Solutions for homework 3

2.2. Solutions to Separable Equations

In Exercises 1-12, find the general solution of the indicated differential equation. If possible, find an explicit solution.

3. $y' = e^{x-y}$.

The independent variable is x and y is the unknown function.
Using $e^{x-y} = \frac{e^x}{e^y}$, we separate the variables and integrate:

$$\frac{dy}{dx} = e^{x-y},$$

$$e^y dy = e^x dx, \qquad \int e^y dy = \int e^x dx,$$

$$e^y = e^x + C.$$

Finally, to find y, we take the natural log of both sides and obtain the general solution

$$y(x) = \ln(e^x + C).$$

5. $y' = xy + y$.

As above, y is the unknown function and the independent variable is x. To separate the variables, we factor out y

$$\frac{dy}{dx} = y(x+1),$$

$$\frac{1}{y}dy = (x+1)dx,$$

and integrate

$$\int \frac{1}{y}dy = \int (x+1)dx,$$

$$\ln|y| = \frac{1}{2}x^2 + x + C.$$

To solve for y, we take the natural log of both sides

$$|y| = e^{\frac{1}{2}x^2+x+C}$$

Using the definition of absolute value and $e^{a+b} = e^a e^b$

$$y(x) = \pm e^C e^{\frac{1}{2}x^2+x}.$$

Defining the constant $D = \pm e^C$, allowed to take both positive and negative values, we can write the general solution as

$$y = De^{\frac{1}{2}x^2+x}.$$

9. $xy' = y\ln y - y'$

The unknown function is y and the independent variable is x. First we need a little algebra, and start by regrouping the terms containing y':

$$x^2 y' = y \ln y - y'$$

$$(x^2+1)y' = y \ln y$$

Now separate the variables

$$\frac{1}{y \ln y} dy = \frac{1}{x^2+1} dx.$$

Let rewrite this as

$$\underbrace{\frac{1}{\ln y}}_{=\frac{1}{u}} \underbrace{\frac{1}{y} dy}_{=du} = \frac{1}{x^2+1} dx. \tag{0.1}$$

Remark that with the change of variables

$$u = \ln y, \tag{0.2}$$

we have

$$du = \frac{1}{y} dy,$$

and equation (0.1) writes

$$\frac{1}{u} du = \frac{1}{x^2+1} dx.$$

Now integrate

$$\int \frac{1}{u} du = \int \frac{1}{x^2+1} dx$$

to obtain

$$\ln |u| = \tan^{-1} x + C.$$

Solve for u:

$$|u(x)| = e^{\tan^{-1} x + C},$$

use the definition of absolute value

$$u(x) = \underbrace{\pm e^C}_{:=D} e^{\tan^{-1} x},$$

define a new constant

$$D := \pm e^C,$$

replace u with $\ln y$ (see (0.2)), and solve for y:

$$\ln y(x) = D e^{\tan^{-1} x},$$

$$y(x) = e^{D\tan^{-1}x}.$$

33. *A murder victim is discovered at midnight and the temperature of the body is recorded at* $31°C$*. One hour later, the temperature of the body is* $29°C$*. Assume that the surrounding air temperature remains constant at* $21°C$*. Use Newton's law of cooling to calculate the victim's time of death.*

`Note: the "normal" temperature of a living human being is approximately` $37°C$.

Let $T(t)$ denote the temperature of the body, and $t = 0$ correspond to midnight. Thus $T(0) \equiv T_0 = 31°C$.

Because the temperature of the surrounding medium is $A = 21°C$, we can use Newton's law of cooling in the form

$$T(t) = A + (T_0 - A)e^{-kt},$$

and write

$$T(t) = 21 + (31 - 21)e^{-kt} = 21 + 10e^{-kt}.$$

At $t = 1, T = 29°C$, which can be used to calculate k. From the relation above we get

$$29 = 21 + 10e^{-k\cdot 1}$$

which yields

$$k = -\ln(0.8)$$

$$k \approx 0.2231$$

Thus the victim's body temperature has the following formula

$$T(t) = 21 + 10e^{-0.2231\cdot t}.$$

To find the time of death, enter "normal" body temperature,

$$T(t_{\text{death}}) = 37°C$$

in the expression above and solve for t_{death}. Namely,

$$37 = 21 + 10e^{-0.2231\cdot t_{\text{death}}}$$

$$t_{\text{death}} = \frac{\ln(1.6)}{-0.2231}$$

$$t_{\text{death}} \approx -2.1067hrs$$

Thus, the murder occurred at approximately 9 : 54 PM.

