Solutions for homework 6

1 Section 4.5 Inhomogeneous equations; the method of undetermined coefficients

15. *Use the technique shown in Example 5.17 to find a particular solution for the given differential equation.*

$$y'' + 6y' + 8y = 2t - 3.$$

The RHS is a linear polynomial, so we seek

$$y_p(t) = at + b.$$

Therefore

$$y_p'(t) = a, \qquad y_p''(t) = 0,$$

and substituting in the equation

$$0 + 6 \cdot a + 8 \cdot (at + b) = 2t - 3, \qquad \forall t$$

we have

$$\begin{aligned} &8a = 2, \\ &6a + 8b = -3. \end{aligned}$$

The particular solution is then

$$y_p(t) = \frac{1}{4}t - \frac{9}{16}.$$

19. *Use the technique of Section 4.3 to find a solution to the associated homogeneous equation; then use the technique of this section to find a particular solution. Use Theorem 5.2 to form a general solution. Then find the solution satisfying the given initial condition.*

$$y'' - 4y' - 5y = 4e^{-2t}, \qquad y(0) = 0, \qquad y'(0) = -1.$$

The homogeneous equation

$$y_h'' - 4y_h' - 5y_h = 0$$

gives the characteristic polynomial

$$\lambda^2 - 4\lambda - 5 = 0,$$

with roots

$$\lambda_1 = -1, \qquad \lambda_2 = 5,$$

and therefore

$$y_h(t) = C_1 e^{-t} + C_2 e^{5t}.$$

To find a particular solution, we seek

$$y_p(t) = a e^{-2t},$$

therefore

$$y_p'(t) = -2a e^{-2t}, \qquad y_p''(t) = 4a e^{-2t},$$

and substituting in the equation

$$\big(1 \cdot 4a - 4 \cdot (-2a) - 5 \cdot a\big) = 4e^{-2t}.$$

Then

$$y_p(t) = \frac{4}{7} e^{-2t},$$

and the general solution is

$$y(t) = y_h(t) + y_p(t) \equiv C_1 e^{-t} + C_2 e^{5t} + \frac{4}{7} e^{-2t}.$$

Using the first initial condition

$$y(0) = 0$$

gives

$$C_1 + C_2 = -\frac{4}{7}.$$

For the second initial condition, we need the derivative of the solution

$$y'(t) = -C_1 e^{-t} + 5C_2 e^{5t} - \frac{8}{7} e^{-2t}$$

therefore

$$-C_1 + 5C_2 - \frac{8}{7} = -1, \qquad -C_1 + 5C_2 = \frac{1}{7}.$$

Finally,

$$C_1 = -\frac{1}{2}, \qquad C_2 = -\frac{1}{14},$$

and the solution to the IVP is

$$y(t) = y_h(t) + y_p(t) \equiv -\frac{1}{2} e^{-t} - \frac{1}{14} e^{5t} + \frac{4}{7} e^{-2t}.$$

2 Section 4.6 Variation of parameters

Find a particular solution to the given second-order differential equation.

1. $$y'' + 9y = \tan 3t.$$

We will use the method of variation of parameters. To find the fundamental set of solutions, we use the characteristic equation $\lambda^2 + 9 = 0$ to get

$$y_1(t) = \cos 3t, \qquad y_2(t) = \sin 3t,$$

and note that the Wronskian (the determinant of the linear system in v_1', v_2' is)

$$W(t) = \begin{vmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{vmatrix} \equiv \begin{vmatrix} \cos 3t & 3\sin 3t \\ -\sin 3t & 3\cos 3t \end{vmatrix} = 3.$$

Now we look for a particular solution

$$y_p(t) = v_1(t)y_1(t) + v_2(t)y_2(t),$$

where $v_1(t), v_2(t)$ are to be determined.
The system in v_1', v_2' is

$$\begin{cases} v_1'y_1 & +v_2'y_2 & = 0 \\ v_1'y_1' & +v_2'y_2' & = g(t) \end{cases},$$

i.e., in this case

$$\begin{cases} v_1' \cos 3t & +v_2' \sin 3t = & 0 \\ v_1'(-3\sin 3t) & -v_2'(3\cos 3t) = & \tan 3t \end{cases}$$

and has the solution

$$v_1'(t) = \frac{\begin{vmatrix} 0 & y_2(t) \\ g(t) & y_2'(t) \end{vmatrix}}{W(t)} = \frac{-gy_2}{W} \equiv \frac{\tan 3t \sin 3t}{3} = -\frac{1}{3}(\sec 3t - \cos 3t)$$

$$v_2'(t) = \frac{\begin{vmatrix} y_1(t) & 0 \\ y_1'(t) & g(t) \end{vmatrix}}{W(t)} = \frac{y_1 g}{W} = \frac{\cos 3t \tan 3t}{3} = \frac{1}{3}\sin 3t.$$

