Solutions for homework 9

1 Section 5.4 Using the Laplace Transform to solve Differential Equations

7. *Use the Laplace transform to solve the first-order initial value problem*

$$y' + 8y = e^{-2t}\sin t, \qquad y(0) = 0.$$

We compute the Laplace tranform of the LHS using the linearity of the LT, formula (2.2) and the initial condition:

$$\mathcal{L}(y' + 8y)(s) = \mathcal{L}(y' + 8y)(s) = \mathcal{L}(y')(s) + 8\mathcal{L}(y)(s) = \underbrace{s\mathcal{L}(y)(s) - y(0)}_{=\mathcal{L}(y')(s)} + 8\mathcal{L}(y)(s)$$

$$= (s+8)\mathcal{L}(y)(s) - \underbrace{y(0)}_{=0} = (s+8)\mathcal{L}(y)(s).$$

The Laplace transform of the RHS, by Proposition 2.12 or formula 6 in the Table 1 (with $a = -2, b = 1$), writes:

$$\mathcal{L}\{e^{-2t}\sin t\}(s) = \frac{1}{(s+2)^2 + 1},$$

and therefore the initial value problem reduces to the algebraic equation:

$$(s+8)\mathcal{L}(y)(s) = \frac{1}{(s+2)^2 + 1},$$

i.e.,

$$\mathcal{L}(y)(s) = \frac{1}{\big((s+2)^2 + 1\big)(s+8)}.$$

Now we use partial fractions

$$\frac{1}{\big((s+2)^2 + 1\big)(s+8)} = \frac{A}{s+8} + \frac{Bs + C}{(s+2)^2 + 1} = \frac{A[(s+2)^2 + 1] + (Bs + C)(s+8)}{\big((s+2)^2 + 1\big)(s+8)}$$

in order to find coefficients A, B, C such that:

$$1 = A[(s+2)^2 + 1] + (Bs + C)(s+8).$$

With the **substitution method**:

$$s = -8 \Rightarrow A = \frac{1}{37}$$

and

$$s = -2 + i \Rightarrow 1 = \big(B(-2+i) + C\big)(6+i)$$

$$
\begin{aligned}
&= (-2B + C + iB)(6 + i) \\
&= -12B + 6C - B + i(-2B + C + 6B) \\
&= -13B + 6C + i(4B + C)
\end{aligned}
$$

equivalently

$$
\begin{cases} -13B & + 6C & = 1 \\ \quad 4B & + \quad C & = 0 \end{cases}, \qquad B = -\frac{1}{37}, C = \frac{4}{37}.
$$

So the Laplace transfrom is

$$
\begin{aligned}
\mathcal{L}(y)(s) &= \frac{1}{37(s+8)} + \frac{-s+4}{37\big((s+2)^2+1\big)} \\
&= \frac{1}{37}\frac{1}{s+8} + \frac{1}{37}\frac{-(s+2)+6}{(s+2)^2+1} \\
&= \frac{1}{37}\frac{1}{s+8} - \frac{1}{37}\frac{s+2}{(s+2)^2+1} + \frac{6}{37}\frac{1}{(s+2)^2+1}
\end{aligned}
$$

and using again Table 1 for the inverse Laplace transform we obtain:

$$
y(t) = \frac{1}{37}e^{-8t} - \frac{1}{37}e^{-2t}\cos t + \frac{6}{37}e^{-2t}\sin t.
$$

11. *Use the Laplace transform to solve the second-order initial value problem*

$$
y'' - 4y = e^{-t}, \qquad y(0) = -1, y'(0) = 0.
$$

The Laplace transform of LHS, using (2.5) is:

$$
\begin{aligned}
\mathcal{L}(y'' - 4y)(s) = \mathcal{L}(y'') - 4\mathcal{L}(y)(s) &= s^2\mathcal{L}(y)(s) - s\underbrace{y(0)}_{-1} - \underbrace{y'(0)}_{0} - 4\mathcal{L}(y)(s) \\
&= (s^2 - 4)\mathcal{L}(y)(s) + s.
\end{aligned}
$$

The Laplace transform of LHS, using Table 1 is:

$$
\mathcal{L}\{e^{-t}\}(s) = \frac{1}{s+1}.
$$

Therefore

$$
\mathcal{L}(y)(s) = \frac{1}{(s+1)(s^2-4)} - \frac{s}{s^2-4} = \frac{-s^2 - s + 1}{(s+1)(s-2)(s+2)} = \frac{A}{s+1} + \frac{B}{s-2} + \frac{C}{s+2}.
$$

