Solutions for homework 1

1 Introduction to Differential Equations

1.1 Differential Equation Models

The phrase "y is proportional to x" implies that y is related to x via the equation $y = kx$, where k is a constant. In a similar manner, "y is proportional to the square of x" implies $y = kx^2$, "y is proportional to the product of x and z" implies $y = kxz$, and "y is inverse proportional to the cube of x" implies $y = \frac{k}{x^3}$. for example, when Newton proposed that the force of attraction of one body on another is proportional to the product of masses and inversely proportional to the square of the distance between them, we can immediately write

$$F = \frac{GMm}{r^2},$$

where G is the constant of proportionality, usually known as the universal gravitational constant. In the following Exercises, use these ideas to model each application with a different equation. All rates are assumed to be with respect to time.

1. *The rate of growth of bacteria in a petri dish is proportional to the number of bacteria in the dish.*

Let $y(t)$ be the number of bacteria at time t. The rate of change of the number of bacteria is $y'(t)$. Since this rate of change is given to be proportional to $y(t)$, the resulting differential equation is $y'(t) = ky(t)$. Note that k is a positive constant since $y'(t)$ must be positive (i.e., the number of bacteria is growing).

2. *The rate of growth of a population of field mice is inversely proportional to the square root of the population.*

Let $y(t)$ be the number of the field mice at time t. Then rate of change of the number of mice is $y'(t)$. Since this rate of change is given to be inversely proportional to the square root of $y(t)$, the resulting differential equation is $y'(t) = k/\sqrt{y(t)}$. Note that k is a positive constant since $y'(t)$ must be positive (i.e. the number of mice is growing).

5. *The rate of decay of a certain substance is inversely proportional to the amount of substance remaining.*

Let $y(t)$ be the quantity of material at time t. The rate of change is $y'(t)$. Since this rate of change (decay) is given to be inversely proportional to $y(t)$, the resulting differential equation is $y'(t) = -k/y(t)$. Note that k is a positive constant since $y'(t)$ must be negative (i.e. the quantity of material is decreasing).

7. *A thermometer is placed in a glass of ice water and allowed to cool for an extended period of time. The thermometer is removed from the ice water and placed in a room having temperature $77°F$. The rate at which the thermometer warms is proportional to the difference in the room temperature and the temperature of the thermometer.*

Let $y(t)$ be the temperature of the thermometer at time t. The rate of change of the temperature is $y'(t)$. Since this rate of change is given to be proportional to the difference between the thermometer's temperature and that of the surrounding room (i.e. $77 - y(t)$), the resulting differential equation is $y'(t) = k(77 - y(t))$. Note that k is a positive constant since $y'(t)$ must be positive (i.e. the thermometer is warming) and since $77 - y(t) > 0$ (i.e. the thermometer is cooler than the surrounding room).

11 *The voltage drop across an inductor is proportional to the rate at which the current is changing with respect to time.*

Let $V(t)$ be the voltage drop across the inductor and $I(t)$ be the current at time t. The rate of change of the current is $I'(t)$. Since the voltage drop is proportional to the rate of change of I, we obtain the differential equation $V(t) = kI'(t)$, where k is a constant.

2 First-Order Equations

2.1 Differential Equations and Solutions

1. *Given the function ϕ, place the ordinary differential equation $\phi(t, y, y') = 0$ in normal form.*

$\phi(t, y, y') := t^2y' + (1 + t)y = 0$ must be solved for y'. We get

$$y' = -\frac{(1 + t)y}{t^2}.$$

In exercises 3 and 5, show that the given solution is a general solution of the differential equation. Use a computer or calculator to sketch the solutions foe the given values of the arbitrary constant. Experiment with different interval for t until you have a plot that shows what you consider to be the most important behavior of the family.

3. *Show that the equation $y' = -ty$ has the general solution $y(t) = Ce^{-\frac{1}{2}t^2}$, C = -3, -2, . . . ,3*

LHS: $y'(t) = -Cte^{-(1/2)t^2}$,

and

RHS: $-ty(t) = -tCe^{-(1/2)t^2}$, so $y' = -ty$, i.e. LHS≡RHS.

