Solutions for homework 13

1. Section 12.1 FOURIER SERIES: COMPUTATION OF FOURIER SERIES.

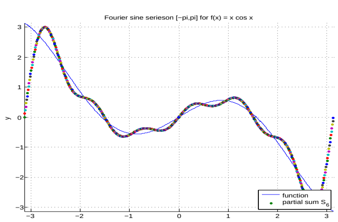
5. *Expand the function*

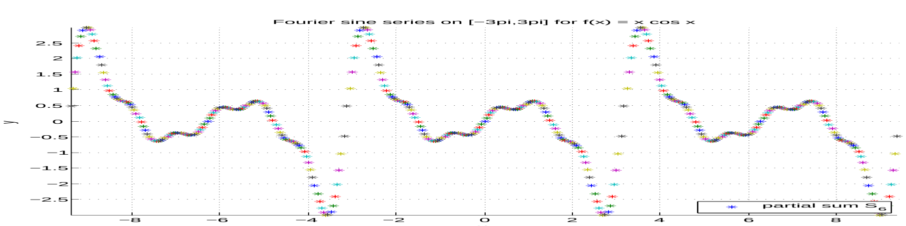
$$f(x) = x\cos x$$

in a Fourier series on the interval $-\pi \le x \le \pi$. *Pot the function and two partial sums of your choice over the interval* $-\pi \le x \le \pi$. *Plot the same partial sums over the interval* $-3\pi \le x \le 3\pi$.

`Solution.` $f(x)$ `is an odd function, and the Fourier series is (see exercise 12.3.31 below)`

$$f(x) = -\frac{1}{2}\sin x + \sum_{n=2}^{\infty} \frac{2n(-1)^n}{n^2-1}\sin nx.$$





7. *Find the Fourier series for the function*

$$f(x) = \begin{cases} 1+x, & \text{for } -1 \le x \le 0, \\ 1, & \text{for } 0 < x \le 1, \end{cases} \qquad \text{on } [-1,1].$$

Plot the function and two partial sums of your choice over the interval.

`Solution. With` $L = 1$ `we have`

$$a_0 = \int_{-1}^{1} f(x)\,dx = \int_{-1}^{0} (1+x)\,dx + \int_{0}^{1} dx = \int_{-1}^{0} (1+x)\,dx + 1 = \frac{3}{2},$$

`while for` $n \geq 1$

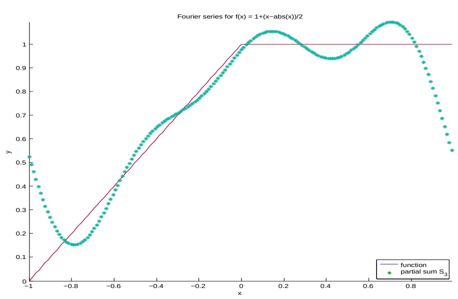
$$\begin{aligned} a_n &= \int_{-1}^{1} f(x)\cos(n\pi x)\,dx = \int_{-1}^{0} (1+x)\cos(n\pi x)\,dx + \int_{0}^{1} \cos(n\pi x)\,dx \\ &= \int_{-1}^{0} \frac{(1+x)}{n\pi} d(\sin n\pi x) + \int_{0}^{1} \frac{1}{n\pi} d(\sin n\pi x) \\ &= \frac{(1+x)}{n\pi} \sin n\pi x \Big|_{-1}^{0} - \frac{1}{n\pi}\int_{-1}^{0} \sin n\pi x\,dx \\ &= \frac{1}{n^2\pi^2} \cos n\pi x \Big|_{-1}^{0} = \frac{1}{n^2\pi^2}(1-(-1)^n), \end{aligned}$$

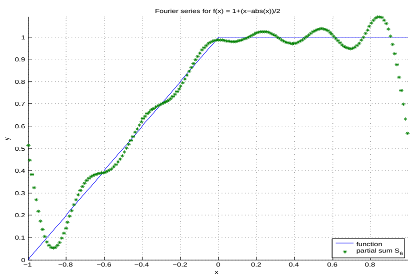
`and`

$$\begin{aligned} b_n &= \int_{-1}^{1} f(x)\sin(n\pi x)\,dx = \int_{-1}^{0} (1+x)\sin(n\pi x)\,dx + \int_{0}^{1} \sin(n\pi x)\,dx \\ &= -\int_{-1}^{0} \frac{(1+x)}{n\pi} d(\cos n\pi x) - \int_{0}^{1} \frac{1}{n\pi} d(\cos n\pi x) \\ &= -\frac{(1+x)}{n\pi} \cos n\pi x \Big|_{-1}^{0} + \frac{1}{n\pi}\int_{-1}^{0} \cos n\pi x\,dx - \frac{1}{n\pi}\cos n\pi x \Big|_{0}^{1} \\ &= -\frac{1}{n\pi} + \frac{1}{n^2\pi^2} \sin n\pi x \Big|_{-1}^{0} - \frac{1}{n\pi}((-1)^n - 1) = \frac{1}{n\pi}(-1)^{n+1}. \end{aligned}$$

