Solutions for homework 8

1 Section 5.1 The definition of the Laplace Transform

7. *Use Definition 1.1 of the Laplace transform to find the Laplace transform of each of the following functions defined for $t > 0$.*

$$f(t) = te^{2t}.$$

Using the definition and the integration by parts formula with $u = t$ and $v = -\frac{1}{s-2}e^{-(s-2)t}$, L'Hôpital rule

$$\begin{aligned}
\mathcal{L}\{te^{2t}\}(s) &= \int_0^\infty te^{2t}e^{-st}dt = \lim_{T\to\infty}\int_0^T te^{2t}e^{-st}dt = \lim_{T\to\infty}\int_0^T \underbrace{t}_{=u}\underbrace{e^{-(s-2)t}}_{v'}dt \\
&= \lim_{T\to\infty}\int_0^T \underbrace{t}_{=u}\underbrace{\frac{\left(e^{-(s-2)t}\right)'}{-(s-2)}}_{v'}dt \\
&= \underbrace{\lim_{T\to\infty}\frac{-t}{s-2}e^{-(s-2)t}\Big|_{t=0}^{T}}_{=0} + \lim_{T\to\infty}\frac{1}{s-2}\int_0^T e^{-(s-2)t}dt \\
&= \lim_{T\to\infty}\frac{1}{s-2}\int_0^T e^{-(s-2)t}dt \\
&= \frac{1}{s-2}\lim_{T\to\infty}\frac{e^{-(s-2)t}}{-(s-2)}\Big|_{t=0}^{T} \\
&= \frac{1}{(s-2)^2}.
\end{aligned}$$

13. *Use Definition 1.1 to show that the Laplace transform of the function defined by $f(t) = e^{at}\cos\omega t$ is*

$$F(s) = \frac{s-a}{(s-a)^2+\omega^2}.$$

By definition

$$\mathcal{L}\{e^{at}\cos\omega t\}(s) = \lim_{T\to\infty}\int_0^T e^{at}\cos\omega t\; e^{-st}dt = \lim_{T\to\infty}\int_0^T e^{-(s-a)t}\cos\omega t\, dt$$

and using integration by parts twice we obtain

$$\begin{aligned}
J(s) &= \int_0^T e^{-(s-a)t}\cos\omega t\,dt = \int_0^T \frac{1}{-(s-a)}\left(e^{-(s-a)t}\right)'\cos\omega t\,dt \\
&= \frac{1}{-(s-a)}e^{-(s-a)t}\cos\omega t - \int_0^T \frac{1}{-(s-a)}e^{-(s-a)t}(-\omega\sin\omega t)dt \\
&= \frac{1}{-(s-a)}e^{-(s-a)t}\cos\omega t - \int_0^T \frac{\omega}{s-a}e^{-(s-a)t}\sin\omega t\,dt \\
&= \frac{1}{-(s-a)}e^{-(s-a)t}\cos\omega t - \frac{\omega}{s-a}\int_0^T e^{-(s-a)t}\sin\omega t\,dt \\
&= \frac{1}{-(s-a)}e^{-(s-a)t}\cos\omega t - \frac{\omega}{s-a}\int_0^T \frac{-1}{s-a}\left(e^{-(s-a)t}\right)'\sin\omega t\,dt \\
&= \frac{1}{-(s-a)}e^{-(s-a)t}\cos\omega t + \frac{\omega}{(s-a)^2}\int_0^T \left(e^{-(s-a)t}\right)'\sin\omega t\,dt \\
&= \frac{1}{-(s-a)}e^{-(s-a)t}\cos\omega t + \frac{\omega}{(s-a)^2}\left(e^{-(s-a)t}\sin\omega t - \int_0^T e^{-(s-a)t}\omega\cos\omega t\,dt\right) \\
&= \frac{1}{-(s-a)}e^{-(s-a)t}\cos\omega t + \frac{\omega}{(s-a)^2}e^{-(s-a)t}\sin\omega t - \frac{\omega^2}{(s-a)^2}\int_0^T e^{-(s-a)t}\cos\omega t\,dt \\
&= \frac{1}{-(s-a)}e^{-(s-a)t}\cos\omega t + \frac{\omega}{(s-a)^2}e^{-(s-a)t}\sin\omega t - \frac{\omega^2}{(s-a)^2}J(s)
\end{aligned}$$

and therefore

$$\begin{aligned}
\left(1+\frac{\omega^2}{(s-a)^2}\right)J(s) &= \frac{1}{-(s-a)}e^{-(s-a)t}\cos\omega t + \frac{1}{(s-a)^2}e^{-(s-a)t}\sin\omega t \\
&= \frac{1}{(s-a)^2}e^{-(s-a)t}\left(-(s-a)\cos\omega t + \sin\omega t\right)
\end{aligned}$$

yielding

$$J(s) = \frac{1}{\omega^2+(s-a)^2}e^{-(s-a)t}\left(-(s-a)\cos\omega t + \sin\omega t\right).$$

