Solutions for homework 12

1. Section 9.3 LINEAR SYSTEMS WITH CONSTANT COEFFICIENTS: PHASE PLANE PORTRAITS.

1. *For the* 2×2 *matrix*

$$A = \begin{pmatrix} -10 & -25 \\ 5 & 10 \end{pmatrix}$$

use $p(\lambda) = \lambda^2 - T\lambda + D$, *where* $T = \operatorname{tr}(A)$ *and* $D = \det(A)$, *to compute the characteristic polynomial. Then use* $p(\lambda) = \det(A - \lambda I)$ *to calculate the characteristic polynomial a second time and compare the results.*

`Solution.`

$$\lambda^2 - Tr(A)\lambda + \det(A) = \lambda^2 + 25$$

and

$$\det \begin{pmatrix} -10-\lambda & -25 \\ 5 & 10-\lambda \end{pmatrix} = (-10-\lambda)(10-\lambda) + 125 = \lambda^2 + 25.$$

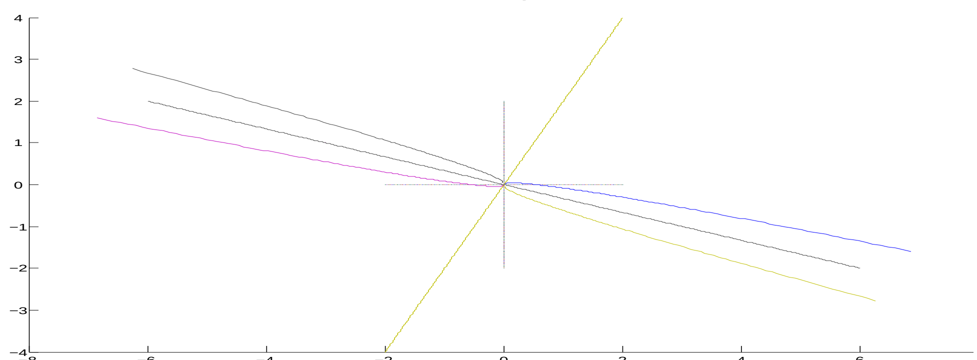
11. *The general solution of* $\mathbf{y}' = A\mathbf{y}$ *is*

$$\mathbf{y}(t) = C_1 e^{t} \begin{pmatrix} -1 \\ -2 \end{pmatrix} + C_2 e^{2t} \begin{pmatrix} 3 \\ -1 \end{pmatrix}.$$

Without the help of a computer or calculator, sketch the half-line solutions generated by each exponential term of the solution. Then, sketch a rough approximation of a solution in each region determined by the half-line solutions. Use arrows to indicate the direction of motion on all solutions. Classify the equilibrium point as a saddle, a nodal sink, or a nodal source.

`Solution.`

Both eigenvalues are real and positive, therefore the **origin** is a nodal source.



13. *The general solution of* $\mathbf{y}' = A\mathbf{y}$ *is*

$$\mathbf{y}(t) = C_1 e^{-3t} \begin{pmatrix} -4 \\ 1 \end{pmatrix} + C_2 e^{-t} \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

Without the help of a computer or calculator, sketch the half-line solutions generated by each exponential term of the solution. Then, sketch a rough approximation of a solution in each region determined by the half-line solutions. Use arrows to indicate the direction of motion on all solutions. Classify the equilibrium point as a saddle, a nodal sink, or a nodal source.

`Solution.` Both eigenvalues are real and negative, therefore the **origin** is a nodal sink.

15. *The general solution of* $\mathbf{y}' = A\mathbf{y}$ *is*

$$\mathbf{y}(t) = C_1 e^{3t} \begin{pmatrix} 4 \\ 1 \end{pmatrix} + C_2 e^{t} \begin{pmatrix} 1 \\ 5 \end{pmatrix}.$$

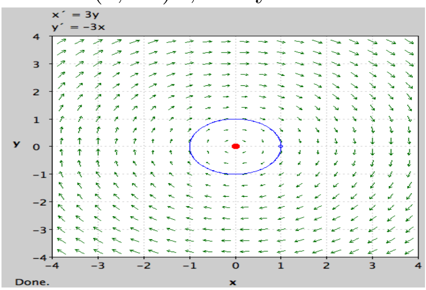
Without the help of a computer or calculator, sketch the half-line solutions generated by each exponential term of the solution. Then, sketch a rough approximation of a solution in each region determined by the half-line solutions. Use arrows to indicate the direction of motion on all solutions. Classify the equilibrium point as a saddle, a nodal sink, or a nodal source.