`The Fourier series is`

$$f(x) \approx \frac{3}{4} + \sum_{m=0}^{\infty} \frac{2}{(2m+1)^2\pi^2} \cos(2m+1)\pi x + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n\pi} \sin n\pi x.$$





13. *Find the Fourier series for the function*

$$f(x) = \begin{cases} \cos \pi x, & \text{for } -1 \le x \le 0, \\ 1, & \text{for } 0 < x \le 1, \end{cases} \qquad \text{on } [-1, 1].$$

Plot the function and two partial sums of your choice over the interval.

Solution. With $L = 1$ we have

$$a_0 = \int_{-1}^{1} f(x)\, dx = \int_{-1}^{0} \cos \pi x \, dx + \int_{0}^{1} dx = 1.$$

Recall that

$$\cos \alpha \cos \beta = \frac{1}{2}\Big(\cos(\alpha - \beta) + \cos(\alpha + \beta) \Big).$$

Then for $n \ge 1$

$$\begin{aligned} a_n &= \int_{-1}^{1} f(x) \cos(n\pi x)\, dx = \int_{-1}^{0} \cos \pi x \cos(n\pi x)\, dx + \int_{0}^{1} \cos(n\pi x)\, dx \\ &= \frac{1}{2}\int_{-1}^{0} \cos(n-1)\pi x \, dx + \frac{1}{2}\int_{-1}^{0} \cos(n+1)\pi x\, dx + \int_{0}^{1} \cos(n\pi x)\, dx \\ &= \int_{-1}^{0} \frac{1}{2(n-1)\pi} d(\sin(n-1)\pi x) + \int_{-1}^{0} \frac{1}{2(n+1)\pi} d(\sin(n+1)\pi x) + \int_{0}^{1} \frac{1}{n\pi} d(\sin n\pi x) \\ &= 0. \end{aligned}$$

Recall also that

$$\cos \alpha \sin \beta = \frac{1}{2}\Big(\sin(\alpha + \beta) - \sin(\alpha - \beta) \Big).$$

Then

$$\begin{aligned} b_1 &= \int_{-1}^{1} f(x) \sin(\pi x)\, dx = \int_{-1}^{0} \cos \pi x \sin(\pi x)\, dx + \int_{0}^{1} \sin \pi x \, dx \\ &= \frac{1}{2}\int_{-1}^{0} \sin 2\pi x \, dx - \int_{0}^{1} \frac{1}{\pi} d(\cos \pi x) = \frac{-1}{4\pi} \cos 2\pi x \Big|_{-1}^{0} - \frac{1}{\pi} \cos \pi x \Big|_{0}^{1} \\ &= \frac{2}{\pi}, \end{aligned}$$

and for $n \geq 2$

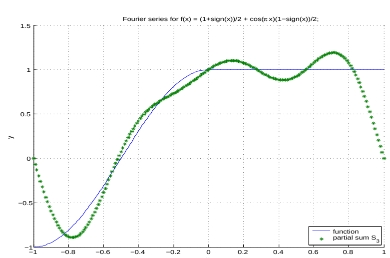
$$
\begin{aligned}
b_n &= \int_{-1}^{1} f(x)\sin(n\pi x)\,dx = \int_{-1}^{0} \cos \pi x \sin(n\pi x)\,dx + \int_{0}^{1} \sin(n\pi x)\,dx \\
&= \frac{1}{2}\int_{-1}^{0} \sin(n+1)\pi x\,dx + \frac{1}{2}\int_{-1}^{0} \sin(n-1)\pi x\,dx - \int_{0}^{1} \frac{1}{n\pi} d(\cos n\pi x) \\
&= \frac{-1}{2(n+1)\pi}\cos(n+1)\pi x\Big|_{-1}^{0} + \frac{-1}{2(n-1)\pi}\cos(n-1)\pi x\Big|_{-1}^{0} - \frac{1}{n\pi}\cos n\pi x\Big|_{0}^{1} \\
&= \frac{-1}{2(n+1)\pi}(1-(-1)^{n+1}) + \frac{-1}{2(n-1)\pi}(1-(-1)^{n-1}) - \frac{1}{n\pi}((-1)^n - 1).
\end{aligned}
$$