Therefore, using L'Hôpital's rule, from above we get

$$\begin{aligned}
F(s) &= \frac{1}{\omega^2+(s-a)^2}\lim_{T\to\infty} e^{-(s-a)t}\left(-(s-a)\cos\omega t + \sin\omega t\right)\Big|_{t=0}^{T} \\
&= \frac{1}{\omega^2+(s-a)^2}(s-a).
\end{aligned}$$

15. *Use the Laplace transforms in Table 1 on page 204 to generate the Laplace transform of the function*

$$f(t) = 3.$$

Check the result with the result generated using Definition 1.1 of the Laplace transform.
Using the linearity and the table:

$$\mathcal{L}\{3\}(s) = 3\mathcal{L}\{1\}(s) = 3\frac{1}{s}.$$

Using the definition:

$$\mathcal{L}\{3\}(s) = \lim_{T\to\infty}\int_0^T 3e^{-st}dt = 3\lim_{T\to\infty}\int_0^T e^{-st}dt = \frac{3}{s}.$$

29. *Engineers frequently use the* **Heaviside function**, *defined by*

$$H(t) = \begin{cases} 0, & \text{if } t < 0, \\ 1, & \text{if } t \geq 0, \end{cases}$$

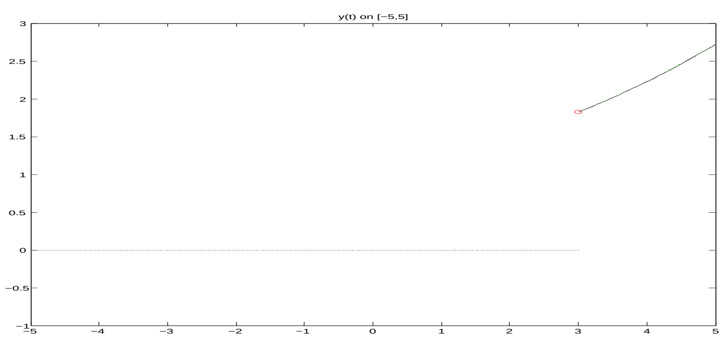
to emulate turning on a switch at a certain instant in time. Sketch the graph of

$$y(t) = H(t-3)e^{0.2t}.$$

Calculate its Laplace transform.

$$y(t) = \begin{cases} 0, & t < 3 \\ e^{0.2t}, & t \geq 3 \end{cases}$$

has the graph on $[-5, 5]$:



and its Laplace transform, for $s > 0$:

$$\begin{aligned}
\mathcal{L}\{y(t)\}(s) &= \int_0^3 0 \cdot e^{-st} dt + \int_3^\infty e^{0.2t} e^{-st} dt \\
&= \lim_{T\to\infty} \int_3^T e^{0.2t} e^{-st} dt = \lim_{T\to\infty} \left. \frac{e^{-(s-0.2)t}}{-(s-0.2)} \right|_3^T \\
&= \frac{e^{-3(s-0.2)}}{s-0.2}.
\end{aligned}$$

5. *Use the linearity of the Laplace transform (Proposition 2.7) and Table 1 of Laplace transforms on page 204 to find the Laplace transform of the function defined on the time domain.*

$$y(t) = -2\cos t + 4\sin 3t.$$

For $s > 0$, from the linearity of the Laplace transform we have:

$$\begin{aligned}
\mathcal{L}\{-2\cos t + 4\sin 3t\}(s) &= -2\mathcal{L}\{\cos t\}(s) + 4\mathcal{L}\{\sin 3t\}(s) \\
&= -2\frac{s}{s^2+1} + 4\frac{3}{s^2+9} \\
&= \frac{-2s(s^2+9) + 12(s^2+1)}{(s^2+1)(s^2+9)} \\
&= \frac{-2s^3 + 12s^2 - 18s + 12}{(s^2+1)(s^2+9)}.
\end{aligned}$$

11. *To test the validity of Proposition 2.1*

(i) compute $\mathcal{L}(y')(s)$ for the given function, and

(ii) compute $s\mathcal{L}(y)(s) - y(0)$ for the given function. Compare this result to that found in part (i) to verify that $\mathcal{L}(y')(s) = s\mathcal{L}(y)(s) - y(0)$.