Using the substitution method:

$$
A(s-2)(s+2) + B(s+1)(s+2) + C(s+1)(s-2) = -s^2 - s + 1
$$

yields

$$
A = -\frac{1}{3}, \quad B = -\frac{5}{12}, \quad C = -\frac{1}{4}.
$$

Using Table 1 to get the inverse Laplace transform from

$$\mathcal{L}(y)(s) = -\frac{1}{3}\frac{1}{s+1} + \frac{1}{12}\frac{1}{s-2} - \frac{1}{4}\frac{1}{s+2}$$

yields

$$y(t) = -\frac{1}{3}e^{-t} - \frac{5}{12}e^{2t} - \frac{1}{4}e^{-2t}.$$

21. *Use the Laplace transform to solve the second-order initial value problem*

$$y'' - y' - 2y = e^{2t}, \qquad y(0) = -1, y'(0) = 0.$$

The Laplace transform of LHS, using (2.2), (2.5) is:

$$\begin{aligned}\mathcal{L}(y'' - y' - 2y)(s) &= \mathcal{L}(y'') - \mathcal{L}(y')(s) - 2\mathcal{L}(y)(s)\\ &= s^2\mathcal{L}(y)(s) - s\underbrace{y(0)}_{=-1} - \underbrace{y'(0)}_{=0} - \Big(s\mathcal{L}(y)(s) - \underbrace{y(0)}_{=-1}\Big) - 2\mathcal{L}(y)(s)\\ &= (s^2 - s - 2)\mathcal{L}(y)(s) + s - 1\\ &= (s+1)(s-2)\mathcal{L}(y)(s) + s - 1.\end{aligned}$$

The Laplace transform of LHS, using Table 1 is:

$$\mathcal{L}\{e^{2t}\}(s) = \frac{1}{s-2}.$$

Therefore

$$\begin{aligned}\mathcal{L}(y)(s) &= \frac{1}{(s+1)(s-2)^2} - \frac{s-1}{(s+1)(s-2)} = \frac{1-(s-1)(s-2)}{(s+1)(s-2)^2}\\ &= \frac{-s^2+3s-1}{(s+1)(s-2)^2} = \frac{A}{s+1} + \frac{Bs+C}{(s-2)^2}.\end{aligned}$$

Using the substitution method:

$$A(s-2)^2 + Bs(s+1) + C(s+1) = -s^2 + 3s - 1$$

yields

$$A = -\frac{5}{9}, \quad B = -\frac{4}{9}, \quad C = \frac{11}{9}.$$

Using Table 1 to get the inverse Laplace transform from

$$\begin{aligned}\mathcal{L}(y)(s) &= -\frac{5}{9}\frac{1}{s+1} + \frac{1}{9}\frac{-4s+11}{(s-2)^2} = -\frac{5}{9}\frac{1}{s+1} + \frac{1}{9}\frac{-4(s-2)+3}{(s-2)^2}\\ &= -\frac{5}{9}\frac{1}{s+1} - \frac{4}{9}\frac{1}{s-2} + \frac{1}{3}\frac{1}{(s-2)^2}\end{aligned}$$

yields

$$y(t) = -\frac{5}{9}e^{-t} - \frac{4}{9}e^{2t} + \frac{1}{3}te^{2t}.$$

2 Section 5.5 Discontinuous forcing terms

1. *Use Proposition 5.6 to find the Laplace transform of*

$$H(t-2)(t-2).$$

Recall first Proposition 5.6:
If $f(t)$ is piecewise continuos of exponential order, and $F(s)$ is the Laplace transform of f. Then, for $c \geq 0$, the Laplace transform of $H(t-c)f(t-c)$ is given by

$$\mathcal{L}\{H(t-c)f(t-c)\}(s) = e^{-cs}F(s).$$

With $f(t) = t$, $F(s) = \frac{1}{s^2}$, and $c = 2$, we have

$$\mathcal{L}\{H(t-2)(t-2)\}(s) = e^{-2s}\frac{1}{s^2}.$$

3. *Use Proposition 5.6 to find the Laplace transform of*

$$H(t-\frac{\pi}{4})\sin 3(t-\frac{\pi}{4}).$$

With $f(t) = \sin 3t$, $F(s) = \frac{3}{s^2+9}$, and $c = \frac{\pi}{4}$, we have

$$\mathcal{L}\{H(t-\frac{\pi}{4})\sin 3(t-\frac{\pi}{4})\} = e^{-\frac{\pi}{4}s}\frac{3}{s^2+9}.$$