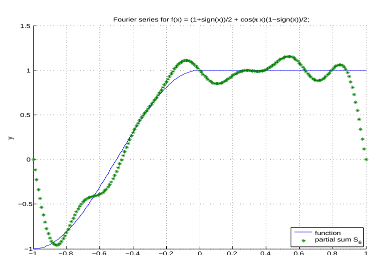
Hence

$$
b_{2m+1} = \frac{2}{(2m+1)\pi}, \qquad b_{2m} = -\frac{1}{\pi}\left(\frac{1}{(2m+1)} + \frac{1}{(2m-1)}\right)
$$

The Fourier series is

$$
f(x) \approx \frac{1}{2} + \frac{1}{2}\cos \pi x + \frac{2}{\pi}\sin \pi x + \sum_{n=2}^{\infty} b_n \sin n\pi x.
$$





17. *Expand the function $f(x) = x^2$ in Fourier series valid on the interval $-\pi \leq x \leq \pi$. Plot both f and the partial sum S_N for $N = 1, 3, 5, 7$. Observe how the graphs of the partial Fourier series approximates the graph of f. Plot the same graphs over the interval $-2\pi \leq x \leq 2\pi$.*

Solution. f is even, hence

$$a_0 = \frac{2}{\pi}\int_0^{\pi} x^2\,dx = \frac{2\pi^2}{3},$$

$$\text{for } n \geq 1: \quad a_n = \frac{2}{\pi}\int_0^{\pi} x^2\cos(nx)\,dx = \frac{2}{n\pi}\int_0^{\pi} x^2 d(\sin nx)$$

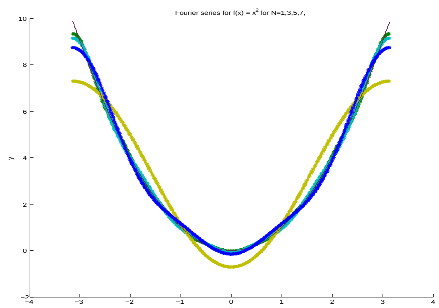
$$= \frac{2}{n\pi}x^2\sin nx\Big|_0^{\pi} - \frac{4}{n\pi}\int_0^{\pi} x\sin nx\,dx$$

$$= \frac{4}{n^2\pi}\int_0^{\pi} xd(\cos nx) = \frac{4}{n^2\pi}x\cos nx\Big|_0^{\pi} - \frac{4}{n^2\pi}\int_0^{\pi}\cos nx\,dx$$

$$= \frac{4}{n^2\pi}\pi(-1)^n = \frac{4(-1)^n}{n^2}.$$

The Fourier series is

$$f(x) \approx \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos(nx)$$



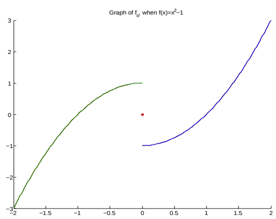
2. Section 12.3 FOURIER SERIES: FOURIER COSINE AND SINE SERIES.

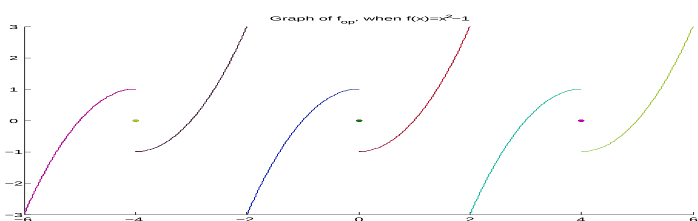
3. *Give a piecewise definition of f_o, the odd extension for f as defined on the given interval. Sketch the graph of f_o. Sketch the graph of f_{op} over three periods.*

$$f(x) = x^2 - 1, \quad [0, 2]$$

`Solution.` `The odd extension is`

$$f_o(x) = \begin{cases} f(x), & x \in [0,2] \\ 0, & x = 0 \\ -f(-x), & x \in [-2,0] \end{cases} = \begin{cases} x^2 - 1, & x \in [0,2] \\ 0, & x = 0 \\ -[(-x)^2 - 1] = -x^2 + 1, & x \in [-2,0] \end{cases}$$



