Solutions for homework 5

1 Section 4.3 Linear Homogeneous Equations with Constant Coefficients

1. *The following equation has distinct, real, characteristic roots. Find the general solution.*

$$y'' - y' - 2y = 0.$$

The characteristic equation is

$$\lambda^2 - \lambda - 2 = 0,$$

with distinct real roots $\lambda_1 = -1, \lambda_2 = 2$. The solutions $y_1(t) = e^{-t}$ and $y_2(t) = e^{2t}$ form a fundamental set of solutions and the general solution is

$$y(t) = C_1 e^{-t} + C_2 e^{2t}.$$

9. *The following equation has complex characteristic roots. Find the general solution.*

$$y'' + y = 0.$$

The characteristic equation is

$$\lambda^2 + 1 = 0,$$

with complex roots $\lambda_1 = -i, \lambda_2 = i$. They form a fundamental set of solutions, but the real solutions are $y_1(t) = e^{0\times t}\cos t = \cos t$, $y_2(t) = e^{0\times t}\sin t = \sin t$ and the general solution is

$$y(t) = C_1 \cos t + C_2 \sin t.$$

17. *The following equation has repeated, real, characteristic roots. Find the general solution.*

$$y'' - 4y' + 4y = 0.$$

The characteristic equation is

$$\lambda^2 - 4\lambda + 4 = 0$$

which has a double root $\lambda = 2$. Therefore $y_1(t) = e^{2t}$ and $y_2 t = te^{2t}$ form a fundamental set of real solutions, and the general solution is

$$y(t) = C_1 e^{2t} + C_2 te^{2t} = (C_1 + C_2 t)e^{2t}.$$

35. *Find the solution of the initial value problem*

$$y'' + 12y' + 36y = 0, \qquad y(1) = 0, \quad y'(1) = -1.$$

The characteristic equation is

$$\lambda^2 + 12\lambda + 36 = 0$$

with double root $\lambda = -6$. This leads to the fundamental set of solutions $y_1(t) = e^{-6t}$ and $y_2(t) = te^{-6t}$, and the general solution

$$y(t) = (C_1 + tC_2)e^{-6t}. \tag{1.1}$$

Using the initial condition

$$y(1) = 0$$

implies $C_1 = -C_2$.
Differentiating the general solution (1.1) we have

$$y'(t) = C_2 e^{-6t} + (C_1 + tC_2)(-6e^{-6t}) = (C_2 - 6C_1 - 6tC_2)e^{-6t},$$

which by $C_1 = -C_2$ yields

$$y'(t) = (C_2 + 6C_2 - 6tC_2)e^{-6t} = C_2(7 - 6t)e^{-6t}.$$

Using the initial condition on y', i.e., $y'(1) = -1$, we get

$$C_2(7 - 6)e^{-6} = -1,$$

and $C_2 = -e^6$, $C_1 = e^6$. Then the solution to the IVP is

$$y(t) = (e^6 - te^6)e^{-6t} = (1 - t)e^{6(1-t)}.$$

2 Section 4.4 Harmonic Motion

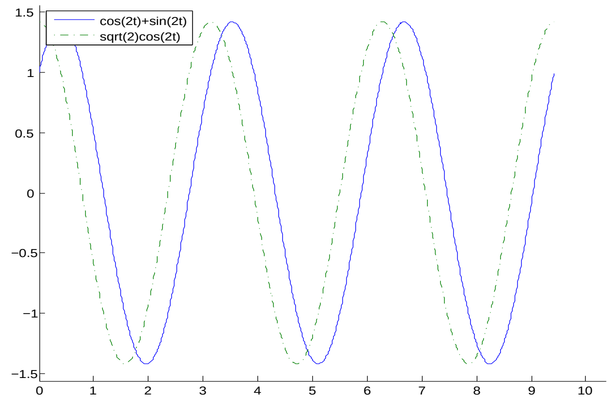
1. In this exercise

(i) use a computer or calculator to plot the graph of the given function, and

(ii) place the solution in the form $y = A\cos(\omega t - \phi)$ and compare the graph of your answer with the plot found in part (i).