2.3. Models of Motion

9. *A ball having mass $m = 0.1$ kg falls from rest under the influence of gravity in a medium that provides a resistance that is proportional to its velocity. For a velocity of 0.2 m/s, the force due to the resistance of the medium is -1 N. [One Newton [N] is the force required to accelerate a 1 kg mass at a rate of $1m/s^2$. Hence, 1 N = 1 kg m/s^2.] Find the terminal velocity of the ball.*

The resistance form has opposite sign to that of the velocity and has the form

$$R(v) = -rv,$$

where r is a positive constant. To find r

$$r = -\frac{R(v)}{v}\left[\frac{kg\,m/s^2}{m/s}\right]$$

take

$$v = 0.2\,[m/s], \qquad R(v) = -1\,[N],$$

and so

$$r = -\frac{-1}{0.2} = \frac{1}{0.2} = 5[kg/s].$$

The terminal velocity

$$v_{\text{terminal}} = -\frac{mg}{r}\left[\frac{kg \cdot m/s^2}{kg/s}\right]$$

(the gravitational constant $g = 9.8\,m/s^2$) is then

$$v_{\text{term}} = -\frac{0.1 \cdot 9.8}{5} = -0.196[m/s].$$

2.4. First Order Linear Equations

5. *Find the general solution of the first-order, linear equation.*

$$x' - \frac{2x}{t+1} = (t+1)^2 \tag{0.3}$$

Rewrite the equation in the form

$$x' = a(t)x + f(t)$$

by taking

$$a(t) = \frac{2}{t+1}, \qquad f(t) = (t+1)^2,$$

i.e.,

$$x' = \underbrace{\frac{2}{t+1}}_{=a(t)} x + (t+1)^2.$$

Therefore the integrating factor is

$$u(t) := e^{-\int a(t)dt} \equiv e^{-\int \frac{2}{t+1} dt} = e^{-2\ln|t+1|} = \frac{1}{e^{\ln(|t+1|^2)}} = \frac{1}{(t+1)^2}$$

Multiply both sides of equation (0.3), or equivalently

$$x' = a(t)x + f(t),$$

by the integrating factor to obtain

$$(ux)' = uf,$$

namely

$$\Big(\frac{1}{(t+1)^2}x\Big)' = 1.$$

Finally we integrate

$$\frac{1}{(t+1)^2}x = t + C,$$

and solve for $x(t)$:

$$x(t) = (t+1)^2\big(t + C\big).$$

15. *Find the solution of the initial value problem*

$$(x^2+1)y' + 3xy = 6x, \qquad y(0) = -1.$$

In order to put the equation in the form

$$y' = a(x)y + f(x)$$

we write it as

$$y' = -\frac{3x}{x^2+1}y + \frac{6x}{x^2+1}, \tag{0.4}$$

so

$$a(x) = -\frac{3x}{x^2+1}.$$

The integrating factor is then

$$u(x) := e^{-\int a(x)dx} \equiv e^{\int \frac{3x}{x^2+1}dx} = e^{\frac{3}{2}\int \frac{2x}{x^2+1}dx} = e^{\frac{3}{2}\ln(x^2+1)} = (x^2+1)^{\frac{3}{2}}.$$

Multiply (0.4) by the integrating factor:

$$\begin{aligned}(x^2+1)^{\frac{3}{2}}y' &= -3x(x^2+1)^{\frac{1}{2}}y + 6x(x^2+1)^{\frac{1}{2}},\\ (x^2+1)^{\frac{3}{2}}y' + 3x(x^2+1)^{\frac{1}{2}}y &= 6x(x^2+1)^{\frac{1}{2}},\\ \left((x^2+1)^{\frac{3}{2}}y\right)' &= 6x(x^2+1)^{\frac{1}{2}}.\end{aligned}$$

