega_\infty(t_k) - \omega| = \lim_{k \to \infty} |\gamma(t_k) - q| = 0\}.
\]

Definition

\( \Omega(s) \) is positively invariant if the following implication holds:

\[
y \in \Omega(s) \quad \Rightarrow \quad w(t; y) \in \Omega(s), \quad \text{all } t \geq 0,
\]

and for all weak solutions \( w(t; y) \).
Proposition. Let $s = s(t; s_0) := (v, \omega_\infty, \gamma)$ be a weak solution corresponding to initial data, $s_0 \in \mathcal{H}$. Then,

a) $\Omega(s)$ is non-empty, compact, and connected,

b) $\Omega(s)$ is positively invariant in the class of weak solutions. Moreover,

c) $\Omega(s) \subset \{(\bar{v}, \bar{\omega}, \bar{\gamma}) \in \mathcal{H} : \bar{v} \equiv 0, \bar{\omega} \times I \cdot \bar{\omega} = \beta^2 e_1 \times \bar{\gamma}, \bar{\omega} \times \bar{\gamma} = 0\}$. 

Steps of the proof.
**Proposition.** Let \( s = s(t; s_0) := (\mathbf{v}, \omega_\infty, \gamma) \) be a weak solution corresponding to initial data, \( s_0 \in \mathcal{H} \). Then,

a) \( \Omega(s) \) is non-empty, compact, and connected,
b) \( \Omega(s) \) is positively invariant in the class of weak solutions. Moreover,
c) \( \Omega(s) \subset \{ (\bar{\mathbf{v}}, \bar{\omega}, \bar{\gamma}) \in \mathcal{H} : \bar{\mathbf{v}} \equiv \mathbf{0}, \bar{\omega} \times \mathbf{I} \cdot \bar{\omega} = \beta^2 \mathbf{e}_1 \times \bar{\gamma}, \bar{\omega} \times \bar{\gamma} = \mathbf{0} \} \).

**Steps of the proof.**

**Step 1.**

The non-emptiness, the compactness and the connectedness of \( \Omega(s) \) follows from the following facts:

- weak solutions corresponding to initial data of finite kinetic energy exist;
- weak solutions satisfy the strong energy inequality;
- \( \omega_\infty, \gamma \in C([0, \infty)) \) and \( \mathbf{v} \to 0 \) as \( t \to \infty \).
**Proposition.** Let \( s = s(t; s_0) := (v, \omega_\infty, \gamma) \) be a weak solution corresponding to initial data, \( s_0 \in \mathcal{H} \). Then,

a) \( \Omega(s) \) is non-empty, compact, and connected,

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c) \( \Omega(s) \subset \{ (\bar{v}, \bar{\omega}, \bar{\gamma}) \in \mathcal{H} : \bar{v} \equiv 0, \bar{\omega} \times I \cdot \bar{\omega} = \beta^2 e_1 \times \bar{\gamma}, \bar{\omega} \times \bar{\gamma} = 0 \} \).

**Steps of the proof.**

**Lemma (Step 2.)**

1. *s is unique in the class of weak solutions in \([t_0, \infty)\);*

2. *s depends continuously upon the data in \([t_0, \infty)\), in the class of weak solutions.*

Moreover,

\[
\lim_{t \to \infty} \| \nabla v \|_2 = 0.
\]
Proposition. Let \( s = s(t; s_0) := (v, \omega_\infty, \gamma) \) be a weak solution corresponding to initial data, \( s_0 \in \mathcal{H} \). Then,

a) \( \Omega(s) \) is non-empty, compact, and connected,

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Steps of the proof.

Step 3.

The dynamics on \( \Omega(s) \) is governed by

\[
\begin{align*}
    v & \equiv 0, \\
    \rho \int_C (\dot{\omega}_\infty \times y) \cdot \psi = 0 \quad & \text{for all } \psi \in D^{1,2}_0(C), \\
    I \cdot \dot{\omega}_\infty + \omega_\infty \times I \cdot \omega_\infty = \beta^2 e_1 \times \gamma, \\
    \dot{\gamma} + \omega_\infty \times \gamma = 0.
\end{align*}
\]
Large-time properties of weak solutions (continued)

**Proposition.** Let \( s = s(t; s_0) := (v, \omega_\infty, \gamma) \) be a weak solution corresponding to initial data, \( s_0 \in \mathcal{H} \). Then,

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c) \( \Omega(s) \subset \{(\bar{v}, \bar{\omega}, \bar{\gamma}) \in \mathcal{H} : \bar{v} \equiv 0, \bar{\omega} \times I \cdot \bar{\omega} = \beta^2 e_1 \times \bar{\gamma}, \bar{\omega} \times \bar{\gamma} = 0\} \).

