

# MATH 2090 NUMERICAL SOLUTIONS OF ORDINARY DIFFERENTIAL EQUATIONS

COMPUTATIONAL PROJECT 4, DUE NOVEMBER 11, 2016

1. Consider a simple pendulum problem, which is given by two coupled nonlinear equations (see [3, 2])

$$\begin{aligned}\frac{d\theta}{dt} &= \frac{v}{L}, \\ \frac{dv}{dt} &= -g \sin \theta,\end{aligned}\tag{SP}$$

where  $\theta, v, L$  and  $g$  denote, respectively, angular displacement, velocity along the arc, length of the pendulum, and the acceleration due to gravity.

Set  $g = 9.8$  and  $L = 49$  to observe the long-time behavior of the numerical solutions.

Choose the initial condition  $(\theta_0, v_0) = (0.9\pi, 0)$  at  $t = 0$  and the time step  $\Delta t = 0.1$ , and then numerically integrate the system using LF-RA ( $\nu = 0.8$ ), LF-RAW ( $\alpha = 0.53, \nu = 0.8$ ), LF-hoRA ( $\beta = 0.4$ ) and AB3 schemes over the time interval  $[0, 200]$ .

Plot the numerical LF-RA, LF-RAW, LF-hoRA and AB3( $\theta$  and  $v$ ) solutions versus time.

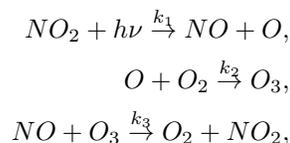
2. (bonus) Consider the Lorenz system:

$$\begin{aligned}\frac{dX}{dt} &= \sigma(Y - X), \\ \frac{dY}{dt} &= -XZ + rX - Y, \\ \frac{dZ}{dt} &= XY - bZ.\end{aligned}\tag{Lorenz}$$

Choose  $\sigma = 12, r = 12, b = 6$ , and the initial condition  $(X_0, Y_0, Z_0) = (-10, -10, 25)$  at  $t = 0$  (see e.g. [1]). Integrate numerically the system over the time interval  $[0, 2.5]$  using LF-RA, LF-RAW, LF-hoRA and AB3 schemes, with time step  $\Delta t = 0.025$  and all other filter-parameters exactly the same as in the previous test.

Plot the  $X$  numerical solutions versus  $t$ .

3. (bonus) Consider an example of reactions between atomic oxygen (O), nitrogen oxides (NO and NO<sub>2</sub>), and ozone (O<sub>3</sub>) (see [1] for more details):



where  $h\nu$  denotes a photon of solar radiation. Let  $c = (c_1, c_2, c_3, c_4)$  represent the concentration in molecules per cubic centimeter of O, NO, NO<sub>2</sub> and O<sub>3</sub>,

respectively. Assuming that the background concentration of  $O_2$  is constant, the reactions are governed by the following system:

$$\begin{aligned}\frac{dc_1}{dt} &= k_1 c_3 - k_2 c_1, \\ \frac{dc_2}{dt} &= k_1 c_3 - k_3 c_2 c_4, \\ \frac{dc_3}{dt} &= k_3 c_2 c_4 - k_1 c_3, \\ \frac{dc_4}{dt} &= k_2 c_1 - k_3 c_2 c_4.\end{aligned}$$

Let

$$\begin{aligned}k_1 &= 10^{-2} \max\{0, \sin(2\pi t/t_d)\} s^{-1}, \\ k_2 &= 10^{-2} s^{-1}, \quad k_3 = 10^{-16} \text{cm}^3 \text{molecule}^{-1} s^{-1},\end{aligned}$$

where  $t_d$  is the length of 1 day in seconds. With initial condition  $c_0 = (0, 0, 5 \times 10^{11}, 8 \times 10^{11})$  molecules  $\text{cm}^{-3}$  at  $t = 0$ , compute the numerical solution using LF-hoRA scheme for  $\beta = 0.4$  and time step  $\Delta t = 40$  s. Plot the LF-hoRA numerical solutions for O, NO,  $\text{NO}_2$ ,  $\text{NO}_3$  versus time.

#### REFERENCES

- [1] D. R. DURRAN, *Numerical methods for fluid dynamics*, vol. 32 of Texts in Applied Mathematics, Springer, New York, second ed., 2010. With applications to geophysics.
- [2] Y. LI AND C. TRENCHIA, *A higher-order Robert–Asselin type time filter*, J. Comput. Phys., 259 (2014), pp. 23–32.
- [3] P. D. WILLIAMS, *A proposed modification to the Robert–Asselin time filter*, Mon. Wea. Rev., 137 (2009), pp. 2538–2546.