$$y = \cos 2t + \sin 2t.$$

(i)



(ii) Let rewrite the function (solution) $y(t)$ in the form

$$y(t) = a \cdot \cos \omega t + b \cdot \sin \omega t \equiv 1 \cdot \cos 2t + 1 \cdot \sin 2t$$

yields $(a = 1, b = 1, \omega = 2)$, hence the **magnitude** A is

$$A := \sqrt{a^2 + b^2} = \sqrt{1^2 + 1^2} = \sqrt{2}$$

and the **polar angle**

$$\phi := \arctan(\tfrac{b}{a}) = \arctan 1 = \frac{\pi}{4}.$$

By factoring out the magnitude and using the trigonometric identity

$$\cos(u - v) = \cos u \cos v + \sin u \sin v,$$

we obtain

$$\begin{aligned} y(t) &= \sqrt{2}\Big(\frac{\sqrt{2}}{2} \cdot \cos 2t + \frac{\sqrt{2}}{2} \cdot \sin 2t\Big) = \sqrt{2}\Big(\cos\phi \cdot \cos 2t + \sin\phi \cdot \sin 2t\Big) \\ &= \sqrt{2}\cos(2t - \phi) = \sqrt{2}\cos(2t - \frac{\pi}{4}) = \sqrt{2}\cos 2(t - \frac{\pi}{8}). \end{aligned}$$

Therefore the solution has amplitude $\sqrt{2}$, period $T = \pi$, and is shifted to the right by $\frac{\pi}{8}$, as seen in the figure.

7. *Place the equation in the form*

$$y = Ae^{-ct} \cos t(\omega t - \phi).$$

Then, on the plot, place the graphs of

$$\begin{aligned} &y = Ae^{-ct} \cos(\omega t - \phi), \\ &y = Ae^{-ct}, \quad \textit{and} \\ &y = -Ae^{-ct}. \end{aligned}$$

For the last two, use a different line style and/or color than for the first.

$$y = e^{-t/2}(\cos 5t + \sin 5t).$$

Again, plot the coefficients (a, b) of

$$\underbrace{1}_{=a} \cdot \cos \underbrace{5}_{=\omega} t + \underbrace{1}_{=b} \cdot \sin 5t$$

in the first quadrant and obtain **magnitude**

$$A = \sqrt{2}$$

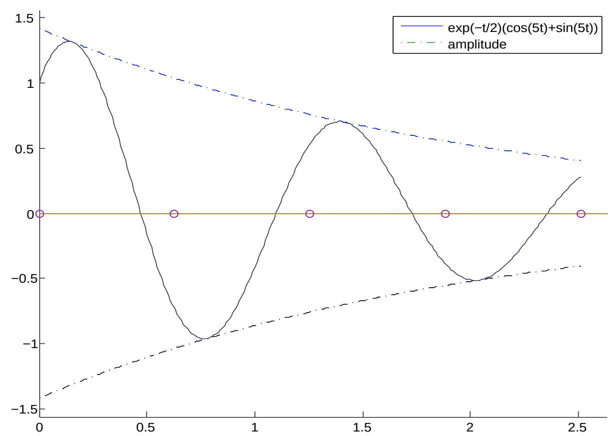
and the **polar angle**

$$\phi = \frac{\pi}{4}.$$

Factoring out the amplitude, we proceed as above

$$\begin{aligned} y(t) &= \sqrt{2}e^{-t/2}\Big(\frac{\sqrt{2}}{2} \cdot \cos 5t + \frac{\sqrt{2}}{2} \cdot \sin 5t\Big) = \sqrt{2}e^{-t/2}\Big(\cos\phi \cdot \cos 5t + \sin\phi \cdot \sin 5t\Big) \\ &= \sqrt{2}e^{-t/2} \cos(5t - \phi) = \sqrt{2}e^{-t/2} \cos(5t - \frac{\pi}{4}) \end{aligned}$$

and the amplitude is $\sqrt{2}e^{-t/2}$.