5. *Show that the equation* $y' + \frac{1}{2}y = 2\cos t$ *has the general solution* $y(t) = \frac{4}{5}\cos t + \frac{8}{5}\sin t + Ce^{-\frac{1}{2}t}$, *$C$ = -5, -4, ...,5*

If $y(t) = (4/5)\cos t + (8/5)\sin t + Ce^{-(1/2)t}$,then

$$y(t)' + (1/2)y(t) = [-(4/5)\sin t + (8/5)\cos t - (C/2)e^{-(1/2)t}] + (1/2)[(4/5)\cos t$$

$$+(8/5)\sin t + Ce^{-(1/2)t}] = 2\cos t.$$

In the Exercises 12-15, use the given general solution to find a solution of the differential equation having the given initial condition. Sketch the solution, the initial condition (IC), and discuss the solution's interval of existence.

12. *The ODE is* $y' + 4y = \cos t$, *with the general solution*

$$y(t) = \frac{4}{17}\cos t + \frac{1}{17}\sin t + Ce^{-4t}, \tag{1}$$

and IC $y(0) = -1$.

Taking $t = 0$ in (**??**) gives the particular solution

$$y(t) = \frac{4}{17}\cos t + \frac{1}{17}\sin t - \frac{21}{17}e^{-4t},$$

and the interval of existence is $(-\infty, \infty)$.

13. *The ODE is* $ty' + y = t^2$, *with the general solution*

$$y(t) = \frac{1}{3}t^2 + \frac{C}{t}, \tag{2}$$

and IC $y(1) = 2$.

Taking $t = 1$ in (**??**) gives the particular solution

$$y(t) = \frac{1}{3}t^2 + \frac{5}{3t}.$$

The interval of existence is $(0, \infty)$. (At zero the solution is not defined and the interval has to contain the time $t = 1$, where the initial condition is defined.)

15. *The ODE is* $y' = y(2 + y)$, *with the general solution*

$$y(t) = \frac{2}{-1 + Ce^{-2t}}, \tag{3}$$

and IC $y(0) = -3$.

Taking $t = 0$ in (**??**) gives the particular solution

$$y(t) = \frac{2}{-1 + \frac{e^{-2t}}{3}}.$$

The interval of existence is $(-\frac{\ln 3}{2}, \infty)$. (At $-\frac{\ln 3}{2}$ the particular solution is not defined and the interval has to contain the time $t = 0$, where the initial condition is defined.)

6. Numerical Methods

3 Section 6.1 Euler's method

3. *Consider the initial value problem*

$$y' = ty, \qquad y(0) = 1.$$

Hand-calculate the first five iterations of Euler's method with step size $h = 0.1$. *Arrange the result in a tabular form.*

k	t_k	y_k	$f(t_k, y_k)$	h	$f(t_k, y_k)h$
0	0.0	1.0	0.0	0.1	0.0
1	0.1	1.0	0.1	0.1	0.01
2	0.2	1.01	0.202	0.1	0.0202
3	0.3	1.0302	0.3091	0.1	0.0309
4	0.4	1.0611	0.4244	0.1	0.04244
5	0.5	1.1036	0.5518	0.1	0.05517

5. *Consider the initial value problem*

$$z' = x - 2z, \qquad z(0) = 1.$$

Hand-calculate the first five iterations of Euler's method with step size $h = 0.1$. *Arrange the result in a tabular form.*

k	x_k	z_k	$f(x_k, z_k)$	h	$f(x_k, z_k)h$
0	0.0	1.0	-2.0	0.1	-0.2
1	0.1	0.8	-1.5	0.1	-0.15
2	0.2	0.65	-1.1	0.1	-0.11
3	0.3	0.54	-0.78	0.1	-0.078
4	0.4	0.462	-0.524	0.1	-0.524
5	0.5	0.4096	-0.3192	0.1	-0.3192