**Steps of the proof.**

**Step 3.**

The dynamics on \( \Omega(s) \) is governed by

\[
\begin{align*}
v & \equiv 0, \quad \dot{\omega}_\infty \times x = \nabla \varphi, \\
I \cdot \dot{\omega}_\infty + \omega_\infty \times I \cdot \omega_\infty & = \beta^2 e_1 \times \gamma, \\
\dot{\gamma} + \omega_\infty \times \gamma & = 0.
\end{align*}
\]
Proposition. Let $s = s(t; s_0) := (v, \omega_\infty, \gamma)$ be a weak solution corresponding to initial data, $s_0 \in \mathcal{H}$. Then,

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c) $\Omega(s) \subset \{(\bar{v}, \bar{\omega}, \bar{\gamma}) \in \mathcal{H} : \bar{v} \equiv 0, \quad \bar{\omega} \times I \cdot \bar{\omega} = \beta^2 e_1 \times \bar{\gamma}, \quad \bar{\omega} \times \bar{\gamma} = 0\}$.

Steps of the proof.

Step 3.

The dynamics on $\Omega(s)$ is governed by

$$v \equiv 0, \quad \dot{\omega}_\infty \times x = \nabla \varphi \Rightarrow \dot{\omega}_\infty = 0,$$

$$I \cdot \dot{\omega}_\infty + \omega_\infty \times I \cdot \omega_\infty = \beta^2 e_1 \times \gamma,$$

$$\dot{\gamma} + \omega_\infty \times \gamma = 0.$$
Proposition. Let \( s = s(t; s_0) := (v, \omega_\infty, \gamma) \) be a weak solution corresponding to initial data, \( s_0 \in \mathcal{H} \). Then,

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Steps of the proof.

Step 3.

The dynamics on \( \Omega(s) \) is governed by

\[
\begin{align*}
v & \equiv 0, \quad \dot{\omega}_\infty = 0, \\
\dot{I} \cdot \dot{\omega}_\infty + \omega_\infty \times I \cdot \omega_\infty & = \beta^2 e_1 \times \gamma & \Rightarrow e_1 \times \dot{\gamma} & = 0, \quad (\omega_\infty \times \gamma) \cdot e_1 = 0, \\
\dot{\gamma} + \omega_\infty \times \gamma & = 0.
\end{align*}
\]
**Proposition.** Let $s = s(t; s_0) := (v, \omega_\infty, \gamma)$ be a weak solution corresponding to initial data, $s_0 \in \mathcal{H}$. Then,

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**Steps of the proof.**

**Step 3.**

The dynamics on $\Omega(s)$ is governed by

\[
\begin{align*}
v & \equiv 0, \quad \dot{\omega}_\infty = 0, \\
I \cdot \dot{\omega}_{\infty} + \omega_\infty \times I \cdot \omega_\infty & = \beta^2 e_1 \times \gamma \quad \Rightarrow e_1 \times \dot{\gamma} = 0, \quad (\omega_\infty \times \gamma) \cdot e_1 = 0, \\
\dot{\gamma} + \omega_\infty \times \gamma & = 0 \quad \Rightarrow e_1 \cdot \dot{\gamma} = 0.
\end{align*}
\]
**Proposition.** Let \( s = s(t; s_0) := (v, \omega_\infty, \gamma) \) be a weak solution corresponding to initial data, \( s_0 \in \mathcal{H} \). Then,

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**Steps of the proof.**

**Step 3.**

The dynamics on \( \Omega(s) \) is governed by

\[
\begin{align*}
v &\equiv 0, \quad \dot{\omega}_\infty = 0, \\
\omega_\infty \times I \cdot \omega_\infty &= \beta^2 e_1 \times \gamma, \\
0 &+ \omega_\infty \times \gamma = 0.
\end{align*}
\]
Motion of a liquid-filled pendulum