Integrating, we get

$$v_1(t) = -\frac{1}{9}\ln|\sec 3t + \tan 3t| + \frac{1}{9}\sin 3t, \qquad v_2(t) = -\frac{1}{9}\cos 3t,$$

and the particular solution

$$y_p(t) = -\frac{1}{9}\cos 3t \ln|\sec 3t + \tan 3t|.$$

3. $$y'' - y = t + 3.$$

We will use the method of variation of parameters. To find the fundamental set of solutions, we use the characteristic equation $\lambda^2 - 1 = 0$ to get

$$y_1(t) = e^t, \qquad y_2(t) = e^{-t},$$

and note that the Wronskian (the determinant of the linear system in v_1', v_2' is)

$$W(t) = \begin{vmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{vmatrix} \equiv \begin{vmatrix} e^t & e^{-t} \\ e^t & -e^{-t} \end{vmatrix} = -2.$$

Now we look for a particular solution

$$y_p(t) = v_1(t)y_1(t) + v_2(t)y_2(t),$$

where $v_1(t), v_2(t)$ are to be determined.
The system in v_1', v_2' is

$$\begin{cases} v_1'y_1 & +v_2'y_2 & = 0 \\ v_1'y_1' & +v_2'y_2' & = g(t) \end{cases},$$

i.e., in this case

$$\begin{cases} v_1'e^t + v_2'e^{-t} = 0 \\ v_1'e^t - v_2'e^{-t} = t + 3 \end{cases}$$

and has the solution

$$v_1'(t) = \frac{\begin{vmatrix} 0 & y_2(t) \\ g(t) & y_2'(t) \end{vmatrix}}{W(t)} = \frac{-gy_2}{W} \equiv \frac{-(t+3)e^{-t}}{-2} = \frac{(t+3)e^{-t}}{2},$$

$$v_2'(t) = \frac{\begin{vmatrix} y_1(t) & 0 \\ y_1'(t) & g(t) \end{vmatrix}}{W(t)} = \frac{y_1 g}{W} = \frac{(t+3)e^t}{-2} = -\frac{(t+3)e^t}{2}.$$

Integrating, we get

$$v_1(t) = -\frac{1}{2}e^{-t}(t+3) - \frac{1}{2}e^{-t} = -\frac{1}{2}e^{-t}(t+4),$$
$$v_2(t) = -\frac{1}{2}e^t(t+3) = -\frac{1}{2}e^t(t+2).$$

Finally,

$$y_p(t) = -(t+3).$$

5. $$y'' - 2y' + y = e^t.$$

The homogeneous equation

$$y'' - 2y' + y = 0$$

has the set of fundamental solutions ($\lambda^2 - 2\lambda + 1 = 0$)

$$y_1(t) = e^t, \qquad y_2(t) = te^t,$$

which are linearly independent

$$W(t) = \begin{vmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{vmatrix} \equiv \begin{vmatrix} e^t & te^t \\ e^t & (t+1)e^t \end{vmatrix} = e^{2t} \neq 0.$$

The particular solution is then

$$y_p(t) = v_1(t)y_1(t) + v_2(t)y_2(t),$$

where $v_1(t), v_2(t)$ are to be determined.
The system in v_1', v_2' is

$$\begin{cases} v_1'y_1 & +v_2'y_2 & = 0 \\ v_1'y_1' & +v_2'y_2' & = g(t) \end{cases},$$

i.e., in this case

$$\begin{cases} v_1'e^t + v_2'\,te^t = 0 \\ v_1'e^t + v_2'(t+1)e^t = e^t \end{cases}$$

and has the solution

$$v_1'(t) = \frac{\begin{vmatrix} 0 & y_2(t) \\ g(t) & y_2'(t) \end{vmatrix}}{W(t)} = \frac{-gy_2}{W} \equiv \frac{-e^t\,te^t}{e^{2t}} = -t,$$

$$v_2'(t) = \frac{\begin{vmatrix} y_1(t) & 0 \\ y_1'(t) & g(t) \end{vmatrix}}{W(t)} = \frac{y_1 g}{W} = \frac{e^t\,e^t}{e^{2t}} = 1.$$

Integrating, we get

$$v_1(t) = -\frac{1}{2}t^2,$$
$$v_2(t) = t.$$

Finally,

$$y_p(t) = -\frac{1}{2}t^2\,e^t + t\,te^t = \frac{1}{2}t^2e^t.$$