`Solution.` The eigenvalues are both real positive, hence the **origin** is a nodal source.

17. *Verify that the equilibrium point at the origin is a center by showing that the real parts of the system's complex eigenvalues are zero. Calculate and sketch the vector generated by the RHS of the system at the point (1,0). Use this to help sketch the elliptic solution trajectory for the system passing through the point* $(1, 0)$. *Draw arrows on the solution, indicating the direction of motion. Use you numerical solver to check your result.*

$$\mathbf{y}' = \begin{pmatrix} 0 & 3 \\ -3 & 0 \end{pmatrix} \mathbf{y}.$$

`Solution.` The characteristic equation is $\lambda^2 + 9 = 0$, hence the eigenvalues are $\lambda_{1,2} = \pm 3i$, hence the origin is a center. When $\mathbf{y} = (1, 0)^T$, the RHS is $(0, -3)^T$, easily seen from the pplane plot.



21. *Calculate the eigenvalues to determine whether the equilibrium point is a spiral sink or a source. Calculate and sketch the vector generated by the right-hand side of the system at the point* $(1, 0)$. *Use this to help sketch the solution trajectory for the system passing through the point (1,0). Draw arrows on the solution, indicating the direction of motion. Use your numerical solver to check your result.*

$$\mathbf{y}' = \begin{pmatrix} -1 & 1 \\ -5 & 3 \end{pmatrix} \mathbf{y}$$

`Solution.` The characteristic polynomial is

$$\lambda^2 - Tr(A)\lambda + \det(A) = \lambda^2 - 2\lambda + 2$$

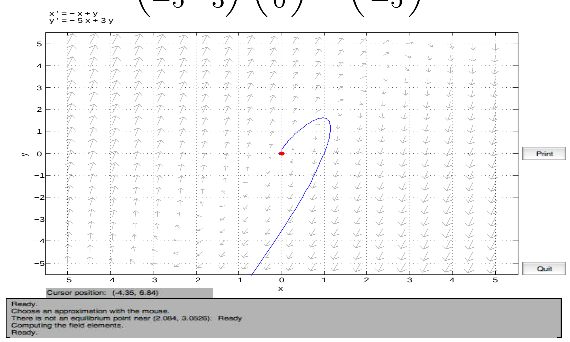
and the eigenvalues are

$$\lambda_{1,2} = 1 \pm i.$$

Since the real part is positive, the equilibrium point $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is a spiral source.

The vector generated by RHS at $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is

$$\begin{pmatrix} -1 & 1 \\ -5 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ -5 \end{pmatrix}.$$



23. *Calculate the eigenvalues to determine whether the equilibrium point is a spiral sink or a source. Calculate and sketch the vector generated by the right-hand side of the system at the point* $(1, 0)$. *Use this to help sketch the solution trajectory for the system passing through the point (1,0). Draw arrows on the solution, indicating the direction of motion. Use your numerical solver to check your result.*

$$\mathbf{y}' = \begin{pmatrix} -3 & 2 \\ -4 & 1 \end{pmatrix} \mathbf{y}$$

`Solution.` The characteristic polynomial is

$$\lambda^2 - Tr(A)\lambda + \det(A) = \lambda^2 + 2\lambda + 2$$

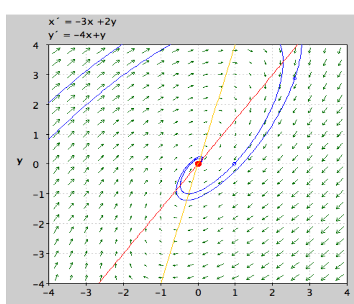
and the eigenvalues are

$$\lambda_{1,2} = -1 \pm i.$$

Since the real part is negative, the equilibrium point, i.e. the **origin** $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is a spiral sink.