11. *A 0.2-kg mass is attached to a spring having a spring constant 5kg/s^2. The system is displaced 0.5 m from its equilibrium position and released from rest. If there is no damping present, find the amplitude, frequency, and the phase of the resulting motion. Plot the solution.*

We are given the mass m, spring constant k, the damping constant μ and external force $f(\cdot)$:

$$m = 0.2, \qquad k = 5, \qquad \mu = 0, \qquad f \equiv 0,$$

therefore the vibrating spring equation

$$mx'' + \mu x' + kx = f(t),$$

is in this case

$$0.2x'' + 5x = 0,$$

equivalently

$$x'' + 25x = 0.$$

Hence the general solution is

$$x(t) = a\cos\omega_0 t + b\sin\omega_0 t,$$

where the **natural frequency**

$$\omega_0 = 5.$$

To find the particular solution, we have to identify a, b using the given initial conditions:

$$\begin{aligned} &x(0) = .5, \\ &x'(0) = 0 \qquad (\textit{the spring is released from rest}). \end{aligned}$$

This gives

$$\begin{aligned} x(0) = .5 \qquad &\Rightarrow \qquad a = .5, \\ x'(0) = 0 \qquad &\Rightarrow \qquad b\omega_0 = 0, \quad \text{so } b = 0, \end{aligned}$$

and finally, the **amplitude** is

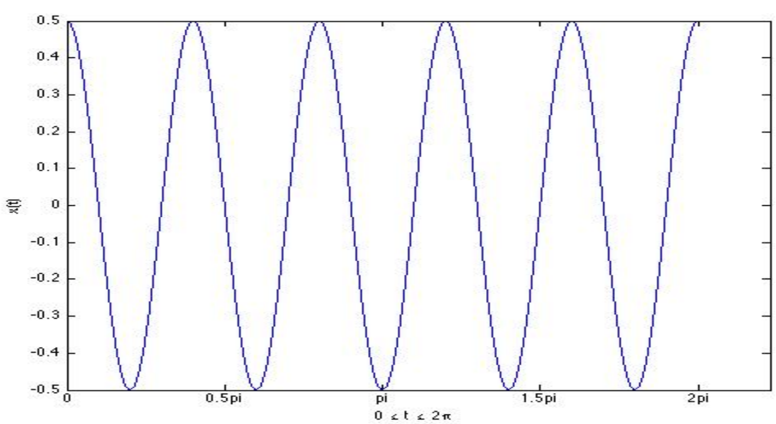
$$A := \sqrt{a^2 + b^2} = .5,$$

while the **phase of the oscillation** is

$$\phi = 0,$$

the solution being

$$x(t) = 0.5\cos 5t.$$



12. *A 0.1-kg mass is attached to a spring having a spring constant 3.6 kg/s^2. The system is allowed to come to rest. Then the mass is given a sharp tap, imparting an instantaneous downward velocity of 0.4 m/s. If there is no damping present, find the amplitude, frequency, and phase of the resulting motion. Plot the solution.*