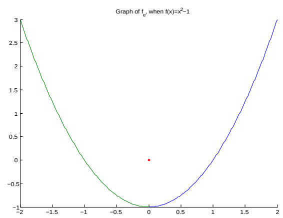


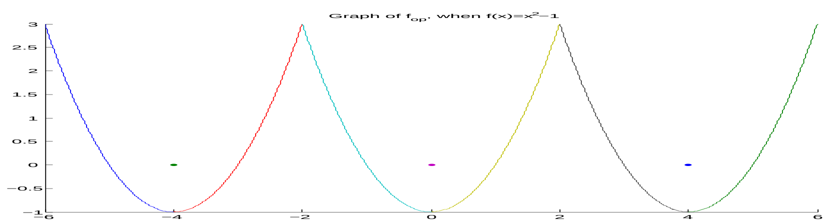
7. *Give a piecewise definition of f_e, the even extension for f as defined on the given interval. Sketch the graph of f_e. Sketch the graph of f_{ep} over three periods.*

$$f(x) = x^2 - 1, \quad [0, 2]$$

`Solution.` `The even extension is`

$$f_e(x) = \begin{cases} f(x), & x \in [0,2] \\ 0, & x = 0 \\ f(-x), & x \in [-2,0] \end{cases} = \begin{cases} x^2 - 1, & x \in [0,2] \\ 0, & x = 0 \\ (-x)^2 - 1 = x^2 - 1, & x \in [-2,0] \end{cases}$$





19. *Expand the given function in a Fourier cosine series valid on the interval $0 \leq x \leq \pi$. Plot the function and two partial sums of your choice over the interval $0 \leq x \leq \pi$. Plot the same partial sums and the function the series converges to over the interval $-3\pi \leq x \leq 3\pi$.*

$$f(x) = x \cos x$$

Solution. The cosine series is

$$f(x) \approx \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(\frac{n\pi x}{L})$$

where

$$a_n = \frac{2}{L} \int_0^L f(x) \cos(\frac{n\pi x}{L}) dx, \qquad n \geq 0.$$

Therefore

$$a_0 = \frac{2}{\pi} \int_0^{\pi} x \cos x dx = \frac{2}{\pi} \int_0^{\pi} x d(\sin x) = \frac{2}{\pi} \Big(x \sin x \Big|_0^{\pi} - \int_0^{\pi} \sin x dx \Big) = \frac{2}{\pi} \cos x \Big|_0^{\pi}$$
$$= -\frac{4}{\pi},$$

and

$$a_n = \frac{2}{\pi} \int_0^{\pi} x \cos x \cos(nx) dx,$$

Using the trigonometric identity

$$cos\alpha \cos\beta = \frac{1}{2}\Big(\cos(\alpha-\beta)+\cos(\alpha+\beta)\Big)$$

we obtain after integration by parts

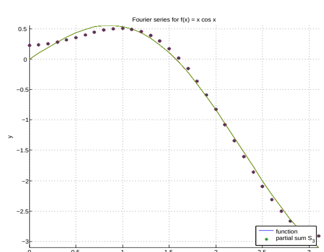
$$a_1 = \frac{1}{\pi}\int_0^\pi x\Big(1+\cos 2x\Big)dx = \frac{\pi}{2}+\frac{1}{2\pi}\int_0^\pi xd(\sin 2x) = \frac{\pi}{2}-\frac{1}{2\pi}\int_0^\pi \sin 2x\,dx$$
$$= \frac{\pi}{2}+\frac{1}{4\pi}\int_0^\pi d(\cos 2x) = \frac{\pi}{2},$$