The vector generated by RHS at $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is

$$\begin{pmatrix} -3 & 2 \\ -4 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -3 \\ -4 \end{pmatrix}.$$



2. Section 10.1 Nonlinear systems: The linearization of a nonlinear system.

3. *Consider the system*

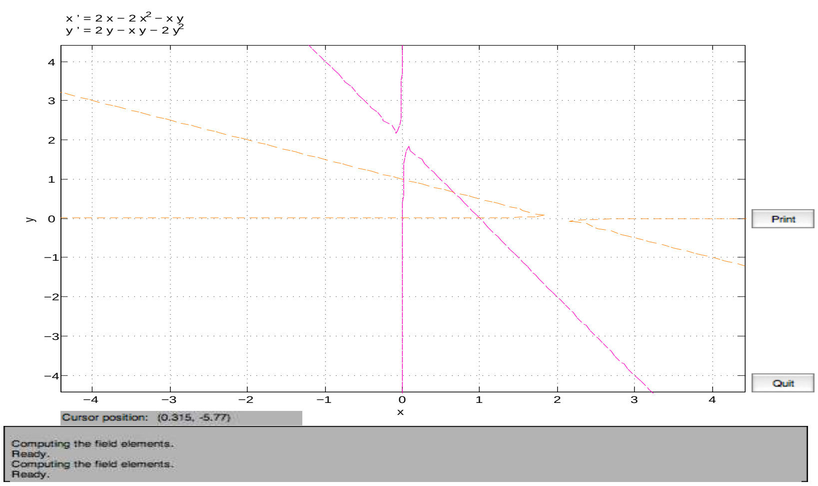
$$x' = 2x - 2x^2 - xy$$
$$y' = 2y - xy - 2y^2$$

(i) Sketch the nullclines. Use a distinctive marking for each nullcline so they can be distinguished.
(ii) Use analysis to find the equilibrium points for the system. Label each equilibrium point on your sketch with its coordinates.
(iii) Use the Jacobian to classify each equilibrium point (spiral source, nodal sink, etc.).

`Solution.`

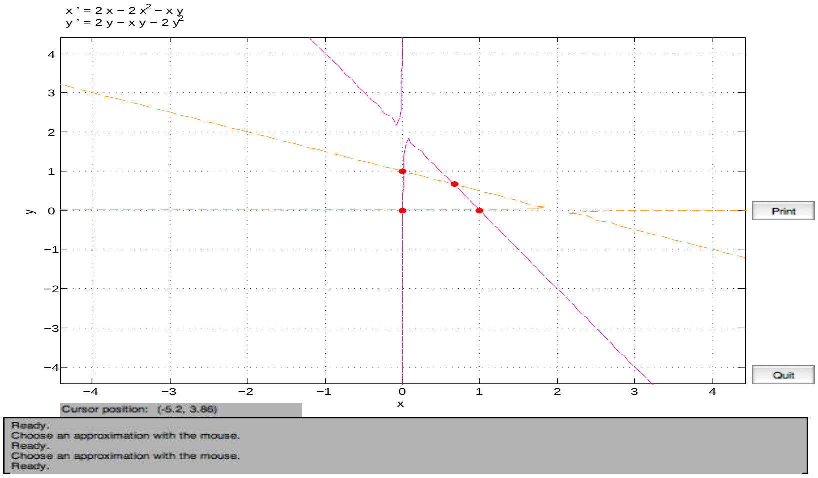
(i) The $x-$ and $y-$nullclines are

$$x = 0 \quad \text{or} \quad y = -2x + 2 \qquad (\ x\text{-nullcline}\ )$$
$$y = 0 \quad \text{or} \quad y = -\frac{1}{2}x + 1 \qquad (\ y\text{-nullcline}\ )$$



(ii) The equilibrium point are the intersection of the nullclines:

$$(0,1), \qquad (\frac{2}{3}, \frac{2}{3}), \qquad (1,0), \qquad (0.0).$$



(iii) The Jacobian is

$$J(x,y) = \begin{pmatrix} 2-4x-y & -x \\ -y & 2-x-4y \end{pmatrix}$$

(A). At $(0,1)$ this is

$$J(0,1) = \begin{pmatrix} 1 & 0 \\ -1 & -2 \end{pmatrix},$$

so the characteristic equation

$$0 = \lambda^2 - Tr(J)\lambda + \det(J) \equiv \lambda^2 + \lambda - 2$$

yields the eigenvalues: one positive, one negative

$$\lambda_1 = -2, \qquad \lambda_2 = 1,$$

and therefore $(0,1)$ is a saddle point.