11. *Use Heaviside function to redefine the piecewise function*

$$f(t) = \begin{cases} 5, & \text{if } 2 \leq t < 4; \\ 0, & \text{otherwise.} \end{cases}$$

Then use Proposition 5.6 to find its Laplace transform.
First we use the interval function H_{24} to rewrite $f(t)$ as

$$f(t) = 5H_{24}(t),$$

then by formula (5.5) with $a = 2, b = 4$ we find the Laplace transform

$$\mathcal{L}\{f(t)\}(s) = \mathcal{L}\{5H_{24}(t)\}(s) = 5\frac{e^{-2s} - e^{-4s}}{s}.$$

17. *Find the inverse Laplace transform of function*

$$G(s) = \frac{e^{-s}}{s-2}.$$

Create a piecewise definition for your solution that doesn't use the Heaviside function.
By Proposition 5.6, with $F(s) = \frac{1}{s-2}, f(t) = e^{2t}, c = 1$, we have

$$e^{-s}\frac{1}{s-2} = \mathcal{L}\{H(t-1)e^{2(t-1)}\}(s)$$

and therefore

$$\mathcal{L}^{-1}\{e^{-s}\frac{1}{s-2}\}(t) = H(t-1)e^{2(t-1)} = \begin{cases} 0, & \text{if } t < 1 \\ e^{2(t-1)}, & \text{if } t \geq 1 \end{cases} .$$

3 Section 5.6 The Delta Function

2. *Find the unit impulse response to the system*

$$y'' - 4y' + 3y = \delta(t), \qquad y(0) = y'(0) = 0.$$

From Theorem 6.10, we have that

$$\begin{aligned}\mathcal{L}\{e(t)\}(s) &= \frac{1}{s^2 - 4s + 3} = \frac{1}{(s-3)(s-1)} = \frac{1}{2}\left(\frac{1}{s-3} - \frac{1}{s-1}\right) \\ &= \frac{1}{2}\frac{1}{s-3} - \frac{1}{2}\frac{1}{s-1},\end{aligned}$$

and therefore

$$e(t) = \frac{1}{2}e^{3t} - \frac{1}{2}e^{t}.$$

3. *Find the unit impulse response to the system*

$$y'' - 4y' - 5y = \delta(t), \qquad y(0) = y'(0) = 0.$$

From Theorem 6.10, we have that

$$\begin{aligned}\mathcal{L}\{e(t)\}(s) &= \frac{1}{s^2 - 4s - 5} = \frac{1}{(s-5)(s+1)} = \frac{1}{6}\left(\frac{1}{s-5} - \frac{1}{s+1}\right) \\ &= \frac{1}{6}\frac{1}{s-5} - \frac{1}{6}\frac{1}{s+1},\end{aligned}$$

and therefore

$$e(t) = \frac{1}{6}e^{5t} - \frac{1}{6}e^{-t}.$$

5. *Find the unit impulse response to the system*

$$y'' - 9y = \delta(t), \qquad y(0) = y'(0) = 0.$$

From Theorem 6.10, we have that

$$\begin{aligned}\mathcal{L}\{e(t)\}(s) &= \frac{1}{s^2 - 9} = \frac{1}{(s-3)(s+3)} = \frac{1}{6}\left(\frac{1}{s-3} - \frac{1}{s+3}\right) \\ &= \frac{1}{6}\frac{1}{s-3} - \frac{1}{6}\frac{1}{s+3},\end{aligned}$$

and therefore

$$e(t) = \frac{1}{6}e^{3t} - \frac{1}{6}e^{-3t}.$$

7. *Find the unit impulse response to the system*

$$y'' + 2y' + 2y = \delta(t), \qquad y(0) = y'(0) = 0.$$

From Theorem 6.10, we have that

$$\mathcal{L}\{e(t)\}(s) = \frac{1}{s^2 + 2s + 2} = \frac{1}{(s+1)^2 + 1}$$

and therefore by Table 1

$$e(t) = e^{-t} \sin t.$$