$$y(t) = e^{-3t}.$$

(i) Using the definition, the Laplace transform of $(e^{-3t})'$ is:

$$\begin{aligned}
\mathcal{L}\{y'(t)\}(s) = \mathcal{L}\{(e^{-3t})'\}(s) = -3\mathcal{L}\{e^{-3t}\}(s) = -3\int_0^\infty e^{-3t}e^{-st}dt \\
= -3\int_0^\infty e^{-(3+s)t}dt = -3\lim_{T\to\infty}\int_0^T e^{-(3+s)t}dt \\
= -3\lim_{T\to\infty} \frac{e^{-(3+s)t}}{-(3+s)}\Bigg|_{t=0}^T = \frac{-3}{3+s}.
\end{aligned}$$

(ii) Using the Laplace transform formula for the exponential we have:

$$\begin{aligned}
s\mathcal{L}(y)(s) - y(0) &= s\mathcal{L}\{e^{-3t}\}(s) - e^{-3\cdot 0} \\
&= s\frac{1}{s-(-3)} - 1 = \frac{s}{s+3} - 1 = \frac{-3}{s+3}.
\end{aligned}$$

19. *Use Propositions 2.1, 2.4 and 2.7 to transform the given initial value problem into an algebraic equation involving $\mathcal{L}(y)$. Solve the resulting equation for the Laplace transform of y.*

$$y' - 5y = e^{-2t}. \qquad y(0) = 1.$$

The Laplace transform of the LHS is (use the linearity of LT, Proposition 2.1, the initial condition):

$$\mathcal{L}(\text{LHS})(s) := \mathcal{L}(y' - 5y)(s) = \mathcal{L}(y')(s) - 5\mathcal{L}(y)(s) = s\mathcal{L}(y)(s) - \underbrace{y(0)}_{=1} - 5\mathcal{L}(y)(s)$$

$$= (s-5)\mathcal{L}(y)(s) - 1$$

The Laplace transform of the RHS is:

$$\mathcal{L}(\text{RHS})(s) := \mathcal{L}\{e^{-2t}\}(s) = \frac{1}{s-(-2)} = \frac{1}{s+2}.$$

With the notation $Y(s) = \mathcal{L}(y)(s)$ we have then:

$$(s-5)Y(s) - 1 = \frac{1}{s+2}$$

and therefore

$$Y(s) = \frac{1}{s-5}\left(1 + \frac{1}{s+2}\right) = \frac{s+3}{(s-5)(s+2)}.$$

29. *Use Proposition 2.12 to find the Laplace transform of the given function*

$$y(t) = e^{-t}(t^2 + 3t + 4).$$

By Proposition 2.12, linearity of LT and formulae (1.8), (1.5), we have

$$\begin{aligned}
\mathcal{L}\{e^{-t}(t^2 + 3t + 4)\}(s) &= \mathcal{L}\{t^2 + 3t + 4\}\big(s - (-1)\big) \\
&= \mathcal{L}\{t^2\}(s+1) + 3\mathcal{L}\{t\}(s+1) + 4\mathcal{L}\{1\}(s+1) \\
&= \frac{2!}{(s+1)^3} + 3\frac{1}{(s+1)^2} + 4\frac{1}{s+1} \\
&= \frac{2 + 3(s+1) + 4(s+1)^2}{(s+1)^3} = \frac{4s^2 + 11s + 9}{(s+1)^3}.
\end{aligned}$$

3 Section 5.3 The Inverse Laplace Transform

3. *Using a table of Laplace transforms, much like using a table of integrals in a calculus class, is not as straightforward as it would seem. Often, one has to make adjustments to the given function in order to match a form in a table. It is the linearity of the Laplace transform and its inverse that makes such adjustments possible. For example, the form* $Y(s) = \frac{1}{2s-3}$ *is not available in Table 1, but if we make the adjustment*

$$Y(s) = \frac{1}{2} \cdot \frac{1}{s - \frac{3}{2}},$$

then by linearity,

$$y(t) = \mathcal{L}^{-1}\left\{\frac{1}{2} \cdot \frac{1}{s - \frac{3}{2}}\right\} = \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{1}{s - \frac{3}{2}}\right\} = \frac{1}{2}e^{\frac{3}{2}t}.$$

Use this technique to find the inverse Laplace transform of the function

$$Y(s) = \frac{1}{s^2 + 4}.$$

We can write

$$Y(s) = \frac{1}{s^2+4} = \frac{1}{s^2+2^2} = \frac{1}{2}\frac{2}{s^2+2^2} = \frac{1}{2}\mathcal{L}\{\sin 2t\}(s)$$

and therefore we have

$$y(t) = \frac{1}{2}\sin 2t.$$

7. *In Exercise 3 we used the fact that*

$$\mathcal{L}^{-1}(\alpha Y) = \alpha\mathcal{L}^{-1}(Y).$$

However, linearity in its more general form demands that

$$\mathcal{L}^{-1}(\alpha X + \beta Y) = \alpha\mathcal{L}^{-1}(X) + \beta\mathcal{L}^{-1}(Y).$$

