Solutions for homework 7

1 Section 4.7 Forced Harmonic Motion

3. *Plot the given function on an appropriate time interval. Use the technique of Exercise 2 to superimpose the plot of the envelope of the beats in a different line style and/or color.*

$$\cos 10t - \cos 11t.$$

Using the trigonometric identity

$$\cos a - \cos b = -2 \sin \frac{a+b}{2} \sin \frac{a-b}{2},$$

for any two given frequencies ω_1, ω_2, we can write

$$\cos \omega_1 t - \cos \omega_2 t = - \sin \big(\frac{\omega_1 + \omega_2}{2} t\big) \sin \big(\frac{\omega_1 - \omega_2}{2} t\big)$$

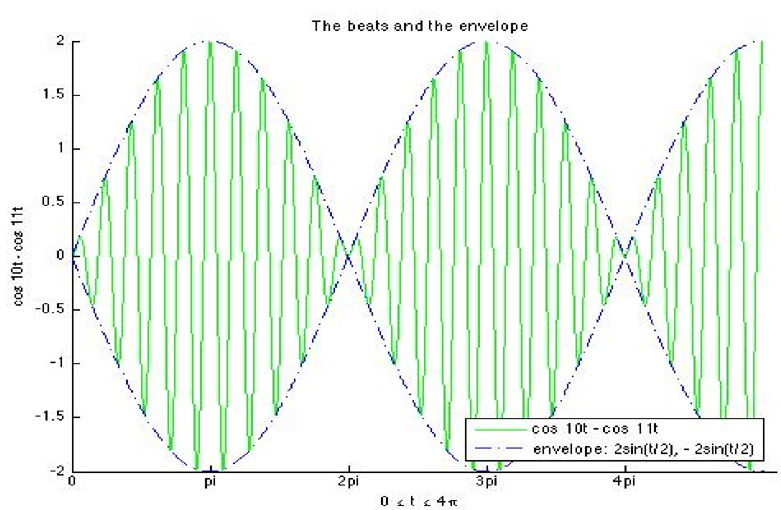
in terms of

$$\textbf{mean frequency}: \quad \overline{\omega} := \frac{\omega_1 + \omega_2}{2},$$

$$\textbf{half difference}: \quad \delta := \frac{\omega_1 - \omega_2}{2}.$$

In our case

$$\cos 10t - \cos 11t = -2 \sin(\frac{21}{2} t) \sin(-\frac{1}{2} t) = 2 \sin \big(\frac{1}{2} t\big) \sin \big(\frac{21}{2} t\big)$$



11. *An inductor (1H) and a capacitor (0.25F) are connected in series with a signal generator that provides an emf* $E(t) = 12\cos\omega t$*. Assume that the system is started from equilibrium (no initial charge on the capacitor, no initial current) and ignore any damping effects.*

(a) Find the current in the system as a function of time. Plot a sample solution assuming that the signal generator provides a driving force at a frequency near the resonant frequency.

(b) Find the current in the system as a function of time, this time assuming that the signal generator provides a driving force at resonant frequency. Plot your solution.

(a) The equation is $E(t) = LI'(t) + \frac{1}{C}Q(t)$, so for the current $E'(t) = LI''(t) + \frac{1}{C}I(t)$, i.e., $I'' + 4I = -12\omega \sin\omega t$.
The solution to the homogeneous equation is $I_h(t) = a\cos 2t + b\sin 2t$.
The particular solution has the form $I_p(t) = \alpha\cos\omega t + \beta\sin\omega t$, where

$$\alpha = 0, \quad \beta = \frac{12\omega}{\omega^2 - 1}.$$

Hence the current is

$$I_{(a)}(t) = I_h(t) + I_p(t) \equiv a\cos 2t + b\sin 2t + \frac{12\omega}{\omega^2 - 4}\sin\omega t.$$

With the initial conditions $I(0) = 0, I'(0) = 0$ we get

$$a = 0, \quad b = -\frac{6\omega^2}{\omega^2 - 4},$$

and therefore

$$I_{(a)}(t) = -\frac{6\omega^2}{\omega^2 - 4}\sin 2t + \frac{12\omega}{\omega^2 - 4}\sin\omega t.$$

(b) With $\omega = 2$, the equation writes $I'' + 4I = -24\sin 2t$, hence a particular solution has the form $I_p(t) = t\big(\gamma\cos 2t + \delta\sin 2t\big)$. Standard calculations yield

$$\delta = 0, \quad \gamma = 6,$$

so

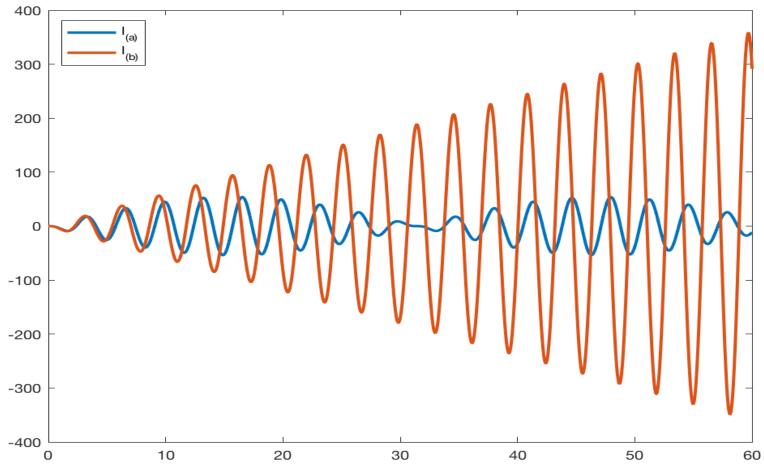
$$I_{(b)}(t) = I_h(t) + I_p(t) \equiv A\cos 2t + B\sin 2t + 6t\cos 2t.$$