(B). At $(\frac{2}{3}, \frac{2}{3})$ this is

$$J\Big(\frac{2}{3}, \frac{2}{3}\Big) = \begin{pmatrix} -\frac{4}{3} & -\frac{2}{3} \\ -\frac{2}{3} & -\frac{2}{3} \end{pmatrix},$$

so the characteristic equation

$$0 = \lambda^2 - Tr(J)\lambda + \det(J) \equiv \lambda^2 + 2\lambda - \frac{4}{9}$$

yields the eigenvalues: both negative

$$\lambda_1 = -1 - \frac{\sqrt{3}}{3} \approx -1.5774, \qquad \lambda_2 = -1 + \frac{\sqrt{3}}{3} \approx -0.4226,$$

and therefore $(\frac{2}{3}, \frac{2}{3})$ is a nodal sink.

(C). At $(1, 0)$ this is

$$J(1,0) = \begin{pmatrix} -2 & -1 \\ 0 & 1 \end{pmatrix},$$

so the characteristic equation

$$0 = \lambda^2 - Tr(J)\lambda + \det(J) \equiv \lambda^2 + \lambda - 2$$

yields the eigenvalues: one negative, one positive

$$\lambda_1 = -2, \qquad \lambda_2 = 1,$$

and therefore $(1, 0)$ is a saddle point.

(D). At $(0, 0)$ this is

$$J(0,0) = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix},$$

so the characteristic equation

$$0 = \lambda^2 - Tr(J)\lambda + \det(J) \equiv \lambda^2 - 4\lambda + 4$$

yields the eigenvalues: both positive

$$\lambda_1 = 2, \qquad \lambda_2 = 2,$$

and therefore $(0, 0)$ is a source.

7. *Consider the system*

$$\begin{aligned} x' &= y \\ y' &= -\sin x - y \end{aligned}$$

(i) Sketch the nullclines. Use a distinctive marking for each nullcline so they can be distinguished.

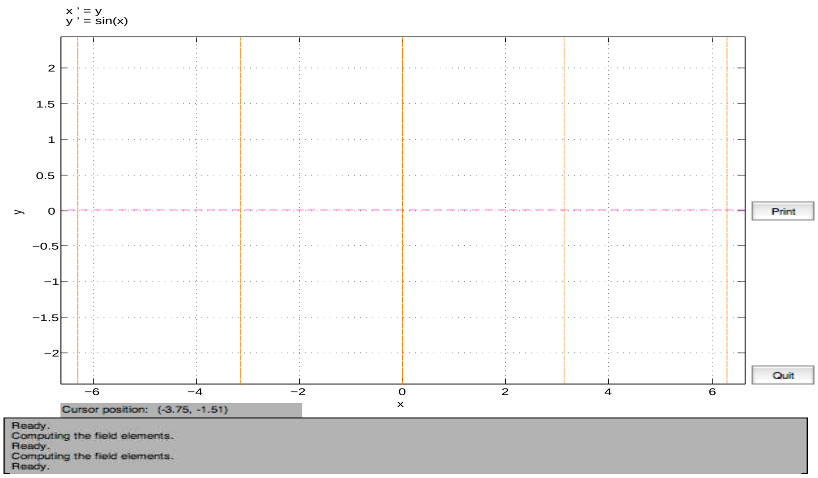
(ii) Use analysis to find the equilibrium points for the system. Label each equilibrium point on your sketch with its coordinates.

(iii) Use the Jacobian to classify each equilibrium point (spiral source, nodal sink, etc.).

`Solution.`

(i) The $x-$ and $y-$nullclines are

$$\begin{aligned} y &= 0 \qquad\qquad (\ x\text{-nullcline}\ ) \\ y &= -\sin x \qquad (\ y\text{-nullcline}\ ) \end{aligned}$$

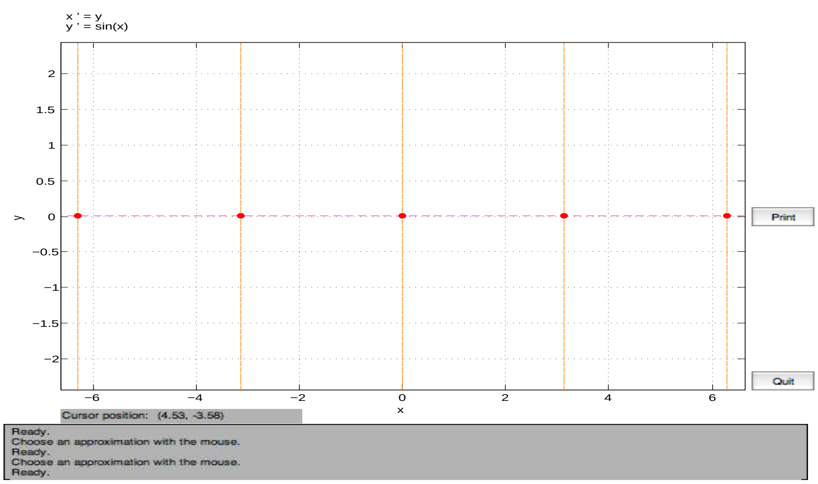


(ii) The equilibrium point are the intersection of the nullclines:

$$\sin(x) = 0 \Rightarrow x_k = k\pi(\text{ for any integer } k\ ), \qquad y = 0,$$

i.e.,

$$(k\pi, 0), \qquad \forall k \in \mathbb{Z}.$$



(iii) The Jacobian is

$$J(x,y) = \begin{pmatrix} 0 & 1 \\ -\cos(x) & -1 \end{pmatrix}$$

(A). At $((2\ell+1)\pi, 0)$ this is

$$J((2\ell+1)\pi, 0) = \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix}$$

so the characteristic equation

$$0 = \lambda^2 - Tr(J)\lambda + \det(J) \equiv \lambda^2 + \lambda - 1$$

yields the eigenvalues: one negative, one positive

$$\lambda_1 = -\frac{1}{2} - \frac{\sqrt{5}}{2} \approx -1.6180, \qquad \lambda_2 = -\frac{1}{2} + \frac{\sqrt{5}}{2} \approx 0.6180,$$

and therefore $((2\ell+1)\pi, 0)$ is a saddle.

(B). At $(2\ell\pi, 0)$ this is

$$J(2\ell\pi, 0) = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}$$

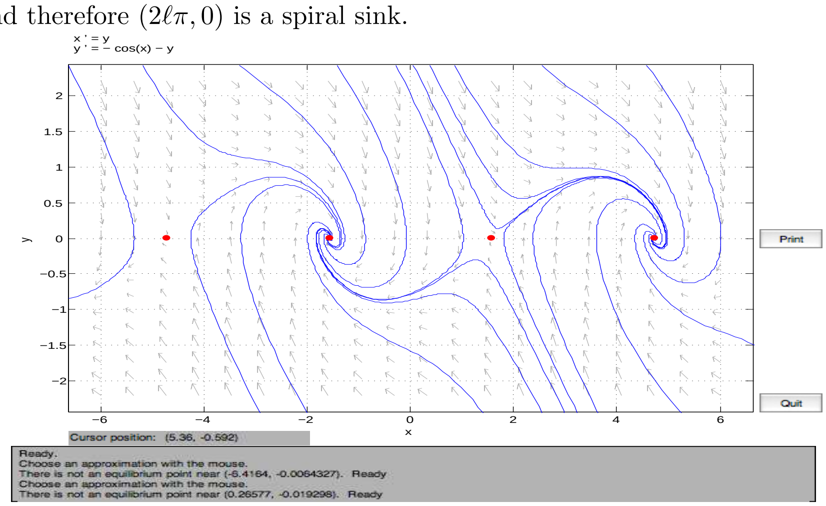
so the characteristic equation

$$0 = \lambda^2 - Tr(J)\lambda + \det(J) \equiv \lambda^2 + \lambda + 1$$

yields the eigenvalues: complex, with negative real part

$$\lambda_1 = -\frac{1}{2} - i\frac{\sqrt{3}}{2}, \qquad \lambda_2 = -\frac{1}{2} + i\frac{\sqrt{3}}{2},$$

and therefore $(2\ell\pi, 0)$ is a spiral sink.



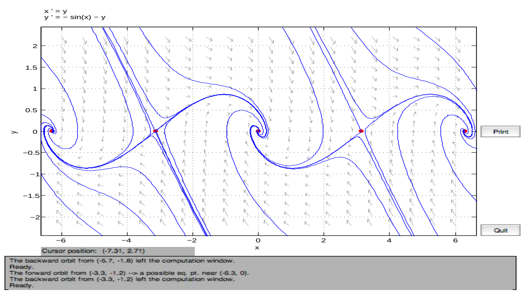
15. *Use your numerical solver to compare the phase portrait of the nonlinear system*

$$x' = y$$
$$y' = -\sin x - y$$

with that of its linearization near the equilibrium point

$$(2\pi, 0).$$

`Solution.`



The linearization at $(2\pi, 0)$ is the system $\mathbf{u}' = J\mathbf{u}$,

$$\mathbf{u}' = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix} \mathbf{u}$$