Now we integrate

$$(x^2+1)^{\frac{3}{2}}y = 2(x^2+1)^{\frac{3}{2}} + C,$$

and solve for y

$$y(x) = 2 + C(x^2+1)^{-\frac{3}{2}}.$$

Finally, using the initial condition

$$y(0) = -1$$

gives

$$-1 = 2 + C, \qquad C = -3,$$

hence the solution to the IVP is

$$y(x) = 2 - 3(x^2+1)^{-\frac{3}{2}}.$$

19. *Find the solution of the initial value problem*

$$(2x+3)y' = y + (2x+3)^{\frac{1}{2}}, \qquad y(-1) = 0.$$

Discuss the interval of existence and provide a sketch of your solution.

From the beginning, notice that the square root function is defined for

$$x \in [-\tfrac{3}{2}, \infty).$$

Rewrite the linear equation

$$y' = \frac{1}{2x+3}y + (2x+3)^{-\frac{1}{2}}, \qquad \forall x \in I := (-\tfrac{3}{2}, \infty),$$

so

$$y' = a(x)y + f(x),$$

where

$$a(x) = \frac{1}{2x+3}, \qquad f(x) = (2x+3)^{-\frac{1}{2}}.$$

The integrating factor is

$$u(x) = e^{-\int \frac{1}{2x+3}dx} = e^{-\frac{1}{2}\ln|2x+3|} = |2x+3|^{-\frac{1}{2}},$$

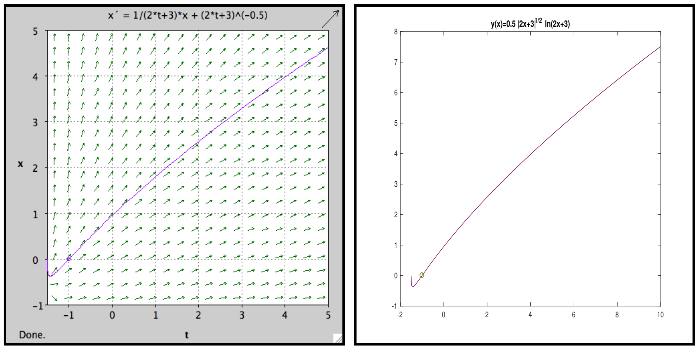

Fig. 0.1: The solution to problem **19**, plotted using dfield and Matlab

Multiply both sides of the equation

$$y' = \frac{1}{2x+3}y + (2x+3)^{-\frac{1}{2}}$$

by the integrating factor to obtain

$$\begin{aligned}|2x+3|^{-\frac{1}{2}}y' &= |2x+3|^{-\frac{1}{2}}\frac{1}{2x+3}y + |2x+3|^{-\frac{1}{2}}(2x+3)^{-\frac{1}{2}},\\ |2x+3|^{-\frac{1}{2}}y' - |2x+3|^{-\frac{3}{2}}y &= |2x+3|^{-1},\\ \left(|2x+3|^{-\frac{1}{2}}y\right)' &= |2x+3|^{-1},\end{aligned}$$

which we integrate

$$|2x+3|^{-\frac{1}{2}}y = \frac{1}{2}\ln|2x+3| + C,$$

and solve for the general solution $y(x)$:

$$y(x) = \frac{1}{2}|2x+3|^{\frac{1}{2}}\ln|2x+3| + C|2x+3|^{\frac{1}{2}}, \qquad \forall x \in (-\tfrac{3}{2}, \infty).$$

Using the initial condition ($-1 \in (-\frac{3}{2}, \infty)$)

$$y(-1) = 0,$$

yields

$$0 = C,$$

and therefore the solution is (see the graph in Figure 0.1)

$$y(x) = \frac{1}{2}|2x+3|^{\frac{1}{2}}\ln|2x+3|, \qquad \forall x \in (-\tfrac{3}{2}, \infty).$$