The form $Y(s) = \frac{2s+5}{s^2+4}$ *is not available in Table 1, but if we make the adjustment*

$$Y(s) = 2 \cdot \frac{s}{s^2+4} + \frac{5}{2} \cdot \frac{2}{s^2+4},$$

then, by linearity,

$$y(t) = 2\mathcal{L}^{-1}\left\{\frac{s}{s^2+4} + \frac{5}{2}\mathcal{L}^{-1}\frac{2}{s^2+4}\right\} = 2\cos 2t + \frac{5}{2}\sin 2t.$$

use this technique to find the inverse Laplace transforms of the function

$$Y(s) = \frac{3s+2}{s^2+25}.$$

Similarly we have

$$\begin{aligned} Y(s) &= \frac{3s+2}{s^2+25} = 3\frac{s}{s^2+25} + 2\frac{1}{s^2+25} = 3\frac{s}{s^2+5^2} + 2\frac{1}{s^2+5^2} \\ &= 3\frac{s}{s^2+5^2} + \frac{2}{5}\frac{5}{s^2+5^2} = 3\mathcal{L}\{\cos 5t\}(s) + \frac{2}{5}\mathcal{L}\{\sin 5t\}(s) \\ &= \mathcal{L}\{3\cos 5t\}(s) + \mathcal{L}\{\frac{2}{5}\sin 5t\}(s) \\ &= \mathcal{L}\{3\cos 5t + \frac{2}{5}\sin 5t\}(s), \end{aligned}$$

and the inverse Laplace transform is:

$$y(t) = 3\cos 5t + \frac{2}{5}\sin 5t.$$

11. *The terminology* **transform pair** *is popular with engineers, and notation such as*

$$y(t) \leftrightarrow Y(s)$$

is used to denote a transform pair. For example $e^{at} \leftrightarrow \frac{1}{s-a}$. *Using this notation, if* $y(t) \leftrightarrow Y(s)$ *is a transform pair, the Proposition 2.12 tells us that* $e^{at}y(t) \leftrightarrow Y(s-a)$ *is a transform pair. For example, because*

$$\cos 2t \leftrightarrow \frac{s}{s^2+4},$$

Prposition 2.12 tells us that

$$e^{3t}\cos 2t \leftrightarrow \frac{s-3}{(s-3)^2+4}.$$

Use this technique to find the inverse Laplace transform of the function

$$Y(s) = \frac{5}{(s+2)^3}.$$

Using the linearity of the inverse Laplace transform, Proposition 2.12, formula (1.8) with $n = 2$, we obtain:

$$\begin{aligned} \mathcal{L}^{-1}\{\frac{5}{(s+2)^3}\}(t) &= 5\mathcal{L}^{-1}\{\frac{1}{(s+2)^3}\}(t) = 5e^{-2t}\mathcal{L}^{-1}\{\frac{1}{s^3}\}(t) = 5e^{-2t}\frac{1}{2}\mathcal{L}^{-1}\{\frac{2}{s^3}\}(t) \\ &= 5e^{-2t}\frac{1}{2}\mathcal{L}^{-1}\{\frac{2}{s^3}\}(t) \\ &= 5e^{-2t}\frac{1}{2}t^2 = \frac{5}{2}t^2e^{-2t}. \end{aligned}$$

19. *Perform the appropriate partial fraction decomposition, and then use the result to find the inverse Laplace transform of*

$$Y(s) = \frac{1}{(s+2)(s-1)}.$$

First we find the partial fraction decomposition:

$$\frac{1}{(s+2)(s-1)} = \frac{A}{s-1} + \frac{B}{s+2} = \frac{A(s+2)+B(s-1)}{(s+2)(s-1)}$$

and using the substitution method, i.e., taking

$$s = -2 \text{ and } s = 1, \text{ respectively}$$

we get:

$$A = \frac{1}{3}, \qquad B = -\frac{1}{3},$$

yielding

$$\frac{1}{(s+2)(s-1)} = \frac{1}{3}\frac{1}{s-1} - \frac{1}{3}\frac{1}{s+2}.$$

Then using the linearity of the inverse Laplace transform, the formula

$$\mathcal{L}\{e^{at}\}(s) = \frac{1}{s-a}$$

with $a = 1$ and $a = -2$, we obtain:

$$\begin{aligned} y(t) &= \mathcal{L}^{-1}\{\frac{1}{3}\frac{1}{s-1} - \frac{1}{3}\frac{1}{s+2}\}(t) \\ &= \frac{1}{3}\mathcal{L}^{-1}\{\frac{1}{s-1}\}(t) - \frac{1}{3}\mathcal{L}^{-1}\{\frac{1}{s+2}\}(t) \\ &= \frac{1}{3}e^{t} - \frac{1}{3}e^{-2t}. \end{aligned}$$