# Solutions for homework 4

## 2.5. Mixing Problems

**5.** *A 50-gal tank initially contains 20 gal of pure water. Salt-water solution containing 0.5 lb of salt per each gallon of water begins entering the tank at a rate of 4 gal/min. Simultaneously, a drain is opened at the bottom of the tank, allowing the salt-water solution to leave the tank at a rate of 2 gal/min.*

*What is the salt content (lb) in the tank at the precise moment that the tank is full of salt-water solution?*

The rate of change in time of the volume of salt-water solution is

$$\frac{dV}{dt} = \{\text{rate volume in}\} - \{\text{rate volume out}\} \equiv 4 - 2 = 2\text{gal/min},$$

hence the volume at time $t$ is

$$V(t) = 2t + V(0) = 2t + 20 \text{ gal.}$$

Therefore the 50-gallon tank will be full in $T = 15$ min.
With

$$x(t) := \text{amount of salt at time } t \text{ in lb},$$
$$c(t) := \text{concentration of salt in the solution at time } t \equiv \frac{x(t)}{V(t)}\text{lb/gal},$$

the rate of change in the amount of salt satisfies

$$\frac{dx}{dt} = \{\text{rate salt in}\} - \{\text{rate salt out}\}$$

where

$$\{\text{rate salt in}\} = 0.5\text{lb/gal} \times 4\text{gal/min} = 2\text{lb/min}$$
$$\{\text{rate salt out}\} = c(t)\text{lb/gal} \times 2\text{gal/min} = 2\frac{x(t)}{V(t)}\text{lb/min}.$$

Therefore

$$\frac{dx}{dt} = 2 - 2\frac{x(t)}{2t+20} \equiv 2 - \frac{x(t)}{t+10} \qquad \text{(lb/min)}.$$

With the integrating factor

$$\exp\Big(\int \frac{1}{t+10}dt\Big) \equiv t + 10$$

the equation writes

$$(t+10)*x' + x = 2*(t+10), \quad ((t+10)*x)' = 2*(t+10), \quad (t+10)*x = t^2+20t+C.$$

Since

$$x(0) = 0$$

the solution is

$$x(t) = \frac{t^2 + 20t}{t + 10},$$

which yields

$$x(15) = \frac{15^2 + 20 * 15}{15 + 10} = 21 \quad \text{lb.}$$

**9.b** *It has been determined that a concentration of over 2% is hazardous for the fish in the lake. Suppose that $r = 50$ $km^3/yr$, and the initial concentration of pollutant in the lake is zero. How long will take the lake to become hazardous to the health of the fish?*

The equation for the concentration being

$$c' + \frac{p+r}{V}c = \frac{p}{V},$$

in this case means $(r = 50, p = 2, V = 100)$

$$c' + \frac{52}{100}c = \frac{2}{100},$$
$$c' + 0.52c = 0.02.$$

With the integrating factor $e^{0.52t}$, the equation write

$$\left(e^{.52t}c\right)' = 0.02e^{0.52t},$$
$$e^{.52t}c = \frac{0.02}{0.52}e^{0.52t} + K,$$
$$c(t) = \frac{1}{26} + Ke^{-0.52t}.$$

The initial condition

$$c(0) = 0$$

implies

$$c(t) = \frac{1}{26}\left(1 - e^{-0.52t}\right).$$

The lake will become hazardous at time $T$ given by

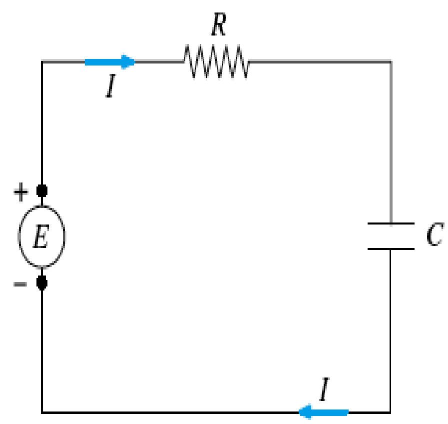
$$0.02 = \frac{1}{26}\left(1 - e^{-0.52T}\right),$$

namely

$$1 - e^{-0.52T} = 0.52, \quad e^{-0.52T} = 0.48, \quad T = -\frac{ln(0.48)}{0.52} = 1.4115 \text{ years.}$$

## 3.4. Electrical Circuits

*A resistor (20 Ω) and capacitor (0.1 F) are joined in series with an electromotive force (emf) $E = E(t)$, as shown in the figure.*




*If there is no charge on the capacitor at time $t = 0$, find the ensuing charge on the capacitor at time $t$ for the given emf in each of the following exercises.*