Reference:

G. P. Galdi and G. Mazzone,
Consider the coupled system constituted by a rigid body, $B$, with an interior cavity, $C$, entirely filled with a Navier-Stokes liquid, satisfying the following properties:

- $B$ is constrained to move **without friction** about a fixed axis, $a$,
Consider the coupled system constituted by a rigid body, $\mathcal{B}$, with an interior cavity, $\mathcal{C}$, entirely filled with a Navier-Stokes liquid, satisfying the following properties:

- $\mathcal{B}$ is constrained to move **without friction** about a fixed axis, $\mathbf{a}$,

- the **angular velocity** is: $\omega(t)e_3$;

- $\gamma = (\cos \varphi, -\sin \varphi, 0)$. 

Asymptotic Behavior

Theorem

Let \((v, \omega, \gamma)\) be a weak solution corresponding to initial data of finite energy. Then,

\[
\lim_{t \to \infty} \left( \|v(t)\|_{2,2} + \left\| \frac{\partial v(t)}{\partial t} \right\|_{2} \right) = 0, \quad \text{and} \quad \lim_{t \to \infty} (\max_{x \in C} |v(x, t)|) = 0,
\]

\[
\lim_{t \to \infty} |\omega(t)| = 0, \quad \lim_{t \to \infty} |\gamma(t) - \alpha e_1| = 0, \quad \text{with } \alpha = 1 \text{ or } \alpha = -1.
\]
Asymptotic Behavior

Theorem

Let \((v, \omega, \gamma)\) be a weak solution corresponding to initial data of finite energy. Then,

\[ \lim_{t \to \infty} \left( \left\| v(t) \right\|_{2,2} + \left\| \frac{\partial v(t)}{\partial t} \right\|_{2} \right) = 0, \quad \text{and} \quad \lim_{t \to \infty} \left( \max_{x \in \Omega} |v(x, t)| \right) = 0, \]

\[ \lim_{t \to \infty} |\omega(t)| = 0, \quad \lim_{t \to \infty} |\gamma(t) - \alpha e_1| = 0, \quad \text{with} \ \alpha = 1 \ \text{or} \ \alpha = -1. \]
Theorem (Attainability)

Let $C$ be of class $C^2$, and let $(v_0, \omega_0, \gamma_0) \in H(C) \times \mathbb{R} \times S^1$ be given with

$$\rho \|v_0\|_2^2 + C (\omega_0 - a(0))^2 < 2C \beta^2 (1 + \gamma_{1,0}).$$

Then all weak solutions corresponding to $(v_0, \omega_0, \gamma_0)$ tend $(v \equiv 0, \omega \equiv 0, \gamma \equiv e_1)$. 

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Attainability & Stability

**Theorem (Attainability)**

Let $C$ be of class $C^2$, and let $(v_0, ω_0, γ_0) ∈ H(C) × ℝ × S^1$ be given with

$$ρ \|v_0\|^2 + C (ω_0 − a(0))^2 < 2 C β^2 (1 + γ_{1,0}).$$

Then all weak solutions corresponding to $(v_0, ω_0, γ_0)$ tend $(v ≡ 0, ω ≡ 0, γ ≡ e_1)$.

**Theorem (Stability)**

- $(v ≡ 0, ω ≡ 0, γ ≡ e_1)$ is stable in the sense of Lyapunov.
- $(v ≡ 0, ω ≡ 0, γ ≡ −e_1)$ is unstable in the sense of Lyapunov.
Theorem (Attainability)

Let $C$ be of class $C^2$, and let $(v_0, \omega_0, \gamma_0) \in H(C) \times \mathbb{R} \times S^1$ be given with

$$\rho \|v_0\|^2_2 + C (\omega_0 - a(0))^2 < 2C \beta^2 (1 + \gamma_{1,0}).$$

Then all weak solutions corresponding to $(v_0, \omega_0, \gamma_0)$ tend $(v \equiv 0, \omega \equiv 0, \gamma \equiv e_1)$.

Stability results:
THANK YOU!