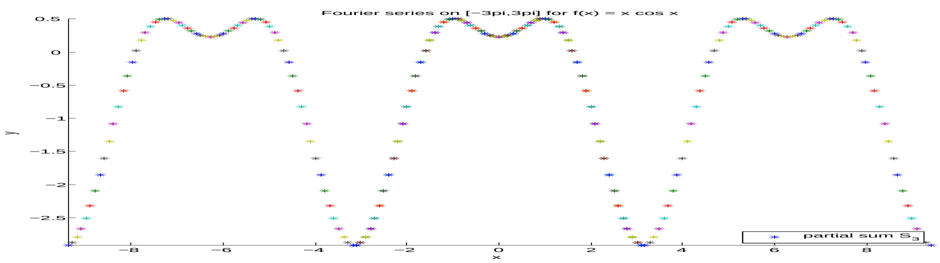
and for $n \geq 2$

$$\begin{aligned}
a_n &= \frac{1}{\pi}\int_0^\pi x\Big(\cos(n-1)x+\cos(n+1)x\Big)dx \\
&= \frac{1}{\pi}\int_0^\pi x\cos(n-1)x\,dx + \frac{1}{\pi}\int_0^\pi x\cos(n+1)x\,dx \\
&= \frac{1}{(n-1)\pi}\int_0^\pi xd\sin(n-1)x + \frac{1}{(n+1)\pi}\int_0^\pi xd\sin(n+1)x \\
&= -\frac{1}{(n-1)\pi}\int_0^\pi \sin(n-1)x\,dx - \frac{1}{(n+1)\pi}\int_0^\pi \sin(n+1)x\,dx \\
&= \frac{1}{(n-1)^2\pi}\int_0^\pi d\cos(n-1)x + \frac{1}{(n+1)^2\pi}\int_0^\pi d\cos(n+1)x \\
&= \frac{1}{(n-1)^2\pi}(\cos(n-1)\pi-1) + \frac{1}{(n+1)^2\pi}(\cos(n+1)\pi-1) \\
&= \frac{1}{(n-1)^2\pi}((-1)^{n+1}-1) + \frac{1}{(n+1)^2\pi}((-1)^{n+1}-1) \\
&= \Big(\frac{1}{(n-1)^2\pi}+\frac{1}{(n+1)^2\pi}\Big)((-1)^{n+1}-1)
\end{aligned}$$

Therefore

$$f(x) = -\frac{2}{\pi}+\frac{\pi}{2}\cos x+\sum_{n=2}^{\infty}\Big(\frac{1}{(n-1)^2\pi}+\frac{1}{(n+1)^2\pi}\Big)((-1)^{n+1}-1)\cos n\pi x$$





31. *Expand the given function in a Fourier sine series valid on the interval* $0 \le x \le \pi$*. Plot the function and two partial sums of your choice over the interval* $0 \le x \le \pi$*. Plot the same partial sums and the function the series converges to over the interval* $-3\pi \le x \le 3\pi$*.*

$$f(x) = x\cos x$$

Solution. The sine series is

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right) \equiv \sum_{n=1}^{\infty} b_n \sin nx$$

where, for $n \ge 1$

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx \equiv \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx = \frac{2}{\pi}\int_0^{\pi} x\cos x\sin nx\,dx.$$

Using the trigonometric identity

$$\cos\alpha\sin\beta = \frac{1}{2}\Big(\sin(\alpha+\beta) - \sin(\alpha-\beta)\Big)$$

we have

$$\begin{aligned} b_1 &= \frac{2}{\pi}\int_0^{\pi} x\cos x\sin x\,dx = \frac{1}{\pi}\int_0^{\pi} x\sin 2x\,dx \\ &= \frac{1}{\pi}\int_0^{\pi}\frac{-x}{2}d(\cos 2x) = \frac{-x}{2\pi}\cos 2x\Big|_0^{\pi} + \frac{1}{2\pi}\int_0^{\pi}\cos 2x\,dx \\ &= -\frac{1}{2}, \end{aligned}$$

and for $n \geq 2$

$$
\begin{aligned}
b_n &= \frac{2}{\pi}\int_0^{\pi} x\cos x \sin nx\, dx = \frac{1}{\pi}\int_0^{\pi} x\big(\sin(n+1)x + \sin(n-1)x\big)\, dx \\
&= \frac{1}{\pi}\int_0^{\pi} x\big(\frac{-1}{n+1} d(\cos(n+1)x) + \frac{-1}{n-1} d(\cos(n-1)x)\big) \\
&= \frac{1}{\pi}\, x\Big(\frac{-1}{n+1}\cos(n+1)x + \frac{-1}{n-1}\cos(n-1)x\Big)\Big|_0^{\pi} \\
&\quad - \frac{1}{\pi}\int_0^{\pi} \big(\frac{-1}{n+1}\cos(n+1)x + \frac{-1}{n-1}\cos(n-1)x\big)\, dx \\
&= \Big(\frac{(-1)^{n+2}}{n+1} + \frac{(-1)^n}{n-1}\Big) \\
&\quad - \frac{1}{\pi}\int_0^{\pi} \big(\frac{-1}{n+1}\cos(n+1)x + \frac{-1}{n-1}\cos(n-1)x\big)\, dx \\
&= (-1)^n \frac{2n}{(n+1)(n-1)} \\
&\quad - \frac{1}{\pi}\int_0^{\pi} \Big(\frac{-1}{(n+1)^2} d(\sin(n+1)x) + \frac{-1}{(n-1)^2} d(\sin(n-1)x)\Big) \\
&= \frac{2n(-1)^n}{n^2-1}.
\end{aligned}
$$

Therefore

$$
f(x) = -\frac{1}{2}\sin x + \sum_{n=2}^{\infty} \frac{2n(-1)^n}{n^2-1}\sin nx.
$$