The initial conditions $I(0) = 0, I'(0) = 0$ give

$$A = 0, \quad B = -3,$$

hence

$$I_{(b)}(t) = -3\sin 2t + 6t\cos 2t.$$



13. *Place the transfer function in the form*

$$H(i\omega) = \frac{1}{R} e^{-i\phi}.$$

Use this result to find the steady-state solution of the given equation.

$$x'' + 2x' + 2x = 3\sin 4t.$$

Recalling the form of the forced harmonic motion equation

$$x'' + 2cx' + \omega_0^2 = A\cos\omega t,$$

we have

$$\begin{cases} c = 1, & \textbf{(damping constant)} \\ \omega_0 = \sqrt{2}, & \textbf{(natural frequency)} \\ \omega = 4, & \textbf{(forcing frequency)} \\ A = 3. & \textbf{(forcing amplitude)} \end{cases}$$

The associated homogeneous equation is

$$\begin{aligned} x_h'' + 2x_h' + 2x_h &= 0 \quad \text{(homogeneous equation)} \\ \lambda^2 + 2\lambda \;\; + 2 \;\; &= 0 \quad \text{(characteristic equation)} \end{aligned}$$

with gives the characteristic roots

$$\lambda_{1,2} = \frac{-2 \pm \sqrt{2^2 - 4\cdot 2}}{2} = -1 \pm i,$$

and the general solution to the homogeneous equation

$$x_h(t) = e^{-t}\big(c_1 \cos t + c_2 \sin t\big).$$

To find a steady (particular) solution to the forced equation

$$x_p'' + 2x_p' + 2x_p = 3\sin 4t,$$

we shall use the method of undetermined coefficients in complex form. Namely, $x_p(t)$ is the *imaginary* part of $z(t)$:

$$x_p(t) := Im(z(t)), \tag{1.1}$$

where $z(t)$ is a particular solution to the complexified equation:

$$z'' + 2z' + 2z = \underbrace{3e^{i4t}}_{=Ae^{i\omega t}}. \tag{1.2}$$

Therefore (recall the method of undetermined coefficients) we seek $z(t)$ of the form

$$z(t) = \alpha e^{i4t}. \tag{1.3}$$

Substituting in (1.2) gives

$$\big(\underbrace{16i^2\alpha + 2(4i\alpha) + 2\alpha}_{=\alpha P(i\omega)}\big)\underbrace{e^{i4t}}_{=e^{i\omega t}}, = \underbrace{3e^{i4t}}_{=Ae^{i\omega t}},$$

hence

$$\alpha = \frac{1}{P(i\omega)}A \equiv \frac{1}{-14+8i}3, \qquad \text{(we have } P(i\omega) := -14+8i\text{)}$$

and from (1.3)

$$z(t) = \underbrace{\frac{1}{-14+8i}}_{H(i\omega):=\frac{1}{P(i\omega)}} 3e^{i4t}, \qquad \Big(\textbf{transfer function } H(i\omega) := \frac{1}{P(i\omega)}\Big). \tag{1.4}$$

Now

$$P(i\omega) := Re^{i\phi}$$
$$P(i\omega) := -14+8i \Rightarrow \begin{cases} R = \sqrt{14^2+8^2} = \sqrt{260}, \\ \phi = \arctan\big(-\frac{4}{7}\big) + \pi. \end{cases}$$

Then (1.4) yields

$$\begin{aligned} z(t) &= H(i\omega)Ae^{i\omega t} \\ &= \frac{1}{Re^{i\phi}}Ae^{i\omega t} = \frac{A}{R}e^{-i\phi}e^{i\omega t} = \frac{A}{R}e^{i(\omega t-\phi)} \\ &\equiv \frac{3}{\sqrt{260}}e^{i\left(4t+\arctan(\frac{4}{7})-\pi\right)} \end{aligned}$$

and finally from (1.1)

$$\begin{aligned} x_p(t) &= Im\Big\{ \frac{3}{\sqrt{260}} e^{i\left(4t+\arctan(\frac{4}{7})-\pi\right)} \Big\} \\ &= Im\Big\{ \frac{3}{\sqrt{260}} \Big[\cos\left(4t+\arctan(\frac{4}{7})-\pi\right) + i \sin\left(4t+\arctan(\frac{4}{7})-\pi\right) \Big] \Big\} \\ &= \frac{3}{\sqrt{260}} \sin\left(4t+\arctan(\frac{4}{7})-\pi\right) \end{aligned}$$