**1.** $E(t) = 100$ V.

If there is no inductor, the model equation is

$$R\frac{dQ}{dt} + \frac{1}{C}Q = E,$$

namely,

$$20\frac{dQ}{dt} + \frac{1}{0.1}Q = 100, \qquad Q' + \frac{1}{2}Q = 5.$$

Using the integrating factor method or the variation of parameters method, one obtains the solution

$$Q(t) = 10(1 - e^{-\frac{t}{2}}).$$

**3.** $E(t) = 100\sin 2t \ \ V$.

As above, the model equation is

$$R\frac{dQ}{dt} + \frac{1}{C}Q = E,$$

and in this case,

$$Q' + \frac{Q}{2} = 5\sin 2t.$$

Using the integrating factor method or the variation of parameters method, one obtains the solution

$$\begin{aligned} Q(t) &= e^{-\frac{t}{2}}Q_0 + e^{-\frac{t}{2}}\int_0^t e^{\frac{s}{2}}5\sin 2s\,ds \\ &= 5e^{-\frac{t}{2}}\int_0^t e^{\frac{s}{2}}\sin 2s\,ds \\ &= 5e^{-\frac{t}{2}}\mathcal{I}_1, \end{aligned}$$

where (using twice integration by parts one gets)

$$\mathcal{I}_1 = \int_0^t e^{\frac{s}{2}} \sin 2s \, ds = \frac{1}{17} e^{2t} \big(2 \sin 2t - 8 \cos 2t + 8\big).$$

Therefore the solution is

$$Q(t) = \frac{1}{17}(10 \sin 2t - 40 \cos 2t + 40 e^{-\frac{t}{2}}).$$

**5.** $E(t) = 100 - t$ V.
The model equation is

$$Q' + \frac{Q}{2} = 5 - \frac{t}{20}.$$

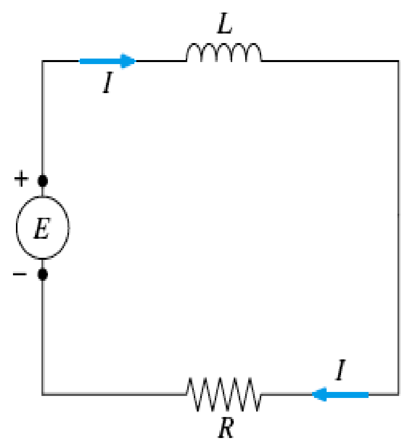
Using variation of parameters and integration by parts we have

$$\begin{aligned}
Q(t) &= e^{-\frac{t}{2}} Q_0 + e^{-\frac{t}{2}} \int_0^t e^{\frac{s}{2}} \big(5 - \frac{s}{20}\big) \, ds \\
&= 5 e^{-\frac{t}{2}} \int_0^t e^{\frac{s}{2}} \, ds - \frac{1}{20} e^{-\frac{t}{2}} \int_0^t e^{\frac{s}{2}} s \, ds \\
&= 10 e^{-\frac{t}{2}} e^{\frac{s}{2}} \Big|_{s=0}^{s=t} - \frac{1}{20} e^{-\frac{t}{2}} \int_0^t 2 \big(e^{\frac{s}{2}}\big)' s \, ds \\
&= 10 e^{-\frac{t}{2}} (e^{\frac{t}{2}} - 1) - \frac{1}{10} e^{-\frac{t}{2}} \Big( (e^{\frac{s}{2}}) s \Big|_{s=0}^{s=t} - \int_0^t e^{\frac{s}{2}} \, ds \Big) \\
&= 10 e^{-\frac{t}{2}} (e^{\frac{t}{2}} - 1) - \frac{1}{10} e^{-\frac{t}{2}} \Big( t e^{\frac{t}{2}} - 2(e^{\frac{t}{2}} - 1) \Big),
\end{aligned}$$

and finally the solution is

$$Q(t) = \frac{51}{5}(1 - e^{-\frac{t}{2}}) - \frac{t}{10}.$$

*An inductor (1 H) and resistor (0.1 $\Omega$) are joined in series with an electromotive force* emf $E = E(t)$*, as shown in the figure.*




*If there is no current in the circuit at time* $t = 0$*, find the ensuing current in the circuit at time t for each given* emf *in each of the following exercises.*

**7.** $E(t) = 1$ V.

The model equation is

$$I' + \frac{I}{10} = 1.$$

Using the variation of parameters and integration by parts, the solution is

$$\begin{aligned} I(t) &= e^{-\frac{1}{10}t} \int_0^t e^{\frac{1}{10}s} ds \\ &= 10e^{-\frac{1}{10}t}\left( e^{\frac{1}{10}s}\Big|_{s=0}^{s=t} \right) = 10e^{-\frac{1}{10}t}(e^{\frac{1}{10}t} - 1) \\ &= 10(1 - e^{-\frac{t}{10}}). \end{aligned}$$