The spring's equation is

$$x'' + 36x = 0$$

with the solution

$$x(t) = a\cos 6t + b\sin 6t,$$

hence the **natural frequency** is

$$\omega_0 = 6.$$

The initial conditions are

$$x(0) = 0,$$
$$x'(0) = 0.4,$$

hence

$$a = 0, \qquad b = \frac{2}{30},$$

and the **amplitude**

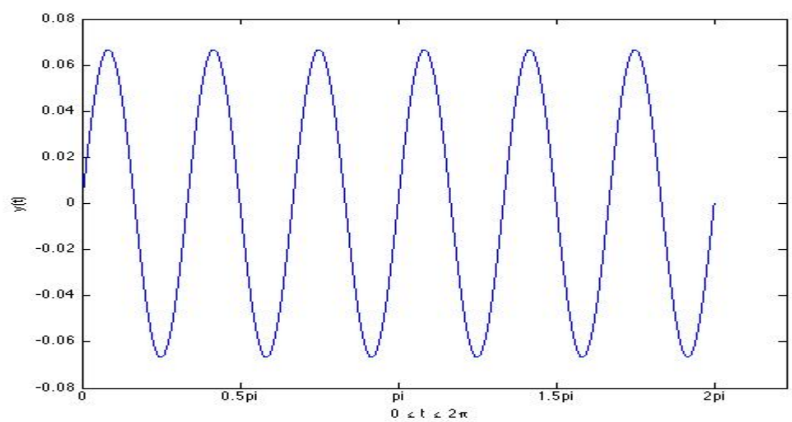
$$A = \frac{2}{30},$$

the solution

$$x(t) = \frac{2}{30}\sin 6t = \frac{2}{30}\cos\left(6t - \tfrac{\pi}{2}\right),$$

and the **phase**

$$\phi = \frac{\pi}{2}.$$



3 Section 4.5 INHOMOGENEOUS EQUATIONS; THE METHOD OF UNDETERMINED COEFFICIENTS

1. *Use the technique demonstrated in Example 5.6 to find a particular solution for the given differential equation.*

$$y'' + 3y' + 2y = 4e^{-3t}.$$

The forcing term is $f = 4e^{-3t}$, hence we look for a particular solution

$$y_p(t) = ae^{-3t},$$

where a is the undetermined coefficient. Consequently

$$y_p'(t) = -3ae^{-3t}, \qquad y_p''(t) = 9ae^{-3t},$$

and inserting these into the LHS of the equation, we obtain

$$\Big(9a + 3 \cdot (-3a) + 2 \cdot a\Big)e^{-3t} = 4e^{-3t},$$

and therefore

$$a = 2 \qquad \Rightarrow \qquad y_p(t) = 2e^{-3t}.$$

5. *Use the form $y_p = a\cos\omega t + b\sin\omega t$, as in Example 5.8, to help find a particular solution for the given differential equation.*

$$y'' + 4y = \cos 3t.$$

We seek a particular solution of the form of the RHS of the equation, namely

$$y_p(t) = a\cos 3t + b\sin 3t.$$

Then

$$y_p'(t) = -3a\sin 3t + 3b\cos 3t, \qquad y_p''(t) = -9a\cos 3t - 9b\sin 3t.$$

Substituting into the LHS of the equation we obtain

$$-9a\cos 3t - 9b\sin 3t + 4 \cdot (a\cos 3t + b\sin 3t) = \cos 3t,$$

and therefore

$$\begin{aligned} -9a + 4a &= 1 \\ -9b + 4b &= 0. \end{aligned}$$

Therefore

$$a = -\frac{1}{5}, \qquad b = 0,$$

and the solution

$$y_p(t) = -\frac{1}{5}\cos 3t.$$

11. *Use the complex method, as in Example 5.12, to find a particular solution for the differential equation.*

$$y'' + 9y = \sin 2t.$$

$y(t)$ is the imaginary part of the function $z(t)$ that is the solution to the equation

$$z'' + 9z = e^{i2t},$$

which has a particular solution of the form

$$z_p(t) = ae^{2it}.$$

This yields

$$z_p'(t) = 2iae^{2it}, \qquad z_p''(t) = -4ae^{2it},$$

and therefore

$$(-4a + 9 \cdot a)e^{2it} = e^{2it},$$

hence

$$a = \frac{1}{5} \qquad \Rightarrow \qquad z_p(t) = \frac{1}{5}e^{2it} = \frac{1}{5}\cos 2t + i\frac{1}{5}\sin 2t,$$

and finally, its imaginary part is our solution

$$y_p(t) = \frac{1}{5}\sin 2t.$$