**11.** $E(t) = 10 - 2t$ V.

The model equation is

$$I' + \frac{I}{10} = 10 - 2t.$$

The solution is

$$\begin{aligned} I(t) &= e^{-\frac{1}{10}t} \int_0^t e^{\frac{1}{10}s}(10 - 2s)\, ds \\ &= 10e^{-\frac{1}{10}t} \int_0^t e^{\frac{1}{10}s}\, ds + 2e^{-\frac{1}{10}t} \int_0^t se^{\frac{1}{10}s}\, ds \\ &= 100e^{-\frac{1}{10}t}(e^{\frac{1}{10}t} - 1) + 2e^{-\frac{1}{10}t} \int_0^t s\big(10e^{\frac{1}{10}s}\big)'\, ds \\ &= 100e^{-\frac{1}{10}t}(e^{\frac{1}{10}t} - 1) + 20e^{-\frac{1}{10}t}\left( \big(se^{\frac{1}{10}s}\big)\Big|_{s=0}^{s=t} - \int_0^t e^{\frac{1}{10}s}\, ds \right) \\ &= 100e^{-\frac{1}{10}t}(e^{\frac{1}{10}t} - 1) + 20e^{-\frac{1}{10}t}\Big( te^{\frac{1}{10}t} - 10\big(e^{\frac{1}{10}t} - 1\big) \Big) \\ &= -20t + 300 - 300e^{-\frac{t}{10}}. \end{aligned}$$

## 4.1. Second-Order Equations. Definitions and examples

*For each of the second-order differential equations decide whether the equation is linear or nonlinear. If the equation is linear, state whether the equation is homogeneous or inhomogeneous.*

**1.**

$$y'' + 3y' + 5y = 3\cos 2t$$

Use the definition and compare

$$\begin{array}{ll} & y'' \quad +3y' \qquad +5y \qquad = 3\cos 2t \\ \text{with} & \\ & y'' \quad +p(t)y' \quad +q(t)y \quad = g(t). \end{array}$$

Hence

$$p(t) = 3, \quad q(t) = 5, \quad g(t) = 3\cos(2t),$$

so the equation is linear, nonhomogeneous.

**3.**

$$t^2 y'' + (1-y)y' = \cos 2t$$

Rewrite the equation in normal form

$$y'' + \frac{1-y}{t^2}y' = \frac{\cos 2t}{t^2},$$

observe that

$$p(t) = \frac{1-y}{t^2}, \quad q(t) = 0, \quad g(t) = \frac{\cos 2t}{t^2},$$

and conclude that the equation is nonlinear.

**9.** *In an experiment, a 2-kg mass is suspended from a spring. The displacement of the spring-mass equilibrium from the spring equilibrium is measured to be 50 cm. If the mass is then displaced 12 cm downward from its spring-mass equilibrium and released from rest, set up (but do not solve) the initial value problem that models this experiment. Assume no damping is present.*

The formula for the spring constant is derived from the balance of forces at rest:

$$-ky + mg = 0,$$

hence

$$k = \frac{mg}{y} = \frac{2KG \times 9.8\frac{m}{s^2}}{0.5m} = 39.2\frac{N}{m}.$$

Newton's law is

$$my'' + \mu y' + ky = F(t),$$

so assuming no damping ($\mu = 0$) and no external force ($F(t) = 0$), i.e.,

$$my'' + ky = 0.$$

Therefore the equation writes:

$$2y'' + 39.2y = 0.$$

The initial conditions:

$$\text{initial displacement} : y(0) = 0.12$$
$$\text{initial velocity (released from rest)} : y'(0) = 0.$$

The IVP is

$$\begin{cases} 2y'' + 39.2y = 0. \\ y(0) = 12, y'(0) = 0. \end{cases}$$

**17.** *Use definition 1.22 to explain why $y_1(t)$ and $y_2(t)$ are linearly independent solutions of the given differential equation. In addition, calculate the Wronskian and use it to explain the independence of the given solutions.*

$$y'' - y' - 2y = 0, \qquad y_1(t) = e^{-t}, \quad y_2(t) = e^{2t}$$

First, note that $y_1(t)$ and $y_2(t)$ are solutions by substitution in the equation. Moreover, taking the ratio

$$\frac{y_1(t)}{y_2(t)} = \frac{e^{-t}}{e^{2t}} = e^{-3t} \neq C,$$

by the definition concluded that $y_1, y_2$ are linearly independent.
Since the Wronskian is nonzero

$$W(t) = \begin{vmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{vmatrix} = \begin{vmatrix} e^{-t} & e^{2t} \\ -e^{-t} & 2e^{2t} \end{vmatrix} = 3e^{t} \neq 0,$$

we conclude that $y_1(t)$ and $y_2(t)$ are linearly independent, forming a